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The Nehari manifold for quasilinear equation with Robin boundary condition

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Abstract. Using the Nehari manifold, We prove the existence of positive solutions of the problem $\Delta_p u = \lambda a(x)u|u|^{\alpha-1} + b(x)u|u|^{\gamma-1}$ for $x \in \Omega$, together with the boundary condition $\alpha u|u|^{p-2} + (1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} = 0$. Exploiting the relationship between the Nehari manifold and Fibering maps (i.e., maps of the form $t \to J_\lambda(u)$), where $J_\lambda(u)$ is the Euler functional associated with the equation), and a condition on $b(x)$, we show how existence results for positive solutions of the equations are linked to properties of the Nehari manifold.

Keywords: variational method, nehari manifold, minimizing sequence, critical sobolev exponent

1 Introduction

In this paper we study the existence of positive solutions for the problem:

$$\begin{cases} -\Delta_p u(x) = \lambda a(x)u|u|^{\alpha-1} + b(x)u|u|^{\gamma-1}, & x \in \Omega, \\ \alpha u|u|^{p-2} + (1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Δ_p is the p-Laplacian operator, i.e., $\Delta_p z = (div(|\nabla z|^{p-2}\nabla z)$, Ω is a smooth bounded domain in $\mathbb{R}^n$,

$$0 < \alpha < 1 < p-1 < \gamma - 1 < p^* - 1,$$
$$\left(p^* = \frac{np}{n-p} \text{ if } p < n,\ p^* = \infty\, if\ p \geq n\right), \lambda > 0$$

is a real parameter and $a, b : \Omega \to \mathbb{R}$ are smooth functions which change sign on Ω.

Similar problems have been studied in [2] by using variational methods and in [1], [4] and [7] by using the Nehari manifold. Furthermore this problem in the case $p = 2$ and Dirichlet boundary condition has been studied by Brown and Wu[3]. In this paper we show how a fairly complete knowledge of all possible forms of the fibering maps provides a very simple and comparatively elementary means of establishing results similar to those proved in [7] and [10] on the existence of multiple solutions for the problem Eq. (1).

The plane of the paper is as follows. In Section 2 we first recall the facts that we shall require about the Nehari manifold and examine carefully the connection between the Nehari manifold and the Fibering maps. In Section 3 we use this information to give a very simple variational proof of the existence of solutions of Eq. (1) for sufficiently small λ.

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2 Notation and preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We will work in a Sobolev space $W := W^{1,p}(\Omega)$ equipped with the norm

$$\|u\| = \left(\int (|\nabla u(x)|^p + u(x)^p)dx\right)^{\frac{1}{p}}.$$

First we introduce the structure of the Nehari manifold. It is well known that the steady state solutions of Eq. (1) correspond to critial points of the functional

$$J_\lambda(u) = \frac{1}{p}M(u) - \frac{\lambda}{\alpha+1}A(u) - \frac{1}{\gamma+1}B(u), \tag{2}$$

where

$$\begin{aligned} M(u) &= \int_\Omega |\nabla u|^p dx - \frac{\alpha}{\alpha-1}\int_{\partial\Omega} |u|^p ds, \\ A(u) &= \int_\Omega a(x)|u|^{\alpha+1}dx, \text{ and} \\ B(u) &= \int_\Omega b(x)|u|^{\gamma+1}dx. \end{aligned}$$

This functional is bounded neither above nor below. In order to obtain existence results in this case we introduce the Nehari manifold $S = \{u \in W; \langle J'_\lambda(u), u\rangle = 0\}$, where $\langle , \rangle$ denotes the usual duality. Thus $u \in S$ if and only if $M(u) - \lambda A(u) - B(u) = 0$.

Clearly S is a smaller set than W and so it is easier to study J_λ on S.

For $u \in S$ we have that

$$\begin{aligned} J_\lambda(u) =& \lambda\left(\frac{1}{p} - \frac{1}{\alpha+1}\right)A(u) + \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)B(u) \\ =& \left(\frac{1}{p} - \frac{1}{\alpha+1}\right)M(u) + \left(\frac{1}{\alpha+1} - \frac{1}{\gamma+1}\right)B(u) \\ =& \left(\frac{1}{p} - \frac{1}{\gamma+1}\right)M(u) + \lambda\left(\frac{1}{\gamma+1} - \frac{1}{\alpha+1}\right)A(u). \end{aligned} \tag{3}$$

The Nehari manifold is closely linked to the behaviour of the functions of the from $\phi_u : t \longmapsto J_\lambda(tu)(t > 0)$. Such maps are known as fibering maps and were introduced by Drabek and Pohozaev in [6] and are also discussed in Brown and Zhang[4]. If $u \in W$, we have

$$\phi_u(t) = \frac{1}{p}t^p M(u) - \lambda\frac{t^{\alpha+1}}{\alpha+1}A(u) - \frac{t^{\gamma+1}}{\gamma+1}B(u), \tag{4}$$

$$\phi'_u(t) = t^{p-1}M(u) - \lambda t^\alpha A(u) - t^\gamma B(u), \tag{5}$$

$$\phi''_u(t) = (p-1)t^{p-2}M(u) - \lambda\alpha t^{\alpha-1}A(u) - \gamma t^{\gamma-1}B(u). \tag{6}$$

It is easy to see that $u \in S$ if and only if $\phi'(1) = 0$ and more generally, that $tu \in S$ if and only if $\phi'(t) = 0$, i.e., elements in S correspond to stationary points of the fibering maps.

Now, we split S into three parts:

$$S^+ = \{u \in S; \phi''_u(1) > 0\},\ S^- = \{u \in S : \phi''_u(1) < 0\},\ S^0 = \{u \in S : \phi''_u(1) = 0\},$$

where corresponding to local minima, local maxima and points of inflexion of fibering maps,respectively. Also, if $u \in S$, we have

$$\begin{aligned}\phi''_u(1) =&\lambda(p-\alpha-1)A(u)+(p-\gamma-1)B(u)\\ =&(p-\alpha-1)M(u)+(\alpha-\gamma)B(u)\\ =&(p-\gamma-1)M(u)+\lambda(\gamma-\alpha)A(u).\end{aligned} \tag{7}$$

Define

$$m_u(t)=t^{p-\alpha-1}M(u)-t^{\gamma-\alpha}B(u).$$

Hence

$$m'_u(t)=(p-\alpha-1)t^{p-\alpha-2}M(u)-(\gamma-\alpha)t^{\gamma-\alpha-1}B(u). \tag{8}$$

So we have

Lemma 1. *$tu\in S$ if and only if t is a solution of $m_u(t)=\lambda A(u)$, for $t>0$.*

Theorem 1. *(1) if $B(u)\le 0$, then m_u is a strictly increasing function. (2) if $B(u)\ge 0$, then $m_u(t)>0$ for t small and positive but $m_u(t)\to-\infty$ as $t\to\infty$, also $m_u(t)$ has a unique maximum stationary point.*

Now suppose that $tu\in S$. It follows from Eq. (7) and Eq. (8) that $\phi''_{tu}(1)=t^{\alpha+2}m'_u(t)$ and so $tu\in S^+(S^-)$ provided $m'_u(t)>0(<0)$. We define $L^+=\{u\in W; A(u)>0\}$, and L^- similary by replacing $>$ by $\le$. We also define $B^+=\{u\in W; B(u)>0\}$, and B^- analogously. So
(1) if $u\in L^-\cap B^-$, then $t\to\phi_u(t)$ is an increasing function of t and no multiple of u lies in S,
(2) if $u\in L^+\cap B^-$, then $t\to\phi_u(t)$ has a local minimum at $t=t(u)$ and $t(u)u\in S^+$,
(3) if $u\in L^-\cap B^+$, then $t\to\phi_u(t)$ has a local maximum at $t=t(u)$ and $t(u)u\in S^-$,
(4) if $u\in L^+\cap B^+$, then
$a)$ if $\lambda>0$ is sufficiently large, $t\to\phi_u(t)$ is a decreasing function and no multiple of u lies in S,
$b)$ If $\lambda>0$ is sufficiently small, $t\to\phi_u(t)$ has exactly two points- a local minimum at $t=t_1(u)$ and local maximum at $t=t_2(u)$, such that $t_1(u)u\in S^+$ and $t_2(u)u\in S^-$.

Theorem 2. *[8] suppose $\alpha\in(0,1)$ or that $\int_\Omega a dx\neq 0$ and $\alpha\in(\alpha_0,0]$ so that Eq. (1) has principal eigenvalues $\lambda^-(\alpha)$ and $\lambda^+(\alpha)$. For any $\lambda\in(\lambda^-(\alpha),\lambda^+(\alpha))$*

$$\|u\|_\lambda=\left\{\int_\Omega[|\nabla u|^p-\lambda a u^p]dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^p ds\right\}^{\frac{1}{p}},$$

defines a norm in W which is equivalent to the usual norm for W. So for $\lambda=0$, $\|u\|_\lambda^p=M(u)$.

Lemma 2. *There exists $\lambda_1>0$ such that ϕ_u takes positive values for all non-zero $u\in W$, when $\lambda<\lambda_1$.*

Proof. If $u\in B^-$, then $\phi_u(t)>0$ for t sufficiently large. Suppose that $u\in B^+$.
Let

$$h_u(t)=\frac{1}{p}t^pM(u)-\frac{t^{\gamma+1}}{\gamma+1}B(u).$$

Then elementary calculus shows that h_u takes a maximum value

$$\frac{\gamma-p+1}{p(\gamma+1)}\left[\frac{M(u)^{\gamma+1}}{B(u)^p}\right]^{\frac{1}{\gamma-p+1}},$$

when

$$t=t_{\max}=\left[\frac{M(u)}{B(u)}\right]^{\frac{1}{\gamma-p+1}}.$$

However

$$\frac{M(u)^{\gamma+1}}{(\int_{\Omega}|u|^{\gamma+1}dx)^p} \geq \frac{1}{(C^{-1}S_{\gamma+1})^{p(\gamma+1)}},$$

where $S_{\gamma+1}$ denotes the Sobolev constant of the embedding of W into $L^{\gamma+1}$ and c is a positive constant. Hence

$$h_u(t_{\max}) \geq \frac{\gamma-p+1}{p(\gamma+1)}\left[\frac{1}{(b^+)^p(C^{-1}S_{\gamma+1})^{p(\gamma+1)}}\right]^{\frac{1}{\gamma-p+1}} = \delta,$$

where δ is independent of u.
We shall now show that there exists $\lambda_1 > 0$ such that $\phi_u(t_{\max}) > 0$, i.e., $h_u(t_{\max}) - \frac{\lambda(t_{\max})^{\alpha+1}}{\alpha+1}A(u) > 0$, for all $u \in W - \{0\}$ provided that $\lambda < \lambda_1$. We have

$$\begin{aligned}\frac{(t_{\max})^{\alpha+1}}{\alpha+1}A(u) &\leq \frac{1}{\alpha+1}\left[\frac{M(u)}{B(u)}\right]^{\frac{\alpha+1}{\gamma-p+1}} C^{-(\gamma+1)}\|a\|_\infty S_{\gamma+1}^{\gamma+1}(M(u))^{\frac{\gamma+1}{p}}\\ &= \frac{1}{\alpha+1}C^{-(\gamma+1)}\|a\|_\infty S_{\gamma+1}^{\gamma+1}\left[\frac{M(u)^{\gamma+1}}{B(u)^p}\right]^{\frac{\alpha+1}{p(\gamma-p+1)}}\\ &= \frac{1}{\alpha+1}C^{-(\gamma+1)}\|a\|_\infty S_{\gamma+1}^{\gamma+1}\left[\frac{p(\gamma+1)}{\gamma-p+1}\right]^{\frac{\alpha+1}{p}} h_u(t_{\max})^{\frac{\alpha+1}{p}}\\ &= kh_u(t_{\max})^{\frac{\alpha+1}{p}},\end{aligned}$$

where k is independent of u. Hence

$$\phi_u(t_{\max}) \geq h_u(t_{\max}) - \lambda k h_u(t_{\max})^{\frac{\alpha+1}{p}} = h_u(t_{\max})^{\frac{\alpha+1}{p}}\left[h_u(t_{\max})^{\frac{1-\alpha}{p}} - \lambda k\right]$$

and so, since $h_u(t_{\max}) \geq \delta$ for all $u \in W - \{0\}$, it follows that

$$\phi_u(t_{\max}) \geq \delta^{\frac{\alpha+1}{p}}\left[\delta^{\frac{\alpha+1}{p}} - \lambda k\right].$$

Thus $\phi_u(t_{\max}) > 0$ for all nonzero u provided that $\lambda < \frac{\delta^{\frac{\alpha+1}{p}}}{k} = \lambda_1$. This completes the proof.

Corollary 1. $S^0 = \emptyset$ *when* $0 < \lambda < \lambda_1$.

Corollary 2. *If* $\lambda < \lambda_1$, *then there exists* $\delta_1 > 0$ *such that* $J_\lambda(u) \geq \delta_1$ *for all* $u \in S^-$.

Proof. Consider $u \in S^-$. Then ϕ_u has a positive global maximum at $t = 1$ and $u \in B^+$. Thus

$$J_\lambda(u) = \phi_u(1) \geq \phi_u(t_{\max}) \geq h_u(t_{\max})^{\frac{\alpha+1}{p}}\left(h_u(t_{\max})^{\frac{1-\alpha}{p}} - \lambda k\right) \geq \delta^{\frac{\alpha+1}{p}}\left(\delta^{\frac{\alpha+1}{p}} - \lambda k\right)$$

and the left hand side is uniformly bounded away from 0 provided that $\lambda < \lambda_1$.

Lemma 3. J_λ *is coercive and bounded below on* S.

Proof. It follows from Eq. 3 and the sobolev embedding theorems that there exist positive constants c_1, c_2 and c_3 such that

$$J_\lambda(u) \geq c_1\|u\|^p - c_2\int_{\Omega}|u|^{\alpha+1}dx \geq c_1\|u\|^p - c_3\|u\|^{\alpha+1}$$

and so J_λ is coercive and bounded below on S.

Also as, proved in Binding, Drabek and Huang[2] or in Brown and Zhang[4], we have the following Lemma.

Lemma 4. *Suppose that* u_0 *is a local maximum or minimum for* J_λ *on* S.
Then, if $u_0 \notin S^0$, u_0 *is a critical point of* J_λ.

3 Existence results

Now we can state our main results.

Theorem 3. *If $\lambda < \lambda_1$, there exists a minimizer of J_λ on S^+.*

Proof. Since J_λ is bounded below on S and so on S^+, there exists a minimizing sequence $\{u_n\} \subseteq S^+$ such that $\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^+} J_\lambda(u)$. Then by Lemma 3 and Rellich-Kondrachov Theorem, we may assume that there exit a subsequence $\{u_n\}$ and $u_0 \in W$ such that $u_n \rightharpoonup u_0$ in W and $u_n \to u_0$ in $L^r(\Omega)$ for $1 < r < \dfrac{np}{n-p}$.

If we choose $u \in L^+$, then there exists $t_1(u)$ such that $t_1(u)u \in S^+$ and $J_\lambda(t_1(u)u) < 0$. Hence, $\inf_{u\in S^+} J_\lambda(u) < 0$. By Eq. (3),

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{\gamma+1}\right) M(u) + \lambda\left(\frac{1}{\gamma+1} - \frac{1}{\alpha+1}\right) A(u)$$

and so

$$-\lambda\left(\frac{1}{\gamma+1} - \frac{1}{\alpha+1}\right) A(u) = \left(\frac{1}{p} - \frac{1}{\gamma+1}\right) M(u) - J_\lambda(u_n).$$

Letting $n \to \infty$, then $A(u_0) > 0$.

Suppose $u_n \not\to u_0$ in W. In this case we shall obtain a contradiction by discussing the fibering map. Since $A(u_n) > 0$, there exists $t_0 > 0$ such that $t_0 u_0 \in S^+$ and $\phi_{u_0}(t_0) = 0$.

Since $u_n \not\to u_0$ in W, then $\|u_0\| < \lim_{n\to\infty} \inf \|u_n\|$. Thus, as

$$\phi'_{u_n}(t) = t^{p-1} M(u_n) - \lambda t^\alpha A(u_n) - t^\gamma B(u_n)$$

and

$$\phi'_{u_0}(t) = t^{p-1} M(u_0) - \lambda t^\alpha A(u_0) - t^\gamma B(u_0),$$

and since $\{u_n\}$ tends to u_0 strongly in L^r, we have

$$\begin{aligned} 0 = \phi'_{u_0}(t_0) &= t_0^{p-1} M(u_0) - \lambda t_0^\alpha A(u_0) - t_0^\gamma B(u_0) \\ &< \lim_{n\to\infty} \inf \left(t_0^{p-1} M(u_n) - \lambda t_0^\alpha A(u_n) - t_0^\gamma B(u_n) \right) \\ &= \phi'_{u_n}(t_0). \end{aligned}$$

it follows that $\phi'_{u_n}(t_0) > 0$ for n sufficiently large. Since $\{u_n\} \subseteq S^+$, by considering the possible fibering maps it is easy to see that $\phi'_{u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n}(1) = 0$ for all n. Hence we must have $t_0 > 1$. But $t_0 u_0 \in M_\lambda^+(\Omega)$ and so

$$J_\lambda(t_0 u_0) = \phi_{u_0}(t_0) < \phi_{u_0}(1) = J_\lambda(u_0) < \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^+} J_\lambda(u)$$

and this is a contradiction. Hence $u_n \to u_0$ in W and so

$$J_\lambda(u_0) = \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^+} J_\lambda(u).$$

Thus u_0 is a minimizer for J_λ on S^+.

Theorem 4. *If $\lambda < \lambda_1$, there exists a minimizer of J_λ on S^-.*

Proof. By Corollary 2 we have $J_\lambda(u) \geq \delta_1 > 0$ for all $u \in S^-$ and so $\inf_{u \in S^-} J_\lambda(u) > \delta$. Hence there exists a minimizing sequence $\{u_n\} \subseteq S^-$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^-} J_\lambda(u) > 0.$$

As in the previous proof, since J_λ is coercive, $\{u_n\}$ is bounded in W and we many assume, without loss of generality, that $u_n \rightharpoonup u_0$ in W and $u_n \to u_0$ in $L^r(\Omega)$ for

$$1 < r < \frac{np}{n-p}.$$

By Eq. (3), we have

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{\alpha+1}\right) M(u) + \left(\frac{1}{\alpha+1} - \frac{1}{\gamma+1}\right) B(u)$$

and, since $\lim_{n\to\infty} J_\lambda(u_n) > 0$ and $\lim_{n\to\infty} B(u_n) = B(u_0)$, we must have that $B(u_0) > 0$. Hence there exists $\hat{t} > 0$ such that $\hat{t}u_0 \in S^-$.

Suppose $u_n \not\to u_0$ in W. Since $u_n \subseteq S^-$ we have $\phi_{u_n}(1) = J_\lambda(u_n) \geq J_\lambda(su_n) = \phi_{u_n}(s)$, for all $s \geq 0$. So

$$\begin{aligned} J_\lambda(\hat{t}u_0) &= \frac{1}{p}\hat{t}^p M(u_0) - \frac{\lambda\hat{t}^{\alpha+1}}{\alpha+1} A(u_0) - \frac{\hat{t}^{\gamma+1}}{\gamma+1} B(u_0) \\ &< \lim_{n\to\infty} \left[\frac{1}{p}\hat{t}^p M(u_n) - \frac{\lambda\hat{t}^{\alpha+1}}{\alpha+1} A(u_n) - \frac{\hat{t}^{\gamma+1}}{\gamma+1} B(u_n)\right] \\ &= \lim_{n\to\infty} J_\lambda(\hat{t}u_n) \\ &\leq \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in S^-} J_\lambda(u), \end{aligned}$$

which is a contradiction. (In the first inequality we use this facts that $\|u_0\| < \lim_{n\to\infty} \inf \|u_n\|$.) Hence $u_n \to u_0$ in W and the proof can be completed as in the previous Theorem.

Corollary 3. *Eq. (1) has at least two positive solutions whenever* $0 < \lambda < \lambda_1$.

Proof. By Theorems 3 and 4, there exist $u^+ \in S^+$ and $u^- \in S^-$ such that $J_\lambda(u^+) = \inf_{u\in S^+} J_\lambda(u)$ and $J_\lambda(u^-) = \inf_{u\in S^-} J_\lambda(u)$. Moreover $J_\lambda(u^\pm) = J_\lambda(|u^\pm|)$ and $|u^\pm| \in S^\pm$ and so we may assume $u^\pm \geq 0$. By Lemma 4, $u^\pm$ are critical points of J_λ on W and hence are weak solutions of Eq. (1). Finally, by the Harnack inequality due to Trudinger[9], we obtain that $u^\pm$ are positive solutions of Eq. (1).

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