ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 10 (2014) No. 1, pp. 27-33

Promising technique for analytic treatment of nonlinear fifth-order equations

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(*Received April 10 2013, Accepted December 21 2013*)

Abstract. In this paper, a new technique called variational iteration method-II (VIM-II) is applied to find the analytical approximate solution of nonlinear fifth-order KDV equations. This method is independent of any small parameters. The algorithm overcomes the difficulty arising in calculating nonlinear intricately terms. Besides, it provides us with a simple way to ensure the convergence of solution series so that we can always get enough accuracy in approximations it also is capable of reducing the size of calculation.

Keywords: variational iteration method-II, analytic solution, fifth-order KDV equation

1 Introduction

Differential equations are widely used to describe physical problems. In most cases, the exact solution of these problems may not be available. In addition, it is much easier computing and analyzing these solutions by means of the numerical methods without wasting time or spending money for experimenting problems. Alternatively, the numerical methods can provide approximate solutions rather than the exact solutions. But most of these methods have low accuracy and are highly time consuming. Reaching to a high accurate approximation for linear and nonlinear equations has always been important while it challenges tasks in science and engineering. Therefore several numbers of approximate methods have been established like inverse scattering method[28], Adomian's decomposition method[1, 4], the δ-expansion method[9] ,variational iteration method and its modifies[8, 12, 23–26], homotopy analysis method [22],variational approach[6, 7], Hamiltonian approach[27], homotopy pertrubation method[2, 13–17]. We consider a new analytical method of nonlinear problems called the variational iteration method-II[11, 18], which in the case of comparing with VIM, not uses Lagrange multiplier as variational methods do and not requires small parameter in equations as the perturbation techniques. VIM-II has been shown to solve a large class of nonlinear problems with approximations converging to solutions rapidly, effectively, easily, and accurately. The method used gives rapidly convergent successive approximations. In this work, we implement the VIM-II for finding the exact solutions of nonlinear fifth-order Korteveg-de Vries (FKdV) partial differential equations. The general form of FKdV equation is [10]:

$$u_t - u_{xxxxx} = f(x, t, u, u^2, u_x, u_{xx}, u_{xxx}), \tag{1}$$

which occurs, for example in the theory of magneto- acoustic waves in plasmas[5] and in the theory of shallow water waves with surface tension[20].

2 Basic concept of VIM-II

In the First of all for clarifying the idea of the proposed method, the basic concept of Variational Iteration Method-II[11, 18] is firstly treated. A general differential equation is considered at the following form.

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$$Lu(x,t) + Nu(x,t) = f(x,t), \tag{2}$$

where L is a linear operator, N is a nonlinear operator and $f(x,t)$ an inhomogeneous term. Suppose x and t_4 are two independent variables; According to the definition of Laplace Transform, we have

$$\ell[u(x,t);s] = \int_0^\infty e^{-st}u(x,t)dt, \tag{3}$$

where ℓ is Laplace operator and

$$U(x,s) = \ell[u(x);s]. \tag{4}$$

To simplify the discussion of RVIM, introducing the new linear function as $h(u(x,t)) = f(x,t) - N(u(x,t))$, Eq. (2) can be rewritten as a correction functional:

$$L(u(x,t)) = h(t,x,u). \tag{5}$$

Now, in the presence of VIM based on the new idea of using the Laplace Transform, by taking Laplace Transform to both sides of the above equation and introducing artificial initial conditions to zero in the usual way, final result is obtained as follows:

$$\ell[L(u(x,t))] = U(x,s)p(s), \tag{6}$$

where $P(s)$ is a polynomial with the degree of the highest order derivative in Eq. (6).

$$\ell[L(u(x,t))] = U(x,s)p(s) = \ell[h(x,t,u)], \tag{7}$$

$$U(x,s) = \frac{\ell[h\{u,x,t\}]}{p(s)}, \tag{8}$$

$$U(x,s) = H(x,s)D(s) = \ell\{(d(t) * h(x,t,u))\}, \tag{9}$$

in which, $D(s) = l/p(s)$ and $\ell[\{h(x,t,u)\}] = H(x,s)$. Now, by applying the Inverse Laplace Transform on both sides of Eq. (9), $u(x,t)$ is determined.

$$u(x,t) = \int_0^t d(t-\varepsilon)h(x,\varepsilon,u)d\varepsilon. \tag{10}$$

Subsequently, the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t d(t-\varepsilon)h(x,\varepsilon,u_n)d\varepsilon, \tag{11}$$

in which, $u_0(x,t)$ is the initial solution. It is to be mentioned that, the Lagrange multiplier in the He's variational iteration method is $\lambda(\varepsilon) = d(t-\varepsilon)$ as shown in [19]. The initial values are usually used for selecting the zeroth approximation u_0. With u_0 determined, then several approximations $u_n n > 0$, follow immediately. Consequently, the exact solution could be obtained as follows:

$$ux, t = \lim_{n\to\infty} u_n.$$

3 Implement of method

To demonstrate the effectiveness of the method, we consider Eq. (1) with given initial condition.

3.1 Example 1.

Consider a special case of the fifth-order KDV equation[21].

$$u_t + u_x + u^2 u_{xx} + u_x u_{xx} - 20u^w u_{xxx} + u_{xxxxx} = 0. \tag{12}$$

With the initial condition:

$$u(x,0) = \frac{1}{x}. \tag{13}$$

Here, auxiliary linear operator is selected as $L_t u(x,t) = u_t$. By using the Eq. (9) we have the following operator form equation:

$$L_t u(x,t) = u_t = \overbrace{-u_x - u^2 u_{xx} - u_x u_{xx} + 20u^2 u_{xxx} - u_{xxxxx}}^{w(u(x,t))}. \tag{14}$$

The VIM-II is implemented to Eq. (12). First, according to the method, by applying Laplace Transform to identify the Lagrange multiplier, we have: $\lambda = 1$.

So, variational iteration algorithm-II is derived:

$$u_{n+1} = u_0 + \int_0^t (-u_{nx} - {u_n}^2 u_{nxx} - u_{nx} u_{nxx} + 20{u_n}^2 u_{nxx} - u_{nxxxx}) d\varepsilon. \tag{15}$$

Therefore we begin with accordingly by the Eq. (15) one can get the higher order approximation of the exact solution as the following relations;

$$\begin{aligned} u_1(x,t) = {} & \frac{1}{x} + \frac{t}{x^2}, \\ u_2(x,t) = {} & \frac{1}{x} + + \frac{t}{x^2} + + \frac{2t^2}{x^3} - \left(\frac{1}{x} + \frac{t}{x^2}\right)^2 \left(\frac{2}{x^3} + \frac{6t}{x^4}\right) t - \\ & \left(-\frac{1}{x^2} - \frac{2t}{x^3}\right)\left(\frac{2}{x^3} + \frac{6t}{x^4}\right) t + 20\left(\frac{1}{x} + \frac{t}{x^2}\right)^2 \left(-\frac{6}{x^4} - \frac{24t}{x^5}\right) t + \frac{120t}{x^6} + \frac{720t^2}{x^7}, \end{aligned}$$

which is exactly the same as obtained by Adomian's decomposition method[21]. To verify numerically whether the proposed VIM-II method leads to higher accuracy, we can evaluate the numerical solutions. Using n^{th} approximation shows the high degree of accuracy and un the n^{th} approximation is accurate for quite low of $n (n = 4)$. The behavior of the solutions obtained by the VIM-II method is shown in comparison with exact solution, Fig. 1.

3.2 Example 2.

Consider an equation with initial condition is given by[21]:

$$u_t + uu_x - uu_{xxx} + u_{xxxxx} = 0, \quad u(x,0) = e^x. \tag{16}$$

At first, auxiliary linear operator is selected as:

$$L_t u(x,t) = u_t = \overbrace{-uu_x + uu_{xxx} - u_{xxxxx}}^{w(u(x,t))}, \tag{17}$$

we obtain the following VIM's iteration formula in t-direction:

$$u_{n+1} = u_0 + \int (-u_n u_{nx} + u_n u_{nxxx} - u_{nxxxxx}) d\varepsilon. \tag{18}$$

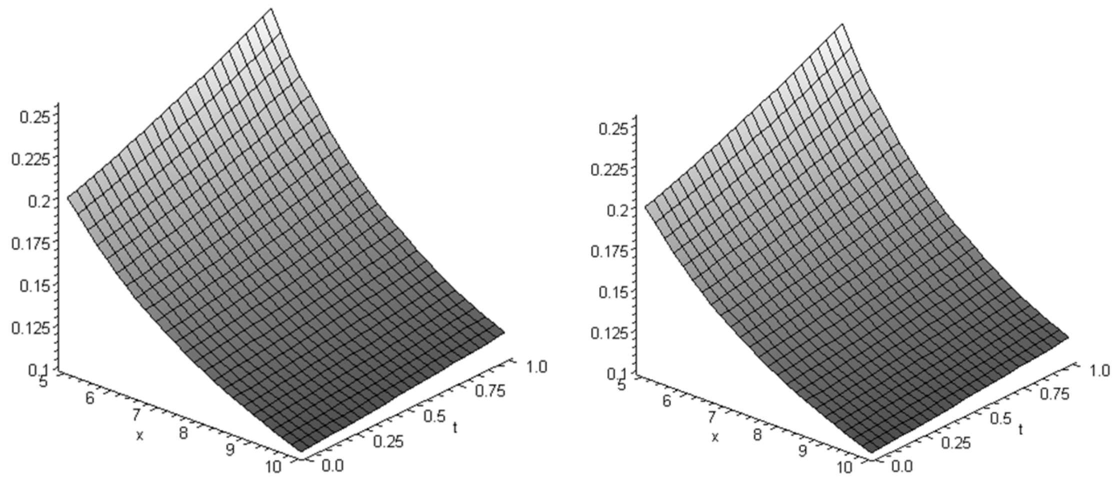

Fig. 1. The surfaces on both columns respectively show the solutions, $u(x,t)$, for VIM-II on the right and ADM

The subscript n indicates the n^th approximation of the solution; we can obtain the other components with selecting the initial approximation as:

$$u_0(x,t) = e^x. \tag{19}$$

So with the iteration formula (18), we obtain the following successive approximations:

$$\begin{aligned} u_1(x,t) =& \ e^x - e^x t, \\ u_2(x,t) =& \ e^x - e^x t + e^x t^2, \\ u_3(x,t) =& \ e^x - e^x t + e^x t^2 - e^3 t^3, \end{aligned}$$

$u(x,t)$ in a closed form is found to be:

$$u(x,t) = e^x \left(1 + 2 + \frac{1}{2!} \dots\right) = e^{x+t}.$$

and so on. Using the above terms, in Fig. 2, $u(x,t)$ and exact solution [21] is drawn.

3.3 Example 3.

At the end, as an example of the application of the self-canceling phenomena[3], let's seek the solution of the inhomogeneous FKdV equation, as follows:

$$u_t - uu_x + u_{xxx} = \cos(x) + 2t\sin(x) + \frac{t^2}{2}\sin(2x), \quad u(x,0) = 0. \tag{20}$$

As said before, auxiliary linear operator which that plays an important role in choosing the initial approximation of the solution, is selected as

$$L_t u(x,t) = u_t = \overbrace{uu_x - u_{xxx} + u_{xxxxx} + \cos(x) + 2t\sin(x) + \frac{t^2}{2}\sin(2x)}^{w(u(x,t))} \tag{21}$$

VIM's iteration formula in t-direction can be readily obtained.

$$u_{n+1} = u_0 + \int_0^t \left(uu_x - u_{xxx} + u_{xxxxx} + + \cos(x) + 2\varepsilon\sin(x) + \frac{\varepsilon^2}{2}\sin(2x)\right) d\varepsilon, \tag{22}$$

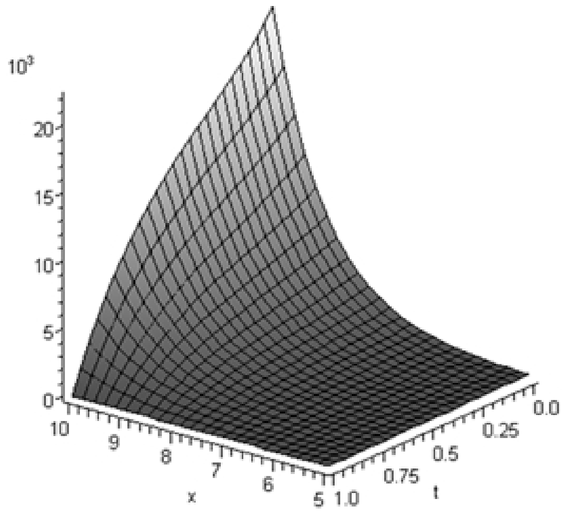

Fig. 2. The surfaces on both columns respectively show the solutions, $u(x,t)$, for VIM-II on the right and ADM on the left

we start with the initial approximation as

$$u_0(x,t) = t\cos(x), \tag{23}$$

and with the iteration Eq. (18), we obtain the following successive approximations

$$u_2(x,t) = t\cos(x), \tag{24}$$

and so on. In the same way the rest of the components of the iteration formula can be obtained. Fig. 3 show the results obtained from VIM-II along with the exact solution, revealing a high level of agreement between the two results shown. It is evident from the curves plotted that the exact solution and the obtained solutions from VIM-II completely overlay each other and the level of agreement between the results is therefore excellent.

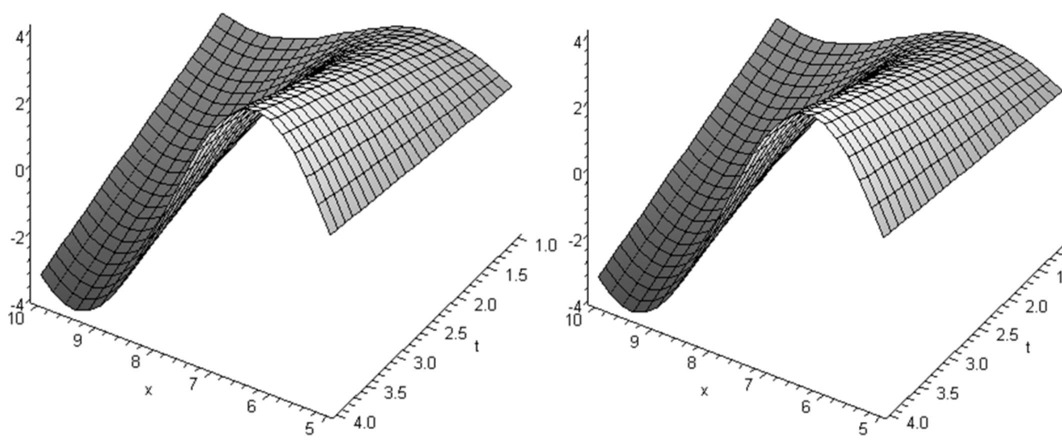

Fig. 3. The surfaces on both columns respectively show the solutions, $u(x,t)$, for VIM-II on the right and ADM on the left

4 Conclusion

In this paper, the new powerful method has been successfully applied to find the solution of the some nonlinear fifth-order Korteweg-de Vries (FKdV) partial differential equations with specified initial conditions to show the power of this method and its significant features. All the examples show that the results of the present method are in excellent accordance with those obtained by the Adomian's decomposition method. Moreover, VIM-II reduces the size of calculations by not requiring the tedious Adomian polynomials, and hence the iteration is direct and straightforward. The solutions obtained by the VIM-II method for appropriate initial conditions, can be, in turn, expressed in a closed form, the exact solution. It is a promising method to solve different types of nonlinear equations in mathematical physics.

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