ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 9 (2013) No. 4, pp. 262-268

Solution of singular two point boundary value problems with forcing function in integral form via differential transform method

Kirubanandam Aruna[1]*, Adivi Sri Venkata Ravi Kanth[2]

[1] Fluid Dynamics Division, School of Advanced Sciences, VIT University, Vellore-632014,India
[2] Department of Mathematics, National Institute of Technology, Kurukshetra-136119, Haryana, India

(*Received February 14 2013, Revised May 1 2013, Accepted July 18 2013*)

Abstract. This paper concerns the solution of singular two point boundary value problems with forcing function in the integral form via the differential transform method. Three examples are given to show ability of the proposed method for solving the singular two point boundary value problems with forcing function in integral form.

Keywords: singular boundary value problems, differential transform method

1 Introduction

We consider the singular two-point boundary value problems with forcing function in integral form

$$y''(x) + \frac{\alpha}{x}y'(x) - \frac{\alpha}{x^2}y(x) = f(x, y, y') + \int_0^1 \gamma(x, s)ds,\ 0 < x < 1,\ 0 < s < 1, \tag{1}$$

subject to the boundary conditions

$$y(0) = 0,\ y(1) = \beta, \tag{2}$$

where β is a finite constant. We assume that $\gamma(x, s)$ is a real valued function of both variables in the range $0 \leq x, s \leq 1$. Let

$$w(x) = \int_0^1 \gamma(x, s)ds \tag{3}$$

and

$$f(x, y, y') + w(x) = g(x, y, y'). \tag{4}$$

Re-writing (1) as

$$y''(x) + \frac{\alpha}{x}y'(x) - \frac{\alpha}{x^2}y(x) = g(x, y, y'),\ 0 < x < 1. \tag{5}$$

Denote

$$\Psi = \{(x, y) : 0 < x < 1, -\infty < y, y' < \infty\}. \tag{6}$$

According to [12], if g, g_y and $g_{y'}$ are continuous on Ψ and $g_y > 0$, $g_{y'} \leq M$, where M is a constant, ensures the existence and uniqueness of the above boundary value problem. In addition we assume that $y(x) \in C^6[0, 1]$ and $\gamma(x, s) \in C^4[0, 1]$. There has been an intensive development of the theory of integral and integro-differential equations[1–4, 8, 13, 14, 17]. Seeking numerical solution for singular boundary value problems with forcing function in integral form are challenging because of singularity behavior at the origin.

* Corresponding author. Tel.: +91-416-2202749.
E-mail address: k.aruna@vit.ac.in

Published by World Academic Press, World Academic Union

Due to this reason, recently singular differential equation has been fascinated towards a number of researchers. Mohanty and Dhall[15] studied the third order accurate variable mesh discretization and application of TAGE iterative method for the nonlinear two point boundary value problems with homogeneous functions in integral form. In [16], a method based on cubic spline approximation and TAGE iterative method for the solution of two point boundary value problems with forcing function in integral form. To the authors knowledge no paper has been reported yet for the solution of singular two point boundary value problems with forcing function in integral form using the differential transform method. In this paper, we introduced the differential transform method as an alternative to existing methods for solving singular two point boundary value problems with forcing function in integral form. The use of the differential transform method in electric circuit analysis was pioneered by Zhou[25]. Since then, differential transform method was successfully applied for large variety of problems. For instance, initial value problems[10], partial differential equations[11], system of ordinary differential equation[5], integro-differential equation[6], differential-difference equation[6], diferential algebraic equations[9], singular boundary value problems[19], linear and nonlinear system of partial differential equations[20], linear and nonlinear Klein-Gordon equations[21], linear and nonlinear Schrodinger equation[22], MHD boundary layer equations[18], nonlinear volterra integro-differential equations[24], Hirota-Satsuma coupled KdV equation[23] and so on. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor's series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally intensive and inefficient especially for the differential equations of high orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. The paper is organized in four sections. Basic concept of the differential transform method is discussed in Section 2. The present method has been applied to the singular boundary value problems with forcing function in integral form in Section 3. In Section 4 the method is applied to three test problems and numerical results are given to demonstrate the applicability and the effectiveness of the proposed method.

2 Basic ideas of the differential transform method

An arbitrary function $y(x)$ can be expanded in Taylor series about a point $x = 0$ as

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left| \frac{d^k y(x)}{dx^k} \right|_{x=0}. \tag{7}$$

The differential transform of $y(x)$ is defined[25] as

$$Y(k) = \frac{1}{k!} \left| \frac{d^k y(x)}{dx^k} \right|_{x=0}, \tag{8}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function.
The differential inverse transform is

$$y(x) = \sum_{k=0}^{\infty} Y(k) x^k. \tag{9}$$

In actual applications, the function $y(x)$ is expressed by a finite series and Eq. (9) can be rewritten as follows

$$y(x) = \sum_{k=0}^{n} Y(k) x^k, \tag{10}$$

which means that $y(x) = \sum_{k=n+1}^{\infty} Y(k) x^k$ is small, negligibly. Usually the value of n are decided by convergence of the series coefficients. The fundamental mathematical operations performed by the differential transform method are listed in Tab. 1.

Table 1. The fundamental operations of the differential transform method

Original function	Transformed function
$y(x) = g(x) \pm h(x)$	$Y(k) = G(k) \pm H(k)$
$y(x) = \alpha g(x)$	$Y(k) = \alpha G(k)$
$y(x) = \frac{dg(x)}{dx}$	$Y(k) = (k+1)G(k+1)$
$y(x) = \frac{d^2 g(x)}{dx^2}$	$Y(k) = (k+1)(k+2)G(k+2)$
$y(x) = g(x)h(x)$	$Y(k) = \sum_{l=0}^{k} G(l)H(k-l)$
$y(x) = x^m$	$Y(k) = \delta(k-l) = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}$
$f(y(x)) = y^n(x)$	$F(Y(k)) = \sum_{l=0}^{k} Y(k-l) \sum_{m=0}^{l} Y(l-m) \ldots \sum_{r=0}^{n} Y(n-r)$
$y(x) = \int_{x_0}^{x_1} g(t)$	$Y(k) = \frac{G(k-1)}{k}$, where $k \geq 1,\ Y(0) = 0$

3 Description of the method

The differential transform of (5) is

$$\sum_{l=0}^{k} \delta(l-2)(k-l+1)(k-l+2)Y(k-l+2) + \alpha \sum_{l=0}^{k} \delta(l-1)(k-l+1)Y(k-l+1) - \alpha Y(k)$$
$$= \sum_{l=0}^{k} \delta(l-2)G(k-l), \tag{11}$$

where $G(k)$ is the transformed version of $g(x, y, y')$ where $g(x, y, y') = f(x, y, y') + w(x)$.
The transformed conditions (2) are

$$Y(0) = 0,\ \sum_{k=0}^{n} Y(k) = \beta. \tag{12}$$

Substituting (12) into (11) and by recursive method we can calculate all values of $Y(k)$. Hence substituting $Y(k)$'s in (10) we get the series solution. The unknown value A can be determined by incorporating the boundary condition $y(1) = \beta$.

4 Numerical results

In this section, we applied the proposed technique to linear and nonlinear singular two point boundary value problems with forcing function in integral form. In order to assess the reliability and efficiency of the proposed method we compared our results with existing results in the literature. The Root mean square errors:

$$\text{(RMS errors)} = \sqrt{\frac{\sum (y_i - \tilde{y}_i)^2}{N}},$$

where N is the total number of nodal points, y_i is the exact solution and $\tilde{y}_i$ is the approximate solution.
Example 1: First, we consider the linear singular two point boundary value problem with forcing function in integral form [15]

$$y''(x) + \frac{\alpha}{x} y'(x) - \frac{\alpha}{x^2} y(x) = \alpha \sinh x + (4+\alpha)x \cosh x + \int_0^1 \left[6 \left(\frac{xs}{1+s^2} \right)^2 \sinh x + 2x \cosh xs \right] ds, \tag{13}$$

subject to the boundary conditions

$$y(0) = 0,\ y(1) = \sinh 1. \tag{14}$$

Table 2. Error estimates for Example 1

x	Error estimate for $\alpha = 0$	Error estimate for $\alpha = 1$	Error estimate for $\alpha = 2$
0	0	0	0
0.1	7.67385×10^{-14}	7.67385×10^{-14}	7.67385×10^{-14}
0.2	1.53478×10^{-13}	1.53478×10^{-13}	1.53478×10^{-13}
0.3	2.30219×10^{-13}	2.30219×10^{-13}	2.30219×10^{-13}
0.4	3.06949×10^{-13}	3.06949×10^{-13}	3.06949×10^{-13}
0.5	3.83693×10^{-13}	3.83693×10^{-13}	3.83693×10^{-13}
0.6	4.60326×10^{-13}	4.60326×10^{-13}	4.60326×10^{-13}
0.7	5.35350×10^{-13}	5.35350×10^{-13}	5.35350×10^{-13}
0.8	5.96634×10^{-13}	5.96634×10^{-13}	5.96634×10^{-13}
0.9	5.62772×10^{-13}	5.62772×10^{-13}	5.62772×10^{-13}
1.0	2.22045×10^{-16}	2.22045×10^{-16}	2.22045×10^{-16}

Table 3. Comparisons of RMS errors for Example 1 obtained by the present method and the method in [15]

N	Proposed method RMS errors	RMS errors in [15] (nonuniform mesh)	RMS errors in [15] (uniform mesh)
		$\alpha = 1$	
11	$3.707e{-}13$	$6.230e{-}05$	$5.136e{-}05$
21	$3.397e{-}13$	$3.030e{-}06$	$6.628e{-}05$
31	$3.826e{-}13$	$6.527e{-}07$	$1.850e{-}06$
41	$3.841e{-}13$	$3.494e{-}07$	$7.302e{-}07$
51	$3.850e{-}13$	$2.740e{-}07$	$3.509e{-}07$
71	$3.861e{-}13$	$2.207e{-}07$	$9.147e{-}08$
81	$3.864e{-}13$	$2.057e{-}07$	$6.349e{-}08$
		$\alpha = 2$	
11	$3.707e{-}13$	$4.146e{-}04$	$2.828e{-}05$
21	$3.397e{-}13$	$1.527e{-}04$	$3.357e{-}06$
31	$3.826e{-}13$	$1.025e{-}04$	$9.010e{-}07$
41	$3.841e{-}13$	$8.329e{-}05$	$3.470e{-}07$
51	$3.850e{-}13$	$7.295e{-}05$	$1.636e{-}07$
71	$3.861e{-}13$	$6.116e{-}05$	$3.242e{-}08$
81	$3.864e{-}13$	$5.720e{-}05$	$1.322e{-}08$

The exact solution of this problem is $y(x) = x^2 \sinh x$. Tab. 2 exhibits the error estimates for different values of α obtained by the 15 terms differential transform method solution and the exact solution. Tab. 3 shows the comparison between the RMS errors obtained by the proposed method and the method in [15].

Example 2: Next, we consider the nonlinear singular two point boundary value problem with forcing function in integral form [15]

$$\begin{aligned} y''(x) + \frac{\alpha}{x}y'(x) - \frac{\alpha}{x^2}y(x) =& y(x)y'(x) + [x^2 + (\alpha+4)x + \alpha]e^x - (x^4 + 2x^3)e^{2x} + 4 \\ &+ \int_0^1 e^{xs}\left[\frac{xs^3 - 3s^2 + x}{(1+s^3)^2}\right] ds, \end{aligned} \tag{15}$$

subject to the boundary conditions

$$y(0) = 0,\ y(1) = e. \tag{16}$$

The exact solution of this problem is $y(x) = x^2 e^x$. Errors obtained by 15 terms differential transform method solution and exact solution for different values of α are given in Tab. 4. In Tab. 5, we compared the proposed method RMS errors with the method in [15].

Example 3: : Finally, we consider the nonlinear singular two point boundary value problem with forcing function in integral form [16]

Table 4. Errors estimates for Example 2

x	Error estimate for $\alpha = 0$	Error estimate for $\alpha = 1$	Error estimate for $\alpha = 2$
0	0	0	0
0.1	6.90174×10^{-13}	7.57758×10^{-13}	8.11160×10^{-13}
0.2	1.38322×10^{-12}	1.51806×10^{-12}	1.62459×10^{-12}
0.3	2.08784×10^{-12}	2.28860×10^{-12}	2.44726×10^{-12}
0.4	2.82163×10^{-12}	3.08520×10^{-12}	3.29331×10^{-12}
0.5	3.61544×10^{-12}	3.93519×10^{-12}	4.18759×10^{-12}
0.6	4.51805×10^{-12}	4.88176×10^{-12}	5.16831×10^{-12}
0.7	5.58520×10^{-12}	5.97000×10^{-12}	6.27265×10^{-12}
0.8	6.73372×10^{-12}	7.09766×10^{-12}	7.38321×10^{-12}
0.9	6.87117×10^{-12}	7.13363×10^{-12}	7.33791×10^{-12}
1.0	0	0	0

Table 5. Comparisons of RMS errors for Example 2 obtained by the present method and the method in [15]

N	Proposed method RMS errors	RMS errors in [15] (nonuniform mesh)	RMS errors in [15] (uniform mesh)
		$\alpha = 0$	
11	$5.476e{-12}$	$3.795e{-04}$	$2.470e{-05}$
21	$5.621e{-12}$	$2.021e{-04}$	$2.140e{-06}$
31	$5.663e{-12}$	$1.614e{-04}$	$4.744e{-07}$
41	$5.684e{-12}$	$1.398e{-04}$	$1.592e{-07}$
51	$5.696e{-12}$	$1.253e{-04}$	$6.764e{-08}$
61	$5.704e{-12}$	$1.146e{-04}$	$2.935e{-07}$
		$\alpha = 1$	
11	$4.225e{-12}$	$1.606e{-03}$	$4.950e{-05}$
21	$4.332e{-12}$	$7.669e{-04}$	$6.622e{-06}$
31	$4.366e{-12}$	$5.976e{-04}$	$1.867e{-06}$
41	$4.383e{-12}$	$5.152e{-04}$	$7.397e{-07}$
51	$4.393e{-12}$	$4.613e{-04}$	$3.561e{-07}$
61	$4.400e{-12}$	$4.217e{-04}$	$1.145e{-07}$
		$\alpha = 2$	
11	$4.418e{-12}$	$4.521e{-03}$	$2.945e{-05}$
21	$4.529e{-12}$	$2.790e{-03}$	$3.494e{-06}$
31	$4.564e{-12}$	$2.272e{-03}$	$9.356e{-07}$
41	$4.582e{-12}$	$1.974e{-03}$	$3.594e{-07}$
51	$4.593e{-12}$	$1.770e{-03}$	$1.695e{-07}$
61	$4.600e{-12}$	$1.618e{-03}$	$6.109e{-08}$

$$y''(x) + \frac{\alpha}{x}y'(x) - \frac{\alpha}{x^2}y(x) = y(x)y'(x) + (2+\alpha+x^2)\cosh x - x^3(x\sinh x + 2\cosh 2x)\cosh x + (4+\alpha)\int_0^1 x^2 \cosh xs ds, \quad (17)$$

subject to the boundary conditions

$$y(0) = 0,\ y(1) = \cosh 1. \quad (18)$$

The exact solution of this problem is $y(x) = x^2 \cosh x$. Tab. 6 shows the errors obtained by the proposed method and the exact solution for different values of α. Comparison of RMS errors between the proposed method and the method[16] are given in Tab. 7.

Table 6. Errors estimates for Example 3

x	Error estimate for $\alpha = 0$	Error estimate for $\alpha = 1$	Error estimate for $\alpha = 2$
0	0	0	0
0.1	8.15868×10^{-13}	8.67126×10^{-13}	9.05076×10^{-13}
0.2	1.63464×10^{-12}	1.73672×10^{-12}	1.81230×10^{-12}
0.3	2.46407×10^{-12}	2.61541×10^{-12}	2.72744×10^{-12}
0.4	3.31818×10^{-12}	3.51516×10^{-12}	3.66096×10^{-12}
0.5	4.21913×10^{-12}	4.45483×10^{-12}	4.62930×10^{-12}
0.6	5.19795×10^{-12}	5.46069×10^{-12}	5.65498×10^{-12}
0.7	6.27731×10^{-12}	6.54743×10^{-12}	6.74705×10^{-12}
0.8	7.31837×10^{-12}	7.56384×10^{-12}	7.74503×10^{-12}
0.9	7.17448×10^{-12}	7.34235×10^{-12}	7.46581×10^{-12}
1.0	2.22045×10^{-16}	2.22045×10^{-16}	2.22045×10^{-16}

Table 7. Comparisons of RMS errors for Example 3 obtained by the present method and the method in [15]

N	Proposed method RMS errors	RMS errors in [15] (nonuniform mesh)	RMS errors in [15] (uniform mesh)
		$\alpha = 0$	
11	$4.392e{-}12$	$1.315e{-}04$	$1.039e{-}05$
21	$4.501e{-}12$	$7.164e{-}05$	$9.062e{-}07$
31	$4.536e{-}12$	$5.732e{-}05$	$2.011e{-}07$
41	$4.554e{-}12$	$4.969e{-}05$	$6.755e{-}08$
51	$4.564e{-}12$	$4.454e{-}05$	$2.868e{-}08$
		$\alpha = 1$	
11	$4.251e{-}12$	$8.507e{-}04$	$1.079e{-}05$
21	$4.344e{-}12$	$3.669e{-}04$	$1.077e{-}06$
31	$4.375e{-}12$	$2.837e{-}04$	$2.540e{-}07$
41	$4.391e{-}12$	$2.443e{-}04$	$8.834e{-}08$
51	$4.401e{-}12$	$2.187e{-}04$	$3.839e{-}08$
		$\alpha = 2$	
11	$4.695e{-}12$	$2.346e{-}03$	$1.130e{-}05$
21	$4.809e{-}12$	$1.413e{-}03$	$1.071e{-}06$
31	$4.846e{-}12$	$1.149e{-}03$	$2.457e{-}06$
41	$4.865e{-}12$	$9.989e{-}04$	$8.410e{-}08$
51	$4.876e{-}12$	$8.955e{-}04$	$3.615e{-}08$

5 Conclusions

In this paper, the differential transform method has been applied successfully for solving linear and nonlinear singular two point boundary value problems with forcing function in integral form. Three test examples are given to demonstrate the efficiency of the proposed method. It is observed from the tables that the proposed method is more efficient than the methods in [15, 16]. Also, it is observed that the proposed solution method is very good agreement with the exact solution. Thus, it is concluded that the proposed method can be applied directly to a wide class of linear and nonlinear problems without requiring linearization, discretization or perturbation.

References

[1] R. Agarwal, D. Regan. *Integral and integro-differential equations: theory, method and applications*. Gordon and Breach, London, 2000.

[2] K. Atkinson. *The numerical solution of integral equations of the second kind*. Cambridge University Press, Cambridge, 1997.

[3] K. Atkinson. A survey of numerical methods for solving nonlinear integral equation. *Journal of integral Equations and Applications*, 1992, **4**: 15–40.

[4] K. Atkinson, J. Flores. The discrete collocation method for nonlinear integral equations. *IMA Journal of Numerical Analysis*, 1993, **13**:195–213.

[5] F. Ayaz. Solution of the system of differential equations by differential transform method. *Applied Mathematics and Computation*, 2004, **147**:547–567.

[6] A. Arikoglu, I. Ozkol. Solution of boundary value problems for integro-differential equations by using differential transform method. *Applied Mathematics and Computation*, 2005, **168**:1145–1158.

[7] A. Arikoglu, I. Ozkol. Solution of differential-difference equations by using differential transform method. *Applied Mathematics and Computation*, 2006, **181**(1): 153–162.

[8] M. Dehghan, A. Saadatmandi. Chebyshev finite difference method for Fredholm integro-differential equation. *International Journal of Computer Mathematics*, 2008, **85**: 123–130.

[9] H. Liu, Y. Song. Differential transform method applied to high index differential-algebraic equations. *Applied Mathematics and Computation*, 2007, **184**(2): 748–753.

[10] M. Jang, C. Chen. On Solving the initial-value problems using the differential transformation method. *Applied Mathematics and Computation*, 2000, **115**(2-3): 145–160.

[11] M. Jang, C. Chen, Y. Liu. Two-dimensional differential transform for partial differential equations. *Applied Mathematics and Computation*, 2001, **121**: 261–270.

[12] H. Keller. *Numerical methods for two-point boundary value problems*. Waltham Mass, Blaisdel, New York, 1968.

[13] V. Lakshmikantham, M. Rao. *Theory of integro-differential equations*. Gordon and Breach, London,1995.

[14] P. Linz. A method for approximate solution of linear integro-differential equations. *SIAM Journal of Numerical Analysis*, 1974, **11**: 137–144.

[15] R. Mohanty, D. Dhall. Third order accurate variable mesh discretization and application of TAGE iterative method for the non-linear two-point boundary value problems with homogeneous functions in integral form. *Applied Mathematics and Computation*, 2009, **215**: 2024–2034.

[16] R. Mohanty, M. Jain, D. Dhall. A cubic spline approximation of TAGE iterative method for the solution of two point boundary value problems with forcing function in integral form. *Applied Mathematics and Modeling*, 2011, **35**: 3036–3047.

[17] M. Rashed. Numerical solution of a special type of integro-differential equations. *Applied Mathematics and Computation*, 2003, **143**: 73–88.

[18] M. Rashidi. The modified differential transform method for solving MHD boundary-layer equations. *Computer Physics Communications*, 2009, **180**(11): 2210–2217.

[19] A. Ravi Kanth, K. Aruna. Solution of singular two-point boundary value problems using differential transformation method. *Physics Letters A*, 2008, **372**: 4671–4673.

[20] A. Ravi Kanth, K. Aruna. Differential transform method for solving linear and non-linear systems of partial differential equations. *Physics Letters A*, 2008, **372**(46): 6896–6898.

[21] A. Ravi Kanth, K. Aruna. Differential transform method for solving the linear and nonlinear Klein-Gordon equation. *Compuer Physics Communications*, 2009, **180**(5): 708–711.

[22] A. Ravi Kanth, K. Aruna. Two-dimensional differential transform method for solving linear and non-linear Shrodinger equations. *Chaos, Solitons and Fractals*, 2009, **41**(5): 2277–2281

[23] R. Abazari, M. Abazari. Numerical simulation of generalized Hirota-Satsuma coupled KdV equation by RDTM and comparison with DTM. *Communications in Nonlinear Science and Numerical Simulation*, 2012, **17**(2): 619–629.

[24] A. Tari, S. Shahmorad. Differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations. *Computer Mathematics with Applications*, 2011, **61**(9): 2621–2629.

[25] J. Zhou. *Differential transform and its Applications for Electrical Circuits*. Huazhong University Press: Wuhan, China 1986.