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Approximate solution of linear integro-differential equations by using modified Taylor expansion method

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Abstract. In this study we developed and modified Taylor expansion method for approximating the solution of linear Fredholm and Volterra integro-differential equations. Via Taylor's expansion of the unknown function at an arbitrary point, the integro-differential equations to be solved is approximately transformed into a system of linear equations for the unknown and its derivatives which can be dealt with in an easy way. This method gives a simple and closed form solution for a linear integro-differential equation. This method also enable us to control truncation error by adjusting the step size used in the numerical scheme. Some numerical examples are provided to illustrate the accuracy of our approach.

Keywords: linear integro-differential equations, approximate solution, taylor expansion method

1 Introduction

Mathematical modeling of real-life problems usually results in functional equations, e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. In particular, integro-differential equations arise in fluid dynamics, biological models and chemical kinetics. The analytical solutions of some integro-differential equations cannot be found, thus numerical methods are required.

In recent years, the numerical methods for linear integro-differential equations have been extensively studied by many authors. The iterated Galerkin methods have been proposed in [1]. Furthermore, mixed interpolation collocation methods for first- and second-order Volterra linear integro-differential equations have been suggested in [2]. Hu[3] developed the interpolation collocation method for solving Fredholm linear integro-differential equations. Rashed treated a special type of integro-differential equations with derivatives appearing in integrals[4].

Taylor series expansion method is a powerful technique for solving integro-differential equations. The Taylor expansion approach for solving Volterra integral equations has been presented by Kanwal and Liu[5]. The Taylor expansion method for approximate solution of a class of linear integro-differential equations presented by Li[6] and also, a Taylor series method[7–10, 22] was presented by Szer et al. for differential, integral and integro-differential equations. Bessel or Chebyshev polynomial approach are used to solve integro-differential equations in [11–16]. Sahin in [17] applied collection method for numerical solution of Volterra integral equations with variable coefficients. An improved Legandre method[18] is used for solution of integro-differential equations. A numerical method for the solution of integro-differential-difference equation with variable coefficients has been proposed by Saadatmandi[19].

In this paper, the Taylors expansion method[6] is modified such that by this modification the corresponding accuracy is drastically improved. Also, the approximate method described here is very easy to implement.

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2 Volterra integro-differential equations

Consider the Volterra integro-differential equations of the form

$$\sum_{i=0}^{m} \beta_i(x) y^{(i)}(x) = f(x) + \lambda_v \int_{\eta}^{x} k_v(x,t) y(t) dt, \ x,t \in I = [a,b], \tag{1}$$

under the initial conditions

$$\sum_{i=0}^{m-1} c_{ji} y^{(i)}(\eta) = \mu_j, \ j = 0,1,\cdots,m-1, \tag{2}$$

where $y(x)$ is an unknown function to be determined, the known continuous functions $\beta_i(x),\ f(x)$ and $k_v(x,t)$ are defined on the interval I. Here the real coefficients λ_v and c_{ji} are known constants and η is a given point in the spatial domain of the problem.

Remark 1. The following partition is used throughout of the paper

$$\triangle = \{a = x_{\theta_l} < \cdots < x_{-1} < x_0 = \eta < x_1 < \cdots < x_{\theta_r} = b\},$$

is an equidistance partition on $I = [a,b]$, where

$$x_\theta = \eta + \theta h, \text{ for } \theta \in \{-\theta_r, \cdots, -1, 0, 1, \cdots, \theta_r\}$$

and h is a step size of the partition.

Theorem 1. *Consider linear integro-differential Eq. (1)- Eq. (2). Let the functions $\beta_i(x),\ f(x),\ k_v(x,t)$ and the exact solution $y(x)$ are sufficiently differentiable on the interval I and $k_v(x,t)$ be a separable kernel. Then there exist linear independent functions $\varphi_i(\chi)$ and constants c_i for $i = 0,1,\cdots,\alpha$, such that $\mathbf{y}(\chi) = y(\chi + x_\theta)$ is an exact solution of*

$$\sum_{i=0}^{m} \beta_i(\chi + x_\theta) \mathbf{y}^{(i)}(\chi) = f(\chi + x_\theta) + \sum_{i=0}^{\alpha} c_i \varphi_i(\chi) + \lambda_v \int_0^{\chi} k_v(\chi + x_\theta,\ \xi + x_\theta) \mathbf{y}(\xi) d\xi, \tag{3}$$

with initial conditions

$$\mathbf{y}^{(i)}(0) = y^{(i)}(x_\theta),\ i = 0,1,\cdots,m-1, \tag{4}$$

where $x = \chi + x_\theta$.

Proof. The kernel in Eq. (1) is separable, then we have

$$k(x,t) = \sum_{i=0}^{p} u_i(x) v_i(t), \tag{5}$$

by substituting Eq. (5) into Eq. (3) and then change of variables $x = \chi + x_\theta$, $t = \xi + x_\theta$, we have

$$\sum_{i=0}^{m} \beta_i(\chi + x_\theta) y^{(i)}(\chi + x_\theta) = f(\chi + x_\theta) + \sum_{i=0}^{p} d_i u_i(\chi + x_\theta) + \lambda_v \int_0^{\chi} k_v(\chi + x_\theta,\ \xi + x_\theta) y(\xi + x_\theta) d\xi, \tag{6}$$

where $d_i = \int_{\eta - x_\theta}^{0} v_i(\xi + x_\theta) y(\xi + x_\theta) d\xi$.

Now, by simplifying and classifying on the middle terms of R.H.S Eq. (6), we can write

$$\sum_{i=0}^{p} d_i u_i(\chi + x_\theta) = \sum_{i=0}^{\alpha} c_i \varphi_i(\chi), \tag{7}$$

where $\varphi_i(\chi),\ i=0,1,\cdots,\alpha$ are known linear independent functions and $c_i,\ i=0,1,\cdots,\alpha$ are unknown constants. Using Eq. (7), and by denoting $\mathbf{y}(\chi)=y(\chi+x_\theta)$, we can rewrite Eq. (6) and Eq. (2), in the form of Eq. (3) and Eq. (4), respectively.

To obtain c_i, by derivation on Eq. (3) for $\alpha-1$ times at $\chi=0$, the following algebraic system is obtained

$$\begin{cases} \sum\limits_{i=0}^{\alpha} c_i\varphi_i(0)=\sum\limits_{i=0}^{m} g_i(0)-f(x_\theta), \\ \sum\limits_{i=0}^{\alpha} c_i\varphi_i'(0)=\sum\limits_{i=0}^{m} g_i'(0)-f'(x_\theta)-\rho(0), \\ \vdots \\ \sum\limits_{i=0}^{\alpha} c_i\varphi_i^{(\alpha-1)}(0)=\sum\limits_{i=0}^{m} g_i^{(\alpha-1)}(0)-f^{(\alpha-1)}(x_\theta)-\rho^{(\alpha-2)}(0), \end{cases} \tag{8}$$

where $g_i(\chi)=\beta_i(\chi+x_\theta)\mathbf{y}^{(i)}(\chi)$ and $\rho(\chi)=k_v(\chi+x_\theta,\ \chi+x_\theta)\mathbf{y}(\chi)$.

By solving algebraic system Eq. (8), the unknown constants can be determined as follows

$$c_i=\tau_i(x_\theta,\mathbf{y}(0),\mathbf{y}'(0),\cdots,\mathbf{y}^{(m+\alpha-1)}(0)),\ i=0,1,\cdots,\alpha, \tag{9}$$

where τ_i are known functions and $\mathbf{y}^{(i)}(0)=y^{(i)}(x_\theta),\ i=0,1,\cdots,\alpha$.

For solving linear integro-differential equations Eq. (1) and Eq. (2), by modified Taylor expansion method, we assume that $\mathbf{y}(\xi)$ in Eq. (3) can be represented in terms of the n_{th}-order Taylor expansion as

$$\mathbf{y}(\xi)=\mathbf{y}(\chi)+\mathbf{y}'(\chi)(\xi-\chi)+\cdots+\frac{\mathbf{y}^{(n)}(\chi)}{n!}(\xi-\chi)^n+\frac{\mathbf{y}^{(n+1)}(\zeta)}{(n+1)!}(\xi-\chi)^{n+1}, \tag{10}$$

where ζ is between χ and ξ. It is readily shown that the Lagrange remainder

$$\frac{\mathbf{y}^{(n+1)}(\zeta)}{(n+1)!}(\xi-\chi)^n$$

is sufficiently small for a large enough n provided that $\mathbf{y}^{(n+1)}(\chi)$ is a uniformly bounded function for any n on the interval I, so that we neglect the remainder and the truncated Taylor expansions $\mathbf{y}(\chi)$ as

$$\mathbf{y}(\xi)\approx\sum_{k=0}^{n}\frac{\mathbf{y}^{(k)}(\chi)}{k!}(\xi-\chi)^k. \tag{11}$$

It is worth noting that the Lagrange remainder vanishes for a polynomial of degree equal to or less than n, implying that the above n_{th}-order Taylor expansion is exact. Also, n is chosen any positive integer such that $n\geq m$.

Now, we substitute Taylor's expansion Eq. (11) for $\mathbf{y}(\xi)$ into Eq. (3) and find

$$k_{00}(\chi)\mathbf{y}(\chi)+k_{01}(\chi)\mathbf{y}'(\chi)+\cdots+k_{0n}(\chi)\mathbf{y}^{(n)}(\chi)=f_0(\chi), \tag{12}$$

where

$$k_{0i}=\begin{cases} \beta_i(\chi+x_\theta)-\frac{\lambda_v}{i!}\int_0^\chi k_v(\chi+x_\theta,\ \xi+x_\theta)(\xi-\chi)^i d\xi, & i=0,1,\cdots,m \\ -\frac{\lambda_v}{i!}\int_0^\chi k_v(\chi+x_\theta,\ \xi+x_\theta)(\xi-\chi)^i d\xi, & i=m+1,m+2,\cdots,n \end{cases}, \tag{13}$$

and

$$f_0(\chi)=f(\chi+x_\theta)+\sum_{i=0}^{\alpha}c_i\varphi_i(\chi). \tag{14}$$

Thus Eq. (12) becomes a m_{th}-order, linear, ordinary differential equation with variable coefficients for $\mathbf{y}(\chi)$ and its derivations up to n. Instead of solving analytically ordinary differential equation, we will determine $\mathbf{y}(\chi),\mathbf{y}'(\chi),\cdots,\mathbf{y}^{(n)}(\chi)$ by solving linear equations. To this end, other n independent linear equations for $\mathbf{y}(\chi), \mathbf{y}'(\chi), \cdots, \mathbf{y}^{(n)}(\chi)$ are needed. This can be achieved by integrating both sides of Eq. (3) with respect to χ from 0 to χ and changing the order of the integrations. Accordingly, one can get that

$$\sum_{i=0}^{m}\int_0^{\chi}\beta_i(\xi+x_\theta)\mathbf{y}^{(i)}(\xi)d\xi=\int_0^{\chi}f_0(\xi)d\xi+\lambda_v\int_0^{\chi}\int_{\xi}^{\chi}k_v(s+x_\theta,\ \xi+x_\theta)ds\mathbf{y}(\xi)d\xi, \tag{15}$$

where we have replaced variable s to χ for convenience. Applying integration by part, accordingly, one can get that

$$\begin{aligned}\int_0^{\chi}\beta_i(\xi+x_\theta)\mathbf{y}^{(i)}(\xi)d\xi =&\beta_i(\xi+x_\theta)\mathbf{y}^{(i-1)}(\xi)|_0^{\chi}-\beta_i'(\xi+x_\theta)\mathbf{y}^{(i-2)}(\xi)|_0^{\chi}+\cdots\\ &+(-1)^{i-1}\beta_i^{(i-1)}(\xi+x_\theta)\mathbf{y}(\xi)|_0^{\chi}+(-1)^i\int_0^{\chi}\beta_i^{(i)}(\xi+x_\theta)\mathbf{y}(\xi)d\xi.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i=0}^{m}\int_0^{\chi}\beta_i(\xi+x_\theta)\mathbf{y}^{(i)}(\xi)d\xi =&\sum_{i=0}^{m-1}(\sum_{k=i+1}^{m}(-1)^{k-i-1}\beta_k^{(k-i-1)}(\xi+x_\theta))\mathbf{y}^{(i)}(\xi))|_0^{\chi}\\ &+\int_0^{\chi}\sum_{k=0}^{m}(-1)^k\beta_k^{(k)}(\xi+x_\theta)\mathbf{y}(\xi)d\xi,\end{aligned} \tag{16}$$

Applying the Taylor expansion again and substituting Eq. (11) for $\mathbf{y}(\xi)$ into Eq. (16) gives

$$k_{10}(\chi)\mathbf{y}(\chi)+k_{11}(\chi)\mathbf{y}'(\chi)+\cdots+k_{1n}(\chi)\mathbf{y}^{(n)}(\chi)=f_1(\chi), \tag{17}$$

where

$$\begin{aligned}k_{1i}(\chi)=&\sum_{k=i+1}^{m}(-1)^{k-i-1}\beta_k^{(k-i-1)}(\chi+x_\theta)+\frac{1}{i!}\int_0^{\chi}\sum_{k=0}^{m}(-1)^k\beta_k^{(k)}(\xi+x_\theta)(\xi-\chi)^i d\xi\\ &-\frac{\lambda_v}{i!}\int_0^{\chi}\int_{\xi}^{\chi}k_v(s+x_\theta,\ \xi+x_\theta)ds(\xi-\chi)^i d\xi,\ i=0,1,\cdots,m-1,\end{aligned} \tag{18}$$

$$\begin{aligned}k_{1i}(\chi)=&\frac{1}{i!}\int_0^{\chi}\sum_{k=0}^{m}(-1)^k\beta_k^{(k)}(\xi+x_\theta)(\xi-\chi)^i d\xi\\ &-\frac{\lambda_v}{i!}\int_0^{\chi}\int_{\xi}^{\chi}k_v(s+x_\theta,\ \xi+x_\theta)ds(\xi-\chi)^i d\xi,\ i=m,m+1,\cdots,n,\end{aligned}$$

and

$$f_1(\chi)=\sum_{i=0}^{m-1}\left(\sum_{k=i+1}^{m}(-1)^{k-i-1}\beta_k^{(k-i-1)}(x_\theta)\right)\mathbf{y}^{(i)}(0)+\int_0^{\chi}f_0(\xi)d\xi. \tag{19}$$

By repeating the integration on both side of Eq. (15) for r times $r=2,3,\cdots,n$, one can arrive that

$$\begin{aligned}\sum_{i=0}^{m}\int_0^{\chi}(\chi-\xi)^{r-1}\beta_i(\xi+x_\theta)\mathbf{y}^{(i)}(\xi)d\xi=&\int_0^{\chi}(\chi-\xi)^{r-1}f_0(\xi)d\xi\\ &+\lambda_v\int_0^{\chi}\int_{\xi}^{\chi}k_v(s+x_\theta,\ \xi+x_\theta)(\chi-s)^{r-1}ds\mathbf{y}(\xi)d\xi.\end{aligned} \tag{20}$$

By using integration by part, we can obtain

$$\sum_{i=0}^{m}\int_0^{\chi}(\chi-\xi)^{r-1}\beta_i(\xi+x_\theta)\mathbf{y}^{(i)}(\xi)d\xi=\int_0^{\chi}\sum_{k=0}^{m}(-1)^k\frac{\partial^k\beta_k(\xi+x_\theta)^{(k)}(\chi-\xi)^{r-1}}{\partial\xi^k}\mathbf{y}^{(k)}(\xi)d\xi$$
$$+\sum_{i=0}^{m-1}\left(\sum_{k=i+1}^{m}(-1)^{k-i-1}\frac{\partial^{k-i-1}\beta_k(\xi+x_\theta)(\chi-\xi)^{r-1}}{\partial\xi^{k-i-1}}\right)\mathbf{y}(\xi)|_0^{\chi}. \tag{21}$$

Substituting Eq. (11) for $y(\xi)$ in the Eq. (21), we obtain

$$k_{r0}(\chi)\mathbf{y}(\chi)+k_{r1}(\chi)\mathbf{y}'(\chi)+\cdots+k_{rn}(\chi)\mathbf{y}^{(n)}(\chi)=f_r(\chi), \tag{22}$$

where

$$k_{ri}(\chi)=\sum_{k=i+1}^{m}(-1)^{k-i-1}\frac{\partial^{k-i-1}\beta_k(\xi+x_\theta)(\chi-\xi)^{r-1}}{\partial\xi^{k-i-1}}|_{\xi=\chi}+\frac{1}{i!}\int_0^{\chi}\sum_{k=0}^{m}(-1)^k\frac{\partial^k\beta_k(\xi+x_\theta)(\xi-\chi)^{r-1}}{\partial\xi^k}d\xi$$
$$-\frac{\lambda_v}{i!}\int_0^{\chi}\int_{\xi}^{\chi}k_v(s+x_\theta,\ \xi+x_\theta)(\chi-s)^{r-1}ds(\xi-\chi)^i d\xi,\ r=1,\cdots,m-1, \tag{23}$$

$$k_{ri}(\chi)=\frac{1}{i!}\int_0^{\chi}\sum_{k=0}^{m}(-1)^k\frac{\partial^k\beta_k(\xi+x_\theta)(\xi-\chi)^{r-1}}{\partial\xi^k}d\xi$$
$$-\frac{\lambda_v}{i!}\int_0^{\chi}\int_{\xi}^{\chi}k_v(s+x_\theta,\ \xi+x_\theta)(\chi-s)^{r-1}ds(\xi-\chi)^i d\xi,\ i=m,m+1,\cdots,n$$

and

$$f_r(\chi)=\sum_{i=0}^{m-1}\left(\sum_{k=i+1}^{m}(-1)^{k-i-1}\frac{\partial^{k-i-1}\beta_k(\xi+x_\theta)(\chi-\xi)^{r-1}}{\partial\xi^{k-i-1}}\right)\mathbf{y}(\xi)|_{\xi=\eta}+\int_0^{\chi}(\chi-\xi)^{r-1}f_0(\xi)d\xi. \tag{24}$$

Therefore, Eqs. (12), (17) and (22) form a system of $n+1$ linear equation for $n+1$ unknown functions $\mathbf{y}(\chi),\mathbf{y}'(\chi),\cdots,\mathbf{y}^{(n)}(\chi)$. For simplicity, this system can be rewritten in an alternative compact form as

$$\mathbf{K}_{nn}(\chi)\mathbf{Y}_n(\chi)=\mathbf{F}(\chi), \tag{25}$$

where $\mathbf{K}_{nn}(\chi)$ is an $(n+1)\times(n+1)$ square matrix function, $\mathbf{Y}_n(\chi)$ and $\mathbf{F}_n(\chi)$ are two vectors of length $n+1$, and these are defined as

$$\mathbf{K}_{nn}(\chi)=\begin{pmatrix} k_{00}(\chi) & k_{01}(\chi) & \dots & k_{0n}(\chi)\\ k_{10}(\chi) & k_{11}(\chi) & \dots & k_{1n}(\chi)\\ \vdots & \vdots & \ddots & \vdots\\ k_{n0}(\chi) & k_{n1}(\chi) & \dots & k_{nn}(\chi)\end{pmatrix}, \tag{26}$$

$$\mathbf{Y}_n(\chi)=\begin{pmatrix}\mathbf{y}(\chi)\\ \mathbf{y}'(\chi)\\ \vdots\\ \mathbf{y}^{(n)}(\chi)\end{pmatrix},\ \mathbf{F}_n(\chi)=\begin{pmatrix}f_0(\chi)\\ f_1(\chi)\\ \vdots\\ f_n(\chi)\end{pmatrix}. \tag{27}$$

Therefore, by well-known Crammer's formula, we can obtain

$$\mathbf{y}^{(i)}(\chi)=\Psi_i(\chi,x_\theta,c_0,c_1,\cdots,c_\alpha,\mathbf{y}(0),\mathbf{y}'(0),\cdots,\mathbf{y}^{(m-1)}(0)),\ i=0,1,\cdots,m+\alpha-1, \tag{28}$$

which is Taylor expansion of solution Eq. (3) and Eq. (4) for $\mathbf{y}(\chi)$ and its derivations up to $m+\alpha-1$ at $\chi=0$, and by change of variable $\chi=x-x_\theta$, we have

$$y^{(i)}(x) = \Psi_i(x - x_\theta, x_\theta, c_0, c_1, \cdots, c_\alpha, \mathbf{y}(x_\theta), \mathbf{y}'(x_\theta), \cdots, \mathbf{y}^{(m-1)}(x_\theta)),\ i = 0, 1, \cdots, m+\alpha-1, \quad (29)$$

which is Taylor expansion of solution Eq. (1) and Eq. (2) for $y(x)$ and it's derivations up to $m+\alpha-1$ at $x = x_\theta$.

Note that, in the Eq. (28) and Eq. (29), in the any step of proposed method $c_i,\ i = 0, 1, \cdots, \alpha$, and $\mathbf{y}^{(i)}(0),\ i = 0, 1, \cdots, m-1$ are known values by using (9).

2.1 Algorithm of the approach

In this section, we try to propose an algorithm on the basis of the above discussions and suppose that we face with the linear Volterra integro-differential Eq. (1) and Eq. (2), where its kernel satisfies the conditions of Theorem 1. This algorithm is presented in two stages such as initialization step and main steps.

Initialization step:

Choose step size $h > 0$ for equidistance partition $\triangle$ on I. Set $c_j = 0,\ j = 0, 1, \cdots, \alpha$ and $\mathbf{y}^{(j)}(0) = y^{(j)}(\eta),\ j = 0, 1, \cdots, m-1$. Set $\theta = 0$ and go to main steps.

Main steps:

Step 1. Compute by Eq. (29) the following approximate solution

$$y^{(i)}(x) = \Psi_i(x - x_\theta, x_\theta, c_0, c_1, \cdots, c_\alpha, \mathbf{y}(0), \mathbf{y}'(0), \cdots, \mathbf{y}^{(m-1)}(0)),\ i = 0, 1, \cdots, m+\alpha-1, \quad (30)$$

which is Taylor expansion approximate Eq. (1)- Eq. (2) for $y(x)$ and it's derivations up to $m+\alpha-1$ at $x = x_\theta$. Go to Step 2.

Step 2. Set $\theta = \theta + 1$, if $\theta > \theta_r - 1$, stop; Otherwise, using Eq. (30) compute the approximate values

$$\mathbf{y}^{(i)}(0) = y^{(i)}(x_\theta),\ i = 0, 1, \cdots, m+\alpha-1, \quad (31)$$

which are the initial conditions Eq. (1) and Eq. (2) at $x = x_\theta$. Go to next Step.

Step 3. By Eq. (31) compute the following approximate values

$$c_i = \tau_i(x_\theta, \mathbf{y}(0), \mathbf{y}'(0), \cdots, \mathbf{y}^{(m+\alpha-1)}(0)),\ i = 0, 1, \cdots, \alpha, \quad (32)$$

and go to Step 1.

Note that in the Step 2, if we replaced condition $\theta > \theta_l - 1$, with the condition $\theta > \theta_l + 1$ and also $\theta - 1$ to $\theta + 1$, then by applying the above algorithm we obtain Taylor expansion approximate solution of Eq. (1)-Eq. (2) and its derivations at $x = x_\theta,\ \theta = -1, -2, \cdots, \theta_l$.

3 Fredholm integro-differential equations

In this section, our study focuses on a class of linear Fredholm integro-differential equations of the following form

$$\sum_{i=0}^{m} \beta_i(x) y^{(i)}(x) = f(x) + \lambda_f \int_a^b k_f(x,t) y(t) dt,\ x, t \in I = [a, b], \quad (33)$$

subject to the initial conditions

$$\sum_{i=0}^{m-1} c_{ji} y^{(i)}(0) = \mu_j,\ a \le \eta \le b,\ j = 0, 1, \cdots, m-1. \quad (34)$$

We assume that the desired solution $y(t)$ is $n+1$ times continually differentiable on the interval I, i. e. $y \in C^{n+1}[I]$. Consequently, for $y \in C^{n+1}$, $y(t)$ can be represented in terms of the truncated Taylor series expansion as

$$y(t) \approx \sum_{k=0}^{n} \frac{y^{(k)}(h)}{k!} (t-h)^k, \quad (35)$$

where h is step size in the partition Δ.

Substituting the approximate solution Eq. (35) for $y(t)$ into Eq. (33), one can get

$$a_{00}y(h) + a_{01}y'(h) + \cdots + a_{0n}y^{(n)}(h) = f(h), \tag{36}$$

where

$$a_{0i} = \begin{cases} \beta_i(h) - \frac{\lambda_f}{i!}\int_a^b k_f(h,t)(t-h)^i dt, \; i = 0, 1, \cdots, m, \\ -\frac{\lambda_f}{i!}\int_a^b k_f(h,t)(t-h)^i dt, \; i = m+1, m+2, \cdots, n. \end{cases} \tag{37}$$

Using a procedure analogous to the previous section, we can obtain other n linear equations for $y(h), y'(h), \cdots, y^{(n)}(h)$ as follows

$$a_{r0}y(h) + a_{r1}y'(h) + \cdots + a_{rn}y^{(n)}(h) = f_r(h), \tag{38}$$

where

$$\begin{aligned} a_{ri} = &\sum_{k=i+1}^{m} (-1)^{k-i-1}\frac{\partial^{k-i-1}\beta_k(\xi)(h-\xi)^{r-1}}{\partial\xi^{k-i-1}}|_{\xi=h} + \frac{1}{i!}\int_\eta^h \sum_{k=0}^{m}(-1)^k\frac{\partial^k(\beta_k(\xi)(\xi-h)^{r-1})}{\partial\xi^k}d\xi \\ &- \frac{\lambda_f}{i!}\int_\eta^h\int_\xi^h k_f(s,\xi)(h-s)^{r-1}ds(\xi-h)^i d\xi, \; r = 1, \cdots, m-1, \end{aligned} \tag{39}$$

$$\begin{aligned} a_{ri} =&\frac{1}{i!}\int_\eta^h \sum_{k=0}^{m}(-1)^k\frac{\partial^k(\beta_k(\xi)(\xi-h)^{r-1})}{\partial\xi^k}d\xi \\ &- \frac{\lambda_f}{i!}\int_\eta^h\int_\xi^h k_f(s,\xi)(h-s)^{r-1}ds(\xi-h)^i d\xi, \; i = m, m+1, \cdots, n, \end{aligned}$$

and

$$f_r(h) = \sum_{i=0}^{m-1}\left(\sum_{k=i+1}^{m}(-1)^{k-i-1}\frac{\partial^{k-i-1}\beta_k(\xi)(h-\xi)^{r-1}}{\partial\xi^{k-i-1}}\right)y(\xi)|_{\xi=\eta} + \int_\eta^h (h-\xi)^{r-1}f(\xi)d\xi. \tag{40}$$

Therefore Eq. (36) and Eq. (38) form a system of $n+1$ linear equation for $n+1$ unknown values $y(h), y'(h), \cdots, y^{(n)}(h)$. By using Crammer's formula we can obtain

$$y_i = y^{(i)}(h), \; i = 0, 1, \cdots, q-1, q < n, \tag{41}$$

where is initial conditions Eq. (1) and Eq. (2) at $x = h$.

Now let

$$k_f = \sum_{i=0}^{q} u_i(x)v_i(t),$$

by using theorem 1, linear integro-differential equations Eq. (1) converted to the following form

$$\sum_{i=0}^{m}\beta_i(\chi + x_\theta)\mathbf{y}^{(i)}(\chi) = f(\chi + x_\theta) + \sum_{i=0}^{q} I_i u_i(\chi + x_\theta), \tag{42}$$

with initial conditions

$$\mathbf{y}^{(i)}(0) = y^{(i)}(x_\theta), \; i = 0, 1, \cdots, m-1, \tag{43}$$

where $\mathbf{y}(\chi) = y(\chi + x_\theta)$, $x = \chi + x_\theta$ and

$$I_i = \lambda_f \int_a^b v_i(t)y(t)dt$$

is an unknown constant. From (42) the following algebraic equation cab be drived

$$b_{00}(\chi)\mathbf{y}(\chi) + b_{01}(\chi)\mathbf{y}'(\chi) + \cdots + b_{0n}(\chi)\mathbf{y}^{(n)}(\chi) = g_0(\chi), \tag{44}$$

where

$$b_{0i}(\chi) = \begin{cases} \beta_i(\chi + x_\theta), & i = 0, 1, \cdots, m, \\ 0, & i = m+1, m+2, \cdots, n, \end{cases} \tag{45}$$

and

$$g_0(\chi) = f(\chi + x_\theta) + \sum_{i=0}^{q} I_i u_i(\chi + x_\theta). \tag{46}$$

Also by using a procedure analogous to the section 2, we can obtain other n linear equations for $\mathbf{y}(\chi), \mathbf{y}'(\chi), \cdots, \mathbf{y}^{(n)}(\chi)$ as follows

$$b_{r0}(\chi)\mathbf{y}(\chi) + b_{r1}(\chi)\mathbf{y}'(\chi) + \cdots + b_{rn}(\chi)\mathbf{y}^{(n)}(\chi) = g_r(\chi), \tag{47}$$

where

$$\begin{aligned} b_{ri}(\chi) = & \sum_{k=i+1}^{m} (-1)^{k-i-1} \frac{\partial^{k-i-1}\beta_k(\xi + x_\theta)(\chi - \xi)^{r-1}}{\partial \xi^{k-i-1}}|_{\xi=\chi} \\ & + \frac{1}{i!}\int_0^\chi \sum_{k=0}^{m} (-1)^k \frac{\partial^k \beta_k(\xi + x_\theta)(\xi - \chi)^{r-1}}{\partial \xi^k} d\xi, \; r = 1, \cdots, m-1, \end{aligned} \tag{48}$$

$$b_{ri}(\chi) = \frac{1}{i!}\int_0^\chi \sum_{k=0}^{m} (-1)^k \frac{\partial^k \beta_k(\xi + x_\theta)(\xi - \chi)^{r-1}}{\partial \xi^k} d\xi, \; i = m, m+1, \cdots, n,$$

and

$$g_r(\chi) = \sum_{i=0}^{m-1} \left(\sum_{k=i+1}^{m} (-1)^{k-i-1} \frac{\partial^{k-i-1}\beta_k(\xi + x_\theta)(\chi - \xi)^{r-1}}{\partial \xi^{k-i-1}} \right) \mathbf{y}(\xi)|_{\xi=\eta} + \int_0^\chi (\chi - \xi)^{r-1} g_0(\xi) d\xi. \tag{49}$$

Therefore Eq. (44) and Eq. (47) form a system of $n+1$ linear equation for $n+1$ unknown functions $\mathbf{y}(\chi), \mathbf{y}'(\chi), \cdots, \mathbf{y}^{(n)}(\chi)$.

For simplicity, this system can be rewritten in an alternative compact form as

$$\mathbf{B}_{nn}(\chi)\mathbf{Y}_n(\chi) = \mathbf{F}(\chi), \tag{50}$$

where $\mathbf{B}_{nn}(\chi)$ is an $(n+1) \times (n+1)$ square matrix function, $\mathbf{Y}_n(\chi)$ and $\mathbf{F}_n(\chi)$ are two vectors of length $n+1$, and these are defined as

$$\mathbf{B}_{nn}(\chi) = \begin{pmatrix} b_{00}(\chi) & b_{01}(\chi) & \ldots & b_{0n}(\chi) \\ b_{10}(\chi) & b_{11}(\chi) & \ldots & b_{1n}(\chi) \\ \vdots & \vdots & \ddots & \vdots \\ b_{n0}(\chi) & b_{n1}(\chi) & \ldots & b_{nn}(\chi) \end{pmatrix}, \tag{51}$$

$$\mathbf{Y}_n(\chi) = \begin{pmatrix} \mathbf{y}(\chi) \\ \mathbf{y}'(\chi) \\ \vdots \\ \mathbf{y}^{(n)}(\chi) \end{pmatrix}, \; \mathbf{F}_n(\chi) = \begin{pmatrix} g_0(\chi) \\ g_1(\chi) \\ \vdots \\ g_n(\chi) \end{pmatrix}. \tag{52}$$

By well-known Crammer's formula, we have

$$\mathbf{y}^{(i)}(\chi) = \Psi_i(\chi, x_\theta, I_0, I_1, \cdots, I_q, \mathbf{y}(0), \mathbf{y}'(0), \cdots, \mathbf{y}^{(m-1)}(0)),\ i = 0, 1, \cdots, m-1, \tag{53}$$

where $\mathbf{y}^{(i)}(0) = y^{(i)}(x_\theta),\ i = 0, 1, \cdots, m-1$, $\{I_k\}_{k=0}^{q}$ are unknown parameters and will be determined.

Using Eq. (41) and Eq. (53) the following algebraic system is yield

$$\Psi_i(0, \eta + h, I_0, I_1, \cdots, I_q, \mathbf{y}(0), \mathbf{y}'(0), \cdots, \mathbf{y}^{(m-1)}(0)) = y_i,\ i = 0, 1, \cdots, q-1, \tag{54}$$

where $\mathbf{y}^{(i)}(0) = y^{(i)}(x_\theta),\ i = 0, 1, \cdots, m-1$. By solving algebraic system Eq. (54), the unknown constants can be determined as follows

$$I_k = \gamma_k,\ k = 0, 1, \cdots, q. \tag{55}$$

By substituting, the obtained values I_k into Eq. (53), we can obtain

$$\mathbf{y}^{(i)}(\chi) = \Psi_i(\chi, x_\theta, \gamma_0, \gamma_1, \cdots, \gamma_q, \mathbf{y}(0), \mathbf{y}'(0), \cdots, \mathbf{y}^{(m-1)}(0)),\ i = 0, 1, \cdots, m-1, \tag{56}$$

which is Taylor expansion approximate solution Eq. (42) and Eq. (43) for $\mathbf{y}(\chi)$ and its derivations up to $m + \alpha - 1$ at $\chi = 0$, and by change of variable $\chi = x - x_\theta$, we have

$$y^{(i)}(x) = \Psi_i(x - x_\theta, x_\theta, \gamma_0, \gamma_1, \cdots, \gamma_q, \mathbf{y}(0), \mathbf{y}'(0), \cdots, \mathbf{y}^{(m-1)}(0)),\ i = 0, 1, \cdots, m-1, \tag{57}$$

which is Taylor expansion approximate solution Eq. (33) and Eq. (34) for $y(x)$ and it's derivations up to $m + \alpha - 1$ at $x = x_\theta$.

3.1 Algorithm of the approach

In this section, we propose an algorithm on the basis of the above discussions. This algorithm is presented in two stages such as initialization step and main steps.

Initialization step:

Choose step size $h > 0$ for equidistance partition $\triangle$ on I. Set

$$\mathbf{y}^{(j)}(0) = y^{(j)}(0),\ j = 0, 1, \cdots, m-1.$$

Set $\theta = 0$ and go to main steps.

Main steps:

Step 1. By Eq. (53) and Eq. (55), compute the following approximate solution

$$y^{(i)}(x) = \Psi_i(x - x_\theta, x_\theta, \gamma_0, \gamma_1, \cdots, \gamma_q, \mathbf{y}(0), \mathbf{y}'(0), \cdots, \mathbf{y}^{(m-1)}(0)),\ i = 0, 1, \cdots, m-1, \tag{58}$$

which is Taylor expansion approximate Eq. (33) and Eq. (34) for $y(x)$ and it's derivations up to $m - 1$ at $x = x_\theta$. Go to next Step.

Step 2. Set $\theta = \theta + 1$, if $\theta > \theta_r - 1$, stop; Otherwise, using Eq. (58) compute the approximate values

$$\mathbf{y}^{(i)}(0) = y^{(i)}(x_\theta),\ i = 0, 1, \cdots, m-1. \tag{59}$$

which are the initial conditions Eq. (33) and Eq. (34) at $x = x_\theta$ and go to Step 1.

Note that in the Step 2, if we replaced condition $\theta > \theta_l - 1$, with the condition $\theta > \theta_l + 1$ and also $\theta - 1$ to $\theta + 1$, then by applying the above algorithm we obtain Taylor expansion approximate solution of Eq. (33) and Eq. (34) and its derivations at $x = x_\theta,\ \theta = -1, -2, \cdots, \theta_l$.

4 Numerical examples

To illustrate the effectiveness of the proposed method we consider several test examples to carried out the applicability of our approach.

Example 1. Consider the first order linear integro-differential equation[22]

$$y' = 1 + \lambda \int_0^x y(t)dt, -1 \leq x, t \leq 1. \tag{60}$$

We consider this Example in the two different cases: one is that $\lambda = -1,\ y(0) = 0$ and corresponding exact solutions $y(x) = \sin x$ and other is $\lambda = 1,\ y(0) = 1$ and corresponding exact solutions $y(x) = e^x$. In the case of $\lambda = -1,\ y(0) = 0$, we explain the steps of our method for $n = 3$ and $h = 0.1$ as follows.

By using theorem 1 linear integro-differential equations convert to the following form

$$\mathbf{y}' = 1 + c - \int_0^{\chi} \mathbf{y}(\xi)d\xi, \tag{61}$$

with initial condition

$$\mathbf{y}'(0) = y(x_\theta), \tag{62}$$

where $c = 1 - \mathbf{y}'(0)$.

Let $n = 4$, in Eq. (11), by solving corresponding algebraic system Eq. (25) and by using Eq. (28), we can obtain

$$\begin{aligned}\mathbf{y}(\chi) =&\Bigg(\frac{\chi^9}{1440} + \frac{c\,\chi^9}{1440} - \frac{\chi^{11}}{11200} - \frac{c\,\chi^{11}}{11200} + \frac{\chi^{13}}{518400} + \frac{c\,\chi^{13}}{518400} \\ &- \frac{\chi^{15}}{108864000} - \frac{c\,\chi^{15}}{108864000} + \frac{\chi^8\,\mathbf{y}(0)}{1440} - \frac{97\,\chi^{10}\,\mathbf{y}(0)}{302400} \\ &+ \frac{59\,\chi^{12}\mathbf{y}(0)}{3628800} - \frac{\chi^{14}\,\mathbf{y}(0)}{7257600}\Bigg) \times \left(\frac{\chi^8}{1440} + \frac{\chi^{10}}{37800} + \frac{\chi^{12}}{1814400} + \frac{\chi^{16}}{435456000}\right)^{-1}\end{aligned} \tag{63}$$

and

$$\begin{aligned}\mathbf{y}'(\chi) =&\Bigg(\frac{\chi^8}{1440} + \frac{c\,\chi^8}{1440} - \frac{97\,\chi^{10}}{302400} - \frac{97c\,\chi^{10}}{302400} + \frac{59\,\chi^{12}}{3628800} \\ &+ \frac{59c\,\chi^{12}}{3628800} - \frac{\chi^{14}}{7257600} - \frac{c\,\chi^{14}}{7257600} - \frac{\chi^9\,\mathbf{y}(0)}{1440} + \frac{\chi^{11}\mathbf{y}(0)}{11200} \\ &- \frac{\chi^{13}\,\mathbf{y}(0)}{518400}\Bigg) \times \left(\frac{\chi^8}{1440} + \frac{\chi^{10}}{37800} + \frac{\chi^{12}}{1814400} + \frac{\chi^{16}}{435456000}\right)^{-1}.\end{aligned} \tag{64}$$

In the initialization step of algorithms (2.1), by setting $c = 0$, $\theta = 0$, $y(0) = 0$ into Eq. (63), Eq. (64), and by using Eq. (29), we have

$$\begin{aligned}y(x) =&\left(\frac{x^9}{1440} - \frac{x^{11}}{11200} + \frac{x^{13}}{518400} - \frac{x^{15}}{108864000}\right) \\ &\times \left(\frac{x^8}{1440} + \frac{x^{10}}{37800} + \frac{x^{12}}{1814400} + \frac{x^{16}}{435456000}\right)^{-1}\end{aligned} \tag{65}$$

and

$$\begin{aligned}y'(x) =&\left(\frac{x^8}{1440} - \frac{97\,x^{10}}{302400} + \frac{59\,x^{12}}{3628800} - \frac{x^{14}}{7257600}\right) \\ &\times \left(\frac{x^8}{1440} + \frac{x^{10}}{37800} + \frac{x^{12}}{1814400} + \frac{x^{16}}{435456000}\right)^{-1},\end{aligned} \tag{66}$$

which is Taylor expansion approximation solution Eq. (1) and Eq. (2) and its derivative at $x = 0$. In the next step, setting $x = h,\ (\theta = 1)$, into Eq. (65) and Eq. (66), the following approximate values can be obtain

$$\begin{cases} \mathbf{y}(0) = 0.0998334, \\ \mathbf{y}'(0) = 0.995004, \\ c = 0.00499583, \end{cases} \tag{67}$$

where $\mathbf{y}(0) = y(0.1)$ and $\mathbf{y}'(0) = y'(0.1)$.

Now again, by setting the obtained values $c,\ \mathbf{y}(0)$, $\mathbf{y}'(0)$ and $\theta = 1$, into Eq. (63), Eq. (64), and by using Eq. (29), Taylor approximation solution Eq. (1) and Eq. (2) and its derivative at $x = h$ can be obtained.

Repeating the above procedure for $\theta = -2, \pm 3, \cdots, \pm 9$, we can obtain Taylor approximate solution Eq. (60) at $x = \theta h$ in the case of $\lambda = -1,\ y(0) = 0$.

Consequently, by using similarly procedure, we can obtain the approximate solution Eq. (60) for $\lambda = 1,\ y(0) = 1$ in the interval $[-1, 1]$. Also we employ the Taylor series method[6] to determine the three- and five-order approximate solution for $\lambda = -1,\ y(0) = 1$ and $\lambda = 1,\ y(0) = 1$. From numerical results in Tab. 1 and 2, we can find that the accuracy of approximate solution by Modified Taylor series is more accurate than the solution by Taylor eries method, particularly at end points. Also, Tab. 1 and Tab. 2 show that proposed method can be used for broad intervals.

Table 1. Absolute error of Example 1. ($\lambda = -1,\ y(0) = 0$)

x_i	Taylor series[6] n = 3	Present method n = 3	Taylor series[6] n = 5	Present method n = 5
1.0	4.34827×10^{-6}	2.13995×10^{-12}	3.52581×10^{-11}	4.44089×10^{-16}
2.0	0.01889	1.18233×10^{-11}	1.88557×10^{-07}	1.66533×10^{-15}
3.0	0.03201	2.06334×10^{-11}	1.66450×10^{-05}	2.77556×10^{-16}
4.0	0.04338	1.17659×10^{-11}	1.73967×10^{-04}	2.22045×10^{-15}
5.0	–	1.65207×10^{-11}	2.62631×10^{-06}	3.44169×10^{-15}
6.0	–	4.02117×10^{-11}	0.00150	6.77236×10^{-15}
7.0	–	2.97748×10^{-11}	0.04675	4.21885×10^{-15}
8.0	–	1.55588×10^{-11}	–	6.77236×10^{-15}
9.0	–	5.75580×10^{-11}	–	1.71529×10^{-14}
10	–	5.09699×10^{-11}	–	1.44329×10^{-14}

Table 2. Absolute error of Example 1. ($\lambda = 1,\ y(0) = 1$)

x_i	Taylor series[6] n = 3	Present method n = 3	Taylor series[6] n = 5	Present method n = 5
1.0	1.69146×10^{-05}	1.06737×10^{-11}	1.34426×10^{-10}	2.22045×10^{-15}
2.0	0.01058	7.62759×10^{-11}	1.23210×10^{-06}	2.66454×10^{-15}
3.0	–	3.42535×10^{-10}	4.06089×10^{-04}	2.48690×10^{-14}
4.0	–	1.30105×10^{-09}	0.03611	9.94760×10^{-14}
5.0	–	4.54253×10^{-09}	–	2.55795×10^{-13}
6.0	–	1.50826×10^{-08}	–	2.61480×10^{-12}
7.0	–	4.84324×10^{-08}	–	1.04592×10^{-11}
8.0	–	1.5186×10^{-07}	–	4.09273×10^{-11}
9.0	–	4.67726×10^{-07}	–	1.42791×10^{-10}
10	–	1.42071×10^{-06}	–	4.69299×10^{-10}

Example 2. Consider the second order linear integro-differential equation[6, 22]

$$y'' + xy' - xy = e^x - 2\sin x + \int_{-1}^{1} e^{-t} y(t) \sin x dt, \quad -1 \leq x,\ t \leq 1, \tag{68}$$

with initial conditions $y(0) = 1,\ y'(0)$, which is the exact solution $y(x) = e^x$.

Our method is applied in this Example with $h = 0.1$ and with order of approximate $n = 4$. The absolute error in the grid points are given and also compared the absolute error in the solution with [6] are tabulated in Tab. 3. From Tab. 3, we fined that numerical results based on the present method are closer to the exact solution than those in [6].

Table 3. Absolute error of Example 2

x_i	Taylor series[6] n = 4	Present method n = 4
-1.0	0.0011	8.1745×10^{-4}
-0.8	4.2151×10^{-4}	4.5573×10^{-4}
-0.6	1.2297×10^{-4}	2.0505×10^{-4}
-0.4	2.0981×10^{-5}	6.3521×10^{-5}
-0.2	4.5690×10^{-7}	8.1471×10^{-6}
0.2	8.15147×10^{-6}	8.1515×10^{-6}
0.4	1.41856×10^{-4}	6.3792×10^{-5}
0.6	9.19094×10^{-4}	2.0801×10^{-4}
0.8	0.0039	4.7141×10^{-4}
1.0	0.0130	8.7295×10^{-4}

Example 3. Consider the following high order linear Fredholm integro-differential equation

$$y^{(5)} - x^2 y^{(3)} - y' - xy = x^2 \cos x - x \sin x + \int_{-1}^{1} y(t)dt, \ \ -1 \le x, \ t \le 1, \tag{69}$$

with initial condition

$$y(0) = y''(0) = y^{(4)} = 0, \ y'(0) = 1, \ y^{(3)} = -1$$

and exact solution $y(x) = \sin x$.

We approximate the solution by $n = 5, \ h = 0.2$ of our method. Tab. 4 shows that the numerical results of our method are better than results obtained in [6].

Table 4. Absolute error of Example 3

x_i	Taylor method[6] n = 5	Present method n = 5
-1.0	5.1213×10^{-5}	6.8536×10^{-9}
-0.8	9.6014×10^{-6}	2.2810×10^{-9}
-0.6	9.6980×10^{-7}	5.6666×10^{-10}
-0.4	3.5460×10^{-8}	8.3216×10^{-11}
-0.2	1.2546×10^{-10}	6.8536×10^{-9}
0.2	1.2967×10^{-10}	4.1369×10^{-12}
0.4	4.3962×10^{-10}	8.3216×10^{-11}
0.6	1.6841×10^{-6}	5.6666×10^{-10}
0.8	2.5717×10^{-5}	2.2899×10^{-9}
1.0	2.2361×10^{-4}	6.85359×10^{-9}

Example 4. Finally, consider the following third order linear Voltera integro-differential equation

$$y^{(3)} - xy^{(2)} = \frac{4}{7}x^9 - \frac{8}{5}x^7 - x^6 + 6x^2 - 6 + 4\int_0^x x^2 t^3 y(t)dt, \ 0 \le x, \ t \le 1, \tag{70}$$

with the initial conditions $y(0) = 1, \ y'(0) = 2, \ y^{(2)}(0) = 0$. We find the approximate solution by the proposed method for $n = 3$ is the same as $y(x) = -x^3 + 2x + 1$, the exact solution.

5 Conclusion

The approximation by Taylor series method in [7–10, 22] has the appropriate accuracy at the closed neighborhood of the point of expansion, but for the points far from points the accuracy of the approximation reduced considerably, to overcome such difficultly, we modified and develop Taylor expansion approximation to obtain Taylor expansion approximate at grade points $x = x_i$ on the interval $[a, b]$. The comparison of the results obtained by the presented method and conventional Taylor series method reveals that ours method is very effective and convenient. The efficiency of the algorithms are verified by some test problems.

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