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Improvement on delay dependent absolute stability of Lurie control systems with multiple time-delays and nonlinearities

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Abstract. In this paper we study the absolute stability of Lurie control systems with multiple time-delays and nonlinearities, based on the Lyapunov stability theory via the approach of decomposing matrices, by using extended Lyapunov functional, we avoid the use of the stability assumption on the main operator and derive to improve stability criteria, which are strictly less conservative than the criteria in other papers.

Keywords: Lurie control system, absolute stability, delay-dependent, linear matrix inequality.

1 Introduction

As we know, a nonlinear physical system can be expressed as a feedback connection of a linear dynamical system with a nonlinear element, where the nonlinearities always satisfy a sector condition. Based on these class of nonlinear systems, the notion of absolute stability was introduced by Lurie in [4]. In recent years, the problem of absolute stability of Lurie system has been extensively studied in [1, 2, 5, 9, 14].

Since time-delays are frequently encountered in such systems and are often a source of instability, a considerable number of studies have also been done on the stability of Lurie control systems[6, 8]. Generally, abandonment of information on the delay causes conservativeness of the criteria especially when the delay is comparatively small. Employing linear matrix inequality and decomposing the matrices and without using of the stability assumption on the main operator[13], presented a new method of obtaining delay-dependent stability criteria for Lurie control system with multiple time-delays and nonlinearities in the infinite sector. The advantage of this method is that it reduces theorem conditions in [3]. In the paper[7], the authors considered absolute stability of Lurie control system with multiple time delays and nonlinearities. The proposed result is less conservative due to decomposing of the matrix $B_i = B_{i1} + B_{i2} (i = 1, 2, \cdots, m)$ and using the operator $\Gamma(x_t) = x(t) + B\int_{t-\tau}^{t} x(s)ds$ to represent the system in the form of a descriptor system with discrete and distributed delays. Also by adding modulatory matrices and employing the linear matrix inequality. However, their condition is still depend on the stability of $\Gamma(x_t)$. Therefore, the main purpose of this paper is to reduce the stability of $\Gamma(x_t)$. Hence, by using extended Lyapunov functions[3] and avoid the use of the stability assumption on the main operator [13], We will get an improvement on the result of other papers. The results are conservative.

This paper discusses the problem of the existence of a Lyapunov functional in the Lurie form with a negative definite derivative that guarantees the absolute stability of Lurie control system with multiple time-delay and nonlinearities are an unbounded sector. The numerical example demonstrates that our criteria is less conservative than [7, 13].

Notation: Throughout this paper, $Q > 0$ (respectively $Q < 0$) means that the matrix Q is positive (respectively negative) definite and I is an identity matrix of appropriate dimensions, $\|x(t)\| = (\sum_{i=1}^{n} {x_i}^2(t))^{1/2}$, $\mathbb{R}$ denotes

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the real set, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrix. For any real matrix $Q = (q_{ij})_{n\times n}$ and $\|Q\|$ denotes any matrix norm, $\langle x|y\rangle$ denotes inner product for all $x, y \in \mathbb{R}^n$, $\langle x|y\rangle = \sum_{i=1}^{n} x_i y_i$.

2 Problem statement

Consider the following Lurie control system with multiple time-delays and nonlinearities:

$$\begin{cases} \dot{x} = Ax(t) + \sum\limits_{i=1}^{m} B_i x(t-\tau_i) + \sum\limits_{i=1}^{m} \rho_i f(\sigma t - \tau_i) + bf(\sigma(t)) \\ \sigma(t) = c^T x(t) \\ x(\Theta) = \phi(\Theta),\ \theta \in \left[-\max\limits_{1\le i\le m}\{\tau_i\}, 0\right], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector; $A, B_i(i = 1, 2, \cdots, m) \in \mathbb{R}^{n\times n}$ are the coefficient matrices; $b, \rho_i(i = 1, 2, \cdots, m) \in \mathbb{R}^n$ are the coefficient of the nonlinearities; $c \in \mathbb{R}^n$; $\tau_i \ge 0(i = 1, 2, \cdots, m)$ are the time delays; $\phi(\cdot) \in \mathbb{C}([-\max_{1\le i\le m}\{\tau_i\}, 0], \mathbb{R}^n)$ is a continuous vector valued initial function. The nonlinearity function $f(\cdot)$ satisfies the following sector condition:

$$f(\cdot) \in K[0, \infty) = \{f(\cdot)|f(0) = 0,\ 0 < \sigma f(\sigma) < \infty,\ \sigma \ne 0\}. \tag{2}$$

Before stating our results, the following facts and lemma are introduced which will be used in our proof.
Fact 2.1. *The Newton-Leibniz formula:*

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds. \tag{3}$$

Lemma 1. [13]. *For any constant symmetric matrix in $\mathbb{R}^{n\times n}$, $M > 0$, scalar $\tau > 0$, vector function $\dot{x}(\cdot) \in ([-\tau, 0], \mathbb{R}^n)$ such that integrations in the following are well defined, then*

$$\tau \int_0^{\tau} g^T(s)Mg(s)ds \ge \left[\int_0^{\tau} g(s)ds\right]^T M \left[\int_0^{\tau} g(s)ds\right]. \tag{4}$$

Lemma 2. [13]. *For given matrixes A_{11}, A_{12}, A_{22} with appropriate dimensions, $\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} < 0$, holds if and only if $A_{22} < 0$, $A_{11} - A_{12}{A_{22}}^{-1}{A_{12}}^T$.*

Lemma 3. [10, 12] *Define an operator $\Gamma(x_t) = C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ as*

$$\Gamma(x_t) = x(t) + B\int_{t-\tau}^{t} x(s)ds, \tag{5}$$

where $x_t = x(t + s)$, $s \in [-\tau, o]$ and $B \in \mathbb{R}^{n\times n}$ is a constant matrix. The operator (5)is said to be stable if there exist a scalar $0 < \delta < 1$ and positive define symmetric matrix M such that

$$\begin{pmatrix} -\delta M & \tau B^T M \\ * & -M \end{pmatrix} < 0, \tag{6}$$

where $*$ denotes the elements below the main diagonal of a symmetric block matrix.

Lemma 4. [3]. *Assume that $S \in \mathbb{R}^{n\times n}$ is a symmetric positive define matrix. Then for every $Q \in \mathbb{R}^{n\times n}$,*

$$2\langle Qy|x\rangle - \langle Sy|y\rangle \le \langle QS^{-1}Q^T x|x\rangle, \tag{7}$$

If we take $S = I$ then we have $|2\langle Qy|x\rangle| \le \|y\|^2 + \|Qx\|^2$.

We also need the main theorem in [3] and its proof for our main result.

Theorem 1. *Assume Eq. (5) is stable, the Lurie system described by Eq. (1) and Eq. (2) is absolutely stable if there exist matrix* $P = P^T > 0, Q_i = Q_i{}^T > 0, R_i = R_i{}^T > 0, M_i = M_i{}^T > 0 (i = 1, 2, \cdots, m)$, *and* $\alpha > 0, \beta > 0$ *such that the following LMI holds:*

$$X + Y < 0,$$

$$X = \begin{bmatrix} 2PA_0 + N + A^TFA & 2P\tilde{B} + A^TFB & 2\beta A^Tc + 2\alpha c + A^TFb + 2Pb & 0 & 0 & 0 \\ B^TFA & -Q + B^TFB & 2\beta Bc + B^TFb & 0 & 0 & 0 \\ b^TFA & b^TFB & 2\beta b^Tc + b^TFb + \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\tilde{B}^TN - \tilde{B}^TA^TFA & -\tilde{B}^TA^TFB & -\tilde{B}^T(2\beta A^Tc + 2\alpha c + A^TFb) & 0 & 0 & 0 \\ \rho^TFA & \rho^TFB & 2\beta\rho c + \rho^TF^Tb & 0 & 0 & 0 \end{bmatrix}, \quad (8)$$

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & -2PA_0\tilde{B} - N\tilde{B} - A^TF\tilde{B} & 2P_\rho + A^TF_\rho \\ 0 & 0 & 0 & 0 & -B^TFA\tilde{B} & B^TF_\rho \\ 0 & 0 & 0 & 0 & -b^TFA\tilde{B} & b^TF_\rho \\ 0 & 0 & 0 & -M & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{B}^TN\tilde{B} - R + \tilde{B}^TA^TFA\tilde{B} & -\tilde{B}^TA^TF\rho \\ 0 & 0 & 0 & 0 & -\rho^TFA\tilde{B} & \rho^TF\rho - \gamma \end{bmatrix}.$$

where,

$$\begin{aligned} N &= \sum_{i=1}^{m}(Q_i + \tau_i{}^2R_i), B = \sum_{i=1}^{m} B_i, \\ M &= \sum_{i=1}^{m} M_i, Q = \sum_{i=1}^{m} Q_i, \\ \gamma &= \sum_{i=1}^{m} \gamma_i, A_0 = [A + B_{11}, \cdots, A + B_{m1}], \\ \rho &= \sum_{i=1}^{m} \rho_i, F = \sum_{i=1}^{m} \tau_i^2 M_i, \\ \tilde{B} &= [B_{11}, \cdots, B_{m1}]. \end{aligned}$$

Proof: According to the Lyapunov stable theory, we choose the following Lyapunov functional candidate as $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t) + V_6(t)$, such that

$$\begin{aligned} V_1(t) &= \Gamma^T(x_t)P\Gamma(x_t), \\ V_2(t) &= \sum_{i=1}^{m} \int_{t-\tau_i}^{t} x^T(s)Q_ix(s)ds, \\ V_3(t) &= \sum_{i=1}^{m} \tau_i \int_{t-\tau_i}^{0} d\xi \int_{\xi}^{t} x^T(s)R_ix(s)ds, \\ V_4(t) &= 2\beta \int_{0}^{\sigma} f(\sigma)d\sigma, \\ V_5(t) &= \sum_{i=1}^{m} \tau_i \int_{t-\tau_i}^{0} d\xi \int_{\xi}^{t} \dot{x}^T(s)M_i\dot{x}(s)ds, \\ V_6(t) &= \sum_{i=1}^{m} \gamma_i \int_{t-\tau_i}^{t} f^2(\sigma(s))ds. \end{aligned}$$

By proof in [3], if LMI (8) in theorem 1 holds then we have $V \geq 0$, and

$$\dot{V}(t) \leq y^T(t)\Xi y(t), \tag{9}$$

where

$$\begin{aligned}
y(t) &= [\Gamma^T x_t, Z_1^T(t), f^T(\sigma(t)), Z_2^T(t), Z_3^T(t), Z_4^T(t)]^T,\\
Z_1(t) &= [x^T(t-\tau_1), x^T(t-\tau_2), \cdots, x^T(t-\tau_m)]^T,\\
Z_2(t) &= [\int_{t-\tau_1}^{t} \dot{x}^T(s)ds, \int_{t-\tau_2}^{t} \dot{x}^T(s)ds, \cdots, \int_{t-\tau_m}^{t} \dot{x}^T(s)ds]^T,\\
Z_3(t) &= [\int_{t-\tau_1}^{t} x^T(s)ds, \int_{t-\tau_2}^{t} x^T(s)ds, \cdots, \int_{t-\tau_m}^{t} x^T(s)ds]^T,\\
Z_4(t) &= [f^T(\sigma(t-\tau_1)), \cdots, f^T(\sigma(t-\tau_m))],
\end{aligned}$$

and $\Xi < 0$.

This implies that there exists a positive number λ such that

$$\dot{V}(t) < -\lambda\big(\| \Gamma(x_t) \|^2 + \sum_{i=1}^{m} \| x(t-\tau_i) \|^2 + \sum_{i=1}^{m} \| \int_{t-\tau_i}^{t} x^T(s)ds \|^2 + \sum_{i=1}^{m} \| \int_{t-\tau_i}^{t} x^T(s)ds \|^2$$
$$+ \| f(\sigma(t)) \|^2 + \sum_{i=1}^{m} \| f(\sigma t-\tau_i) \|^2 \big) < -\lambda \| \Gamma(x_t) \|^2 .$$

Since $\Gamma(x_t)$ is stable, the system (1) is absolutely stable.

Remark: Since Eq. (5), applying triangular inequality, we have

$$\| x(t) \| \leq \| \Gamma(x_t) \| + \sum_{i=1}^{m} \| B_{i1} \int_{t-\tau_i}^{t} x(s)ds \| .$$

Applying the Bunhiakovoski's inequality [11], we have

$$\| x(t) \|^2 \leq (m+1) \left(\| \gamma(x_t) \|^2 + \sum_{i=1}^{m} \| B_{i1} \|^2 \| \int_{t-\tau_i}^{t} x(s)ds \|^2 \right).$$

This implies that

$$- \| \Gamma(x_t) \|^2 \leq \frac{-1}{(m+1)} \| x(t) \|^{(2)} + \sum_{i=1}^{m} \| B_{i1} \|^2 \| \int_{t-\tau_i}^{t} x(s)ds \|^2 .$$

If $\sum_{i=1}^{m} \| B_{i1} \|^2 < 1$ then we have [10],

$$\dot{V} < -\frac{\lambda}{(m+1)} \big(\| x(t) \|^2 + \sum_{i=1}^{m} \| x(t-\tau_i) \|^2 + \| f(\sigma(t)) \|^2 + \sum_{i=1}^{m} \| \int_{t-\tau_i}^{t} \dot{x}^T(s)ds \|^2 +$$
$$\sum_{i=1}^{m} \| f(\sigma(t-\tau_i)) \|^2 \big).$$

If $\sum_{i=1}^{m} \| B_{i1} \|^2 > 1$ then we have [3],

$$\dot{V} < -\frac{\lambda}{(m+1)\sum_{i=1}^{m} \| B_{i1} \|^2)} \big(\| x(t) \|^2 + \sum_{i=1}^{m} \| x(t-\tau_i) \|^2 + \| f(\sigma(t)) \|^2 + \sum_{i=1}^{m} \| \int_{t-\tau_i}^{t} \dot{x}^T(s)ds \|^2$$
$$+ \sum_{i=1}^{m} \| f(\sigma(t-\tau_i)) \|^2 \big).$$

Thus if LMI (8) holds then there exists a positive number

$$\lambda_0 = \begin{cases} \frac{\lambda}{(m+1)}; & \sum\limits_{i=1}^{m} \| B_{i1} \|^2 < 1, \\ \frac{\lambda}{(m+1)\sum\limits_{i=1}^{m} \|B_{i1}\|^2}; & \sum\limits_{i=1}^{m} \| B_{i1} \|^2 > 1. \end{cases}$$

Such that

$$\dot{V} < -\lambda_0 \big(\| x(t) \|^2 + \sum_{i=1}^{m} \| x(t-\tau_i) \|^2 + \| f(\sigma(t)) \|^2 + \sum_{i=1}^{m} \| \int_{t-\tau_i}^{t} \dot{x}^T(s) ds \|^2 + \sum_{i=1}^{m} \| f(\sigma(t-\tau_i)) \|^2 \big).$$

3 Main result

Using remark 1, we have an improved criterion for absolute stability of system (1) as follows:

Theorem 2. *The system (1) absolutely stable if there exist* $P = P^T > 0, Q_i = Q_i^T > 0, R_i = R_i^T > 0, M_i = M_i^T > 0 (i = 1, 2, \cdots, m)$*, and* $\alpha > 0, \beta > 0$ *such that the LMI (8) holds.*

Proof: Consider the Lyapunov functional

$$V^*(t) = V(t) + V_7(t), \tag{10}$$

where $V_7(t) = \varepsilon \parallel x(t) \parallel^2$, ε is a positive number that will be chosen later.

Since$V(t) \geq 0$, $V^*(t) \geq \varepsilon \parallel x(t) \parallel^2$. The derivative of $V_7(t)$ is given by:

$$\dot{V}_7(t) = 2\varepsilon \dot{x}(t)^T x(t) = 2\varepsilon \left[Ax(t) + \sum_{i=1}^{m} B_i x(t-\tau_i) + bf(\sigma(t)) + \sum_{i=1}^{m} \rho_i f(\sigma(t-\tau_i)) \right]^T x(t). \tag{11}$$

By lemma 4, we have:

$$\dot{V}_7(t) \leq \varepsilon \big(\| A \|^2 + \sum_{i=1}^{m} \| B_i \|^2 + \| b \|^2 + \sum_{i=1}^{m} \| \rho_i \|^2 \big) \| x(t) \|^2 + \varepsilon \big(\| x(t) \|^2 + \sum_{i=1}^{m} \| x(t-\tau_i) \|^2 + \| f(\sigma(t)) \|^2 + \sum_{i=1}^{m} \| f(\sigma(t-\tau_i)) \|^2 \big).$$

Choosing

$$\varepsilon = \frac{\lambda_0}{2} \min \left\{ \frac{1}{\| A \|^2 + \sum\limits_{i=1}^{m} \| B_i \|^2 + \| b \|^2 + \sum\limits_{i=1}^{m} \| \rho_i \|^2}, 1 \right\}.$$

Hence, by using the time derivative of each $V_i(t) (i = 1, 2, \cdots, 6)$ and $\dot{V}_7(t)$, we have $\dot{V}^*(t) \leq 0$.

4 A numerical example

Consider the system (1) with

$$A = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix}, B_1 = \begin{bmatrix} -0.2 & -0.5 \\ 0.5 & -0.2 \end{bmatrix},$$
$$b = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, c = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}.$$

Applying the criteria in [3, 5] the maximum value of $\tau_{\max}$ for absolute stability of system (1) are $\tau_{\max} = 5.2000$, $\tau_{\max} = 5.1587$, respectively. Now we use the criterion in this paper to study the problem. Let us decompose matrix B_1 as $B_1 = B_{11} + B_{12}$, where

$$B_{11} = \begin{bmatrix} -0.01 & -0.08 \\ 0.01 & 0.05 \end{bmatrix}, B_{12} = \begin{bmatrix} -0.19 & -0.42 \\ 0.49 & -0.25 \end{bmatrix}.$$

Solving LMI that we obtain from theorem 3.1, the value of delay for absolute stability of system (1) is τ=5.4000. It is larger than results [7, 13].

τ ref by [10]	τ ref by [9]	τ ref by main result
5.1587	5.2000	5.4000

When $\tau = 5.4000$ the solution

$$Q = \begin{bmatrix} 2.2500 & 0.0650 \\ 0.0650 & 2.1600 \end{bmatrix}, R = \begin{bmatrix} 0.2000 & 0.0480 \\ 0.0480 & 0.2600 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.0260 & -0.0150 \\ -0.0150 & 0.0210 \end{bmatrix}, \rho = \begin{bmatrix} 0.4000 \\ 0.2000 \end{bmatrix}, P = \begin{bmatrix} 4.4000 & -0.8700 \\ -0.8700 & 4.5000 \end{bmatrix}, \ \alpha = 3.3900, \ \beta = 1.2900$$

This example shows that the absolute stability criterion in this paper gives a much less conservative result than these in [7, 13].

5 Conclusion

Absolute stability of Lurie control system with multiple time delays and nonlinearities is studied in this paper. By improving the estimation of derivative of the Lyapunov functional in [7, 13], we propose improved Lyapunov functional and obtain stability criteria, which are strictly less conservative than the criteria in [7, 13].

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