The Nehari manifold for quasilinear equation with Robin boundary condition

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Abstract. Using the Nehari manifold, We prove the existence of positive solutions of the problem

\[ \begin{align*}
-\Delta_p u &= \lambda a(x)u|u|^{\alpha-2} + b(x)u|u|^{\gamma-2}, & x \in \Omega, \\
\alpha u|u|^{p-2} + (1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} &= 0, & x \in \partial\Omega,
\end{align*} \]

for \( x \in \Omega \), together with the boundary condition \( \alpha u|u|^{p-2} + (1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} = 0 \). Exploiting the relationship between the Nehari manifold and Fibering maps (i.e., maps of the form \( t \to J_\lambda(u) \)), where \( J_\lambda(u) \) is the Euler functional associated with the equation), and a condition on \( b(x) \), we show how existence results for positive solutions of the equations are linked to properties of the Nehari manifold.

Keywords: variational method, nehari manifold, minimizing sequence, critical sobolev exponent

1 Introduction

In this paper we study the existence of positive solutions for the problem:

\[ \begin{align*}
-\Delta_p u &= \lambda a(x)u|u|^{\alpha-2} + b(x)u|u|^{\gamma-2}, & x \in \Omega, \\
\alpha u|u|^{p-2} + (1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} &= 0, & x \in \partial\Omega,
\end{align*} \]

where \( \Delta_p \) is the p-Laplacian operator, i.e., \( \Delta_p z = (\text{div}(|\nabla z|^{p-2}\nabla z)) \), \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \),

\[ 0 < \alpha < 1 < p - 1 < \gamma - 1 < p^* - 1, \]

\[ \left( p^* = \frac{np}{n-p} \text{ if } p < n, \; p^* = \infty \text{ if } p \geq n \right), \lambda > 0 \]

is a real parameter and \( a, b : \Omega \to \mathbb{R} \) are smooth functions which change sign on \( \Omega \).

Similar problems have been studied in [2] by using variational methods and in [1], [4] and [7] by using the Nehari manifold. Furthermore this problem in the case \( p = 2 \) and Dirichlet boundary condition has been studied by Brown and Wu [3]. In this paper we show how a fairly complete knowledge of all possible forms of the fibering maps provides a very simple and comparatively elementary means of establishing results similar to those proved in [7] and [10] on the existence of multiple solutions for the problem Eq. (1).

The plane of the paper is as follows. In Section 2 we first recall the facts that we shall require about the Nehari manifold and examine carefully the connection between the Nehari manifold and the Fibering maps. In Section 3 we use this information to give a very simple variational proof of the existence of solutions of Eq. (1) for sufficiently small \( \lambda \).

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2 Notation and preliminaries

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. We will work in a Sobolev space $W := W^{1,p}(\Omega)$ equipped with the norm

$$\|u\| = \left(\int (|\nabla u(x)|^p + u(x)^p)dx\right)^{\frac{1}{p}}.$$

First we introduce the structure of the Nehari manifold. It is well known that the steady state solutions of Eq. (1) correspond to critical points of the functional

$$J_\lambda(u) = \frac{1}{p} M(u) - \frac{\lambda}{\alpha + 1} A(u) - \frac{1}{\gamma + 1} B(u),$$

where

$$M(u) = \int_\Omega |\nabla u|^p dx - \frac{\alpha}{\alpha - 1} \int_{\partial \Omega} |u|^p ds,$$

$$A(u) = \int_\Omega a(x)|u|^\alpha dx,$$

$$B(u) = \int_\Omega b(x)|u|^\gamma dx.$$

This functional is bounded neither above nor below. In order to obtain existence results in this case we introduce the Nehari manifold $S = \{u \in W; (J'_\lambda(u), u) = 0\}$, where $(\cdot, \cdot)$ denotes the usual duality. Thus $u \in S$ if and only if $M(u) - \lambda A(u) - B(u) = 0$.

Clearly $S$ is a smaller set than $W$ and so it is easier to study $J_\lambda$ on $S$.

For $u \in S$ we have that

$$J_\lambda(u) = \lambda \left(\frac{1}{p} - \frac{1}{\alpha + 1}\right) A(u) + \left(\frac{1}{p} - \frac{1}{\gamma + 1}\right) B(u)$$

$$= \left(\frac{1}{p} - \frac{1}{\alpha + 1}\right) M(u) + \lambda \left(\frac{1}{\alpha + 1} - \frac{1}{\gamma + 1}\right) A(u).$$

The Nehari manifold is closely linked to the behaviour of the functions of the from $\phi_u : t \mapsto J_\lambda(tu)(t > 0)$. Such maps are known as fibering maps and were introduced by Drabek and Pohozaev in [6] and are also discussed in Brown and Zhang[4]. If $u \in W$, we have

$$\phi_u(t) = \frac{1}{p} t^p M(u) - \lambda \frac{t^{\alpha + 1}}{\alpha + 1} A(u) - \frac{t^{\gamma + 1}}{\gamma + 1} B(u),$$

$$\phi'_u(t) = p t^{p - 1} M(u) - \lambda t^\alpha A(u) - t^\gamma B(u),$$

$$\phi''_u(t) = (p - 1) t^{p - 2} M(u) - \lambda t^{\alpha - 1} A(u) - \gamma t^{\gamma - 1} B(u).$$

It is easy to see that $u \in S$ if and only if $\phi'(1) = 0$ and more generally, that $tu \in S$ if and only if $\phi'(t) = 0$, i.e., elements in $S$ correspond to stationary points of the fibering maps.

Now, we split $S$ into three parts:

$$S^+ = \{u \in S; \phi''_u(1) > 0\}, \quad S^- = \{u \in S; \phi''_u(1) < 0\}, \quad S^0 = \{u \in S; \phi''_u(1) = 0\},$$

where corresponding to local minima, local maxima and points of inflexion of fibering maps, respectively. Also, if $u \in S$, we have
\[
\phi''_u(1) = \lambda(p - \alpha - 1)A(u) + (p - \gamma - 1)B(u) \\
= (p - \alpha - 1)M(u) + (\alpha - \gamma)B(u) \\
= (p - \gamma - 1)M(u) + \lambda(\gamma - \alpha)A(u). \tag{7}
\]

Define
\[m_u(t) = t^{\gamma - \alpha - 1} M(u) - t^{\gamma - \alpha} B(u).\]

Hence
\[m'_u(t) = (p - \alpha - 1)t^{\gamma - \alpha - 2} M(u) - (\gamma - \alpha)t^{\gamma - \alpha - 1} B(u). \tag{8}\]

So we have

**Lemma 1.** \(tu \in S\) if and only if \(t\) is a solution of \(m_u(t) = \lambda A(u)\), for \(t > 0\).

**Theorem 1.**
1. if \(B(u) \leq 0\), then \(m_u\) is a strictly increasing function.
2. if \(B(u) \geq 0\), then \(m_u(t) > 0\) for \(t\) small and positive but \(m_u(t) \rightarrow -\infty\) as \(t \rightarrow \infty\), also \(m_u(t)\) has a unique maximum stationary point.

Now suppose that \(tu \in S\). It follows from Eq. (7) and Eq. (8) that \(\phi''_{tu}(1) = t^{\alpha + 2}m'_u(t)\) and so \(tu \in S^+(S^-)\) provided \(m'_u(t) > 0(<0)\). We define \(L^+ = \{u \in W; A(u) > 0\}\), and \(L^-\) similarly by replacing \(>\) by \(\leq\). We also define \(B^+ = \{u \in W; B(u) > 0\}\), and \(B^-\) analogously. So
1. if \(u \in L^- \cap B^-\), then \(t \rightarrow \phi_u(t)\) is an increasing function of \(t\) and no multiple of \(u\) lies in \(S\),
2. if \(u \in L^+ \cap B^-,\) then \(t \rightarrow \phi_u(t)\) has a local minimum at \(t = t(u)\) and \(t(u)u \in S^+\),
3. if \(u \in L^- \cap B^+\), then \(t \rightarrow \phi_u(t)\) has a local maximum at \(t = t(u)\) and \(t(u)u \in S^-\),
4. if \(u \in L^+ \cap B^+\), then
   a) if \(\lambda > 0\) is sufficiently large, \(t \rightarrow \phi_u(t)\) is a decreasing function and no multiple of \(u\) lies in \(S\),
   b) If \(\lambda > 0\) is sufficiently small, \(t \rightarrow \phi_u(t)\) has exactly two points- a local minimum at \(t = t_1(u)\) and local maximum at \(t = t_2(u)\), such that \(t_1(u)u \in S^+\) and \(t_2(u)u \in S^-\).

**Theorem 2.** [8] suppose \(\alpha \in (0, 1)\) or that \(\int_{\Omega} adx \neq 0\) and \(\alpha \in (\alpha_0, 0)\) so that Eq. (1) has principal eigenvalues \(\lambda^-(\alpha)\) and \(\lambda^+(\alpha)\). For any \(\lambda \in (\lambda^-(\alpha), \lambda^+(\alpha))\)
\[\|u\|_\lambda = \left\{ \int_{\Omega} \left[ |\nabla u|^p - \lambda |u|^p \right] dx + \frac{\alpha}{1 - \alpha} \int_{\partial \Omega} |u|^p ds \right\}^{\frac{1}{p}},\]
defines a norm in \(W\) which is equivalent to the usual norm for \(W\). So for \(\lambda = 0\), \(\|u\|_\lambda^p = M(u)\).

**Lemma 2.** There exists \(\lambda_1 > 0\) such that \(\phi_u\) takes positive values for all non-zero \(u \in W\), when \(\lambda < \lambda_1\).

**Proof.** If \(u \in B^-\), then \(\phi_u(t) > 0\) for \(t\) sufficiently large. Suppose that \(u \in B^+\). Let
\[h_u(t) = \frac{1}{p} t^p M(u) - \frac{t^{\gamma + 1}}{\gamma + 1} B(u).\]
Then elementary calculus shows that \(h_u\) takes a maximum value
\[\gamma - p + 1 \left[ \frac{M(u)^{\gamma + 1}}{B(u)^p} \right]^{\frac{1}{p-\gamma+1}},\]
when
\[t = t_{\text{max}} = \left[ \frac{M(u)}{B(u)} \right]^{\frac{1}{p-\gamma+1}}.\]

However
Consider suppose that it follows from Eq. \[ \lambda \] we shall now show that there exists \( S \), submit@wjms.org.uk WJMS email for contribution 258 S. Khademloo & M. Kardoost J c and the left hand side is uniformly bounded away from 0 where \( \delta \) is independent of \( u \).

We shall now show that there exists \( \lambda_1 > 0 \) such that \( \phi_u(t_{\text{max}}) > 0 \), i.e., \( h_u(t_{\text{max}}) - \lambda(t_{\text{max}})^\alpha + 1 A(u) > 0 \), for all \( u \in W - \{0\} \) provided that \( \lambda < \lambda_1 \). We have

\[
\frac{(t_{\text{max}})^{\alpha + 1}}{\alpha + 1} A(u) \leq \frac{1}{\alpha + 1} \left[ \frac{M(u)}{B(u)} \right]^{\frac{\alpha + 1}{\gamma - p} + 1} C^{-\gamma + 1} (a) \| \infty \| Sp_{\gamma + 1} (M(u))^{\frac{\alpha + 1}{\gamma - p}} \leq \frac{1}{\alpha + 1} C^{-\gamma + 1} (a) \| \infty \| Sp_{\gamma + 1} \left[ \frac{M(u)^{\gamma - 1}}{B(u)^p} \right]^{\frac{\alpha + 1}{\gamma - p}} \leq \frac{1}{\alpha + 1} C^{-\gamma + 1} (a) \| \infty \| Sp_{\gamma + 1} \left[ \frac{p(\gamma + 1)}{\gamma - p + 1} \right]^{\frac{\alpha + 1}{\gamma - p}} h_u(t_{\text{max}})^{\frac{\alpha + 1}{p}} \leq k h_u(t_{\text{max}})^{\frac{\alpha + 1}{p}},
\]

where \( k \) is independent of \( u \). Hence

\[
\phi_u(t_{\text{max}}) \geq h_u(t_{\text{max}}) - \lambda k h_u(t_{\text{max}})^{\frac{\alpha + 1}{p}} = h_u(t_{\text{max}})^{\frac{\alpha + 1}{p}} \left[ h_u(t_{\text{max}})^{\frac{1 - \alpha}{p}} - \lambda k \right]
\]

and so, since \( h_u(t_{\text{max}}) \geq \delta \) for all \( u \in W - \{0\} \), it follows that

\[
\phi_u(t_{\text{max}}) \geq \delta \left[ \delta^{\frac{\alpha + 1}{p}} - \lambda k \right].
\]

Thus \( \phi_u(t_{\text{max}}) > 0 \) for all nonzero \( u \) provided that \( \lambda < \frac{\delta^{\frac{\alpha + 1}{p}}}{k} = \lambda_1 \). This completes the proof.

Corollary 1. \( S^0 = \emptyset \) when \( 0 < \lambda < \lambda_1 \).

Corollary 2. If \( \lambda < \lambda_1 \), then there exists \( \delta_1 > 0 \) such that \( J_\lambda(u) \geq \delta_1 \) for all \( u \in S^- \).

Proof. Consider \( u \in S^- \). Then \( \phi_u \) has a positive global maximum at \( t = 1 \) and \( u \in B^+ \). Thus

\[
J_\lambda(u) = \phi_u(1) \geq \phi_u(t_{\text{max}}) \geq h_u(t_{\text{max}})^{\frac{\alpha + 1}{p}} \left( h_u(t_{\text{max}})^{\frac{1 - \alpha}{p}} - \lambda k \right) \geq \delta \left[ \delta^{\frac{\alpha + 1}{p}} - \lambda k \right]
\]

and the left hand side is uniformly bounded away from 0 provided that \( \lambda < \lambda_1 \).

Lemma 3. \( J_\lambda \) is coercive and bounded below on \( S \).

Proof. It follows from Eq. 3 and the sobolev embedding theorems that there exist positive constants \( c_1, c_2 \) and \( c_3 \) such that

\[
J_\lambda(u) \geq c_1 \| u \|^p - c_2 \int_\Omega |u|^{\alpha + 1} dx \geq c_1 \| u \|^p - c_3 \| u \|^{\alpha + 1}
\]

and so \( J_\lambda \) is coercive and bounded below on \( S \).

Also as, proved in Binding, Drabek and Huang[2] or in Brown and Zhang[4], we have the following Lemma.

Lemma 4. Suppose that \( u_0 \) is a local maximum or minimum for \( J_\lambda \) on \( S \).

Then, if \( u_0 \notin S^0 \), \( u_0 \) is a critical point of \( J_\lambda \).

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3 Existence results

Now we can state our main results.

Theorem 3. If $\lambda < \lambda_1$, there exists a minimizer of $J_\lambda$ on $S^+$.

Proof. Since $J_\lambda$ is bounded below on $S$ and so on $S^+$, there exists a minimizing sequence $\{u_n\} \subseteq S^+$ such that $\lim_{n \to \infty} J_\lambda(u_n) = \inf_{u \in S^+} J_\lambda(u)$. Then by Lemma 3 and Rellich-Kondrachov Theorem, we may assume that there exist a subsequence $\{u_n\}$ and $u_0 \in W$ such that $u_n \to u_0$ in $W$ and $u_n \to u_0$ in $L^r(\Omega)$ for $1 < r < \frac{np}{n-p}$.

If we choose $u \in L^+$, then there exists $t_1(u)$ such that $t_1(u) \in S^+$ and $J_\lambda(t_1(u)) < 0$. Hence, $\inf_{u \in S^+} J_\lambda(u) < 0$. By Eq. (3),

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{\gamma + 1}\right) M(u) + \lambda \left(\frac{1}{\gamma + 1} - \frac{1}{\alpha + 1}\right) A(u)$$

and so

$$-\lambda \left(\frac{1}{\gamma + 1} - \frac{1}{\alpha + 1}\right) A(u) = \left(\frac{1}{p} - \frac{1}{\gamma + 1}\right) M(u) - J_\lambda(u_n).$$

Letting $n \to \infty$, then $A(u_0) > 0$.

Suppose $u_n \not\to u_0$ in $W$. In this case we shall obtain a contradiction by discussing the fibering map. Since $A(u_n) > 0$, there exists $t_0 > 0$ such that $t_0u_0 \in S^+$ and $\phi_{u_0}(t_0) = 0$.

Since $u_n \not\to u_0$ in $W$, then $\|u_0\| < \lim_{n \to \infty} \inf \|u_n\|$. Thus, as

$$\phi'_{u_0}(t) = t^{p-1}M(u_0) - \lambda t^\alpha A(u_0) - t^\gamma B(u_0)$$

and

$$\phi'_{u_0}(t) = t^{p-1}M(u_0) - \lambda t^\alpha A(u_0) - t^\gamma B(u_0),$$

and since $\{u_n\}$ tends to $u_0$ strongly in $L^r$, we have

$$0 = \phi'_{u_0}(t_0) = t_0^{p-1}M(u_0) - \lambda t_0^\alpha A(u_0) - t_0^\gamma B(u_0)$$

$$< \lim_{n \to \infty} \inf \left(t_0^{p-1}M(u_n) - \lambda t_0^\alpha A(u_n) - t_0^\gamma B(u_n)\right)$$

$$= \phi'_{u_n}(t_0).$$

it follows that $\phi'_{u_n}(t_0) > 0$ for $n$ sufficiently large. Since $\{u_n\} \subseteq S^+$, by considering the possible fibering maps it is easy to see that $\phi'_{u_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n}(1) = 0$ for all $n$. Hence we must have $t_0 > 1$. But $t_0u_0 \in M_+^\lambda(\Omega)$ and so

$$J_\lambda(t_0u_0) = \phi_{u_0}(t_0) < \phi_{u_0}(1) = J_\lambda(u_0) < \lim_{n \to \infty} J_\lambda(u_n) = \inf_{u \in S^+} J_\lambda(u)$$

and this is a contradiction. Hence $u_n \to u_0$ in $W$ and so

$$J_\lambda(u_0) = \lim_{n \to \infty} J_\lambda(u_n) = \inf_{u \in S^+} J_\lambda(u).$$

Thus $u_0$ is a minimizer for $J_\lambda$ on $S^+$.

Theorem 4. If $\lambda < \lambda_1$, there exists a minimizer of $J_\lambda$ on $S^{-}$. 
Proof. By Corollary 2 we have $J_{\lambda}(u) \geq \delta_1 > 0$ for all $u \in S^-$ and so $\inf_{u \in S^-} J_{\lambda}(u) > \delta_1$. Hence there exists a minimizing sequence $\{u_n\} \subseteq S^-$ such that

$$
\lim_{n \to \infty} J_{\lambda}(u_n) = \inf_{u \in S^-} J_{\lambda}(u) > 0. 
$$

As in the previous proof, since $J_{\lambda}$ is coercive, $\{u_n\}$ is bounded in $W$ and we may assume, without loss of generality, that $u_n \to u_0$ in $W$ and $u_n \to u_0$ in $L^r(\Omega)$ for

$$
1 < r < \frac{np}{n-p}. 
$$

By Eq. (3), we have

$$
J_{\lambda}(u_n) = \left(\frac{1}{p} - \frac{1}{\alpha + 1}\right) M(u) + \left(\frac{1}{\alpha + 1} - \frac{1}{\gamma + 1}\right) B(u)
$$

and, since $\lim_{n \to \infty} J_{\lambda}(u_n) > 0$ and $\lim_{n \to \infty} B(u_n) = B(u_0)$, we must have that $B(u_0) > 0$. Hence there exists $\hat{t} > 0$ such that $tu_0 \in S^-$. Suppose $u_n \not\to u_0$ in $W$. Since $u_n \subseteq S^-$ we have $\phi_{u_n}(1) = J_{\lambda}(u_n) \geq J_{\lambda}(su_n) = \phi_{u_n}(s)$, for all $s \geq 0$. So

$$
J_{\lambda}(\hat{t}u_0) = \frac{1}{p} \hat{t}^p M(u_0) - \frac{\lambda \hat{t}^{\alpha+1}}{\alpha + 1} A(u_0) - \frac{\hat{t}^{\gamma+1}}{\gamma + 1} B(u_0)
\leq \lim_{n \to \infty} \left[\frac{1}{p} \hat{t}^p M(u_n) - \frac{\lambda \hat{t}^{\alpha+1}}{\alpha + 1} A(u_n) - \frac{\hat{t}^{\gamma+1}}{\gamma + 1} B(u_n)\right]
\leq \lim_{n \to \infty} J_{\lambda}(\hat{t}u_n)
\leq \lim_{n \to \infty} J_{\lambda}(u_n) = \inf_{u \in S^-} J_{\lambda}(u),
$$

which is a contradiction. (In the first inequality we use this facts that $\|u_0\| < \lim_{n \to \infty} \inf \|u_n\|$.) Hence $u_n \to u_0$ in $W$ and the proof can be completed as in the previous Theorem.

Corollary 3. Eq. (1) has at least two positive solutions whenever $0 < \lambda < \lambda_1$.

Proof. By Theorems 3 and 4, there exist $u^+ \in S^+$ and $u^- \in S^-$ such that $J_{\lambda}(u^+) = \inf_{u \in S^+} J_{\lambda}(u)$ and $J_{\lambda}(u^-) = \inf_{u \in S^-} J_{\lambda}(u)$. Moreover $J_{\lambda}(|u^+|) = J_{\lambda}(|u^+|)$ and $|u^+| \in S^+$ and so we may assume $u^+ \geq 0$. By Lemma 4, $u^\pm$ are critical points of $J_{\lambda}$ on $W$ and hence are weak solutions of Eq. (1). Finally, by the Harnack inequality due to Trudinger[39], we obtain that $u^\pm$ are positive solutions of Eq. (1).

References


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