ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 9 (2013) No. 3, pp. 216-222

Dual-phase-lag heat equation of fractional order of heat transfer within microscale region

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(*Received August 29 2012, Revised May 15, Accepted July 13 2013*)

Abstract. In this article, the author has used fractional calculus to explain the wave nature of heat propagation as well as heat conduction at molecular level with dual phase lag; in this row he proposed dual phase lag heat equation of fractional order. Modified Adomian Decomposition Method and New Iterative Method are applied to obtain the approximate analytical solutions of the proposed model. Numerical results are presented and compared graphically. It is observed that the order of the proposed equation affects the transfer of heat significantly.

Keywords: dual-phase-lag, heat flux gradient, fractional calculus, modified adomian decomposition method, new iterative method

1 Introduction

In many practical problems, classical Fourier's equation of heat conduction

$$\vec{q}(\vec{r},t) = -K\nabla T(\vec{r},t), \tag{1}$$

can be used, although this infinitely speed of heat propagation. However, in situations dealing with heat flow in extremely short periods of time (high frequency heat source such as laser or microwave), very high temperature gradient, very low temperatures approaching absolute zero or for microscale conditions such as heat transport via biofilm, the quasi equilibrium concept implemented in Fourier's law breaks down and the wave nature of heat propagation becomes dominant[15, 21],therefore this theory should be modified. In other words, Fourier's law has the unphysical property that it lacks inertial effects: if a sudden temperature perturbation is applied at one point in a heat conduction medium, it will be felt instantaneously and everywhere at distant points. The rapidly developing technology of the ultrafast pulse-laser heating, in which the nonequilibrium thermodynamic transition and the microstructural effect become significantly associated with the shortening of the response time, also challenges the classical Fourier law[21]. Further-more, due to the wide application of microdevices and the rapid development of modern micro- and nanoscale dimensions emerge in various micromechanical systems. It is well known that the conventional Fourier law leads to an unaccepted result for these microdevices[6, 7]. A better understanding of the heat transport in these tiny devices is required for a further development of nanotechnology[7]. Much effort has been devoted to the modification of the classical Fourier law. Several non-Fourier heat conduction models have been established.

One of these is the Maxwell-Cattaneo (MC) model (also called the CV model),

$$\tau\frac{\partial\vec{q}(\vec{r},t)}{\partial t} + \vec{q}(\vec{r},t) = -K\nabla T(\vec{r},t), \tag{2}$$

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where τ is the time delay. This leads to a hyperbolic heat conduction equation. Therefore, the heat propagates like a wave with a finite speed[15, 21], which has also been demonstrated experimentally[1, 2, 27].

The natural extension of the MC model Eq. (3) is

$$\vec{q}(\vec{r}, t + \tau_q) = -K\,\nabla T(\vec{r}, t), \tag{3}$$

which was proposed by Tzou [21] in a phenomenological manner. Apparently the original MC model can be recovered from the first order Taylor expansion of the left side of Eq. (3) with respect to time t. It was found that model (3) leads to the thermal vibration or the thermal wave phenomenon.

In [21], the non-Fourier model (3) was further extended to the following form:

$$\vec{q}(\vec{r}, t + \tau_q) = -K\,\nabla T(\vec{r}, t + \tau_T), \tag{4}$$

where τ_T and τ_q are the phase lags of the temperature gradient and the heat flux vector, respectively. The former is due to the microstructural interactions such as phonon scattering or phonon-electron interactions. The latter is, on the other hand, interpreted as the relaxation time accounting for the fast-transient effects of thermal inertia. In order to examine whether the heat conduction model (4) leads to the finite speed of heat propagation, the condition for the occurrence of the thermal vibration phenomenon was explored in [3].

Recently, Srivastava and Rai [20] give approximate analytical solution of 3D linear fractional microscale heat equation using modified homotopy perturbation method, but they did not give their analysis of evolution of Fractional Dual Phase Lag microscale heat transfer equation and also they took constant thermal conductivity, while recent researches[13] show that it is not so.

The main aim of this research work is development and analysis of fractional time derivative Dual-Phase-Lag (FrDPL) microscale heat transport equation with the temperature dependent thermal conductivity. This FrDPL is obtained by using both the fractional energy equation and dual phase non-Fourier model in Microscale Thermal Science[21].

The fractional energy equation is obtained by replacing the first order time derivative in energy equation by a fractional derivative of order α satisfying $0 < \alpha \leq 1$, have been a field of growing interest as evident from literature survey. Lots of universal phenomena can be modeled to a greater degree of accuracy by using the property of this evolution equation. Analytical methods used to solve these equations have very restricted applications and the numerical techniques commonly used give rise to rounding of errors. Keeping these in view, the author in this paper has made a sincere effort to solve the nonlinear fractional DPL equation analytically using a very elegant mathematical tool, the Modified Decomposition Method[4] and New Iterative Method.

The decomposition method of Adomian has been applied to solve a wide class of non-linear differential and partial differential equations [Adomian [1–3], Adomian and Rach [4]] etc. Wazwaz [22, 26] made further progress of this method with some modifications. The modification of the Adomian decomposition method will accelerate the rapid convergence of the series solution. This modified technique has been shown to be computationally efficient in doing several problems in applied fields (Wazwaz [23, 24], Kaya and Yokus [12], Saha Ray [18]).

Recently diffusion-wave like models for physical problems have been attracted much attention. Lot of works are done by many researchers like Manolis [16], Wazwaz [26] etc.

Gejji and Bhalekar [5] used this method for solving multi-term linear and non-linear diffusion-wave equations of fractional order.

New Iterative Method (NIM) was first proposed by the Indian mathematicians Daftardar-Gejji and Jafari [7] and subsequently implemented by Daftardar-Gejji and Bhalekar [2, 27] to solve linear and nonlinear partial differential equations and fractional diffusion wave equations. The main advantage of this method is its computational simplicity; the solution obtained by this method is expected to give a better approximation in a straightforward manner.

The most important thing of this research work is to discuss with the temperature dependent thermal conductivity, while previously the research emphasized on constant thermal conductivity.

2 Dual-phase-lag microscale heat transfer equation of fractional order

The following constitutive Eq. (1) is used to describe both the temperature dependent thermal conductivity and the lagging behavior:

$$q(\vec{r}, t+\tau_q) = -K(T(\vec{r}, t+\tau_T)) \cdot \nabla T(\vec{r}, t+\tau_T), \tag{5}$$

where $\vec{r} = (x, y, z)$, the space coordinate and t is the time, K(T) - Thermal conductivity is function of temperature $T(\vec{r}, t)$.

by Taylor's expansion to Eq. (5) with respect to t gives

$$\begin{aligned} & q(\vec{r}, t) + \tau_q \frac{\partial q(\vec{r}, t)}{\partial t} + \frac{\tau_q^2}{2!}\frac{\partial^2 q(\vec{r}, t)}{\partial t^2} + \frac{\tau_q^3}{3!}\frac{\partial^3 q(\vec{r}, t)}{\partial t^3} \cdots \\ & = -K\left\{T(\vec{r}, t) + \tau_T \frac{\partial T(\vec{r}, t)}{\partial t} + \frac{\tau_T^2}{2!}\frac{\partial^2 T(\vec{r}, t)}{\partial t^2} + \frac{\tau_T^3}{3!}\frac{\partial^3 T(\vec{r}, t)}{\partial t^3} \cdots \right\} \\ & \quad \cdot \left\{\nabla T(\vec{r}, t) + \tau_T \frac{\partial}{\partial t}[\nabla T(\vec{r}, t)] + \frac{\tau_T^2}{2!}\frac{\partial^2}{\partial t^2}[\nabla T(\vec{r}, t)] + \frac{\tau_T^3}{3!}\frac{\partial^3}{\partial t^3}[\nabla T(\vec{r}, t)] + \cdots \right\}, \end{aligned}$$

or

$$q(\vec{r}, t) + \tau_q \frac{\partial q(\vec{r}, t)}{\partial t} \cong -K\left(T(\vec{r}, t) + \tau_T \frac{\partial T(\vec{r}, t)}{\partial t}\right) \cdot \left\{\nabla T(\vec{r}, t) + \tau_T \frac{\partial}{\partial t}[\nabla T(\vec{r}, t)]\right\}, \tag{6}$$

where both phase lags τ_q and τ_T are assumed very small, implying that the nonlinear terms in τ_q and τ_T are negligible.

Eq. (6) is now combined with the energy equation:

$$-\nabla \cdot \vec{q}(\vec{r}, t) = C_p \frac{\partial^\alpha T(\vec{r}, t)}{\partial t^\alpha},\ 0 < \alpha \leq 1, \tag{7}$$

where, C_p-Specific Heat Coefficient We refer to Eq. (7) as to the time fractional energy equation $\{0 < \alpha \leq 1\}$. This fractional Eq. (7) is able to explain the mixed nature of heat energy at molecular level.

Using Eq. (6) and Eq. (7) we get

$$C_p \frac{\partial^\alpha T}{\partial t^\alpha} + \tau_q \frac{\partial}{\partial t}\left(C_p \frac{\partial^\alpha T}{\partial t^\alpha}\right) = \nabla \cdot \left(K(T \cdot \nabla T)\right) + \tau_T \nabla \cdot \left(\frac{\partial K(T)}{\partial t}\nabla T\right) + \tau_T \nabla \cdot \left(K(T)\,\frac{\partial(\nabla T)}{\partial t}\right) + \circ(\tau_T^2),$$

or

$$C_p \frac{\partial^\alpha T}{\partial t^\alpha} + \tau_q C_p \frac{\partial^{1+\alpha} T}{\partial t^{1+\alpha}} \cong \nabla \cdot \left(K(T) \cdot \nabla T\right) + \tau_T \nabla \cdot \left(\frac{\partial}{\partial t}(K(T) \cdot \nabla T)\right),$$

or

$$\frac{\partial^\alpha T(\vec{r}, t)}{\partial t^\alpha} + \tau_q \frac{\partial^\beta T(\vec{r}, t)}{\partial t^\beta} = \frac{1}{C_p}\left(\nabla \cdot (K(T) \cdot \nabla T) + \tau_T \nabla \cdot \left(\frac{\partial}{\partial t}(K(T) \cdot \nabla T)\right)\right), \tag{8}$$

where $\beta = 1 + \alpha$.

This Eq. (8) represents nonlinear Dual Phase Lag model for heat transfer of Fractional order in time derivative.

3 Solutions

For the solution we apply Modified Adomian Decomposition Method and New Iterative Method[20] on Eq. (8) which is equivalent to the integral equation

$$T(\vec{r},t) = T(\vec{r},\,0)\left[1 - \frac{1}{\tau_q}\frac{t}{\Gamma(2)}\right] + e^r t - \frac{1}{\tau_q}D_t^{-1}T(\vec{r},t) + \frac{1}{C_p\,\tau_q}D_t^{-\beta}\Big(\nabla\cdot(K(T)\cdot\nabla T)\Big) + \frac{\tau_T}{C_p\,\tau_q}\nabla\cdot\Big(D_t^{-\alpha}(K(T)\cdot\nabla T)\Big). \tag{9}$$

According to [4, 22, 26] we apply MADM on Eq. (9) we get

$$T_0(\vec{r},\,t) = T(\vec{r},\,0)\left[1 - \frac{1}{\tau_q}\frac{t}{\Gamma(2)}\right] + e^r t, \tag{10}$$

$$T_{n+1}(\vec{r},\,t) = -\frac{1}{\tau_q}D_t^{-1}T_n(\vec{r},t) + \frac{1}{C_p\,\tau_q}D_t^{-\beta}\Big(\nabla\cdot(K(T_n)\cdot\nabla T_n)\Big) + \frac{\tau_T}{C_p\,\tau_q}\nabla\cdot\Big(D_t^{-\alpha}(K(T_n)\cdot\nabla T_n)\Big). \tag{11}$$

According to [10, 17, 20] we apply NIM on Eq. (9) we get

$$T_0(\vec{r},\,t) = T(\vec{r},\,0)\left[1 - \frac{1}{\tau_q}\frac{t}{\Gamma(2)}\right] + e^r t,$$

$$\begin{aligned}\bar{T}_{n+1}(\vec{r},t) = &-\frac{1}{\tau_q}D_t^{-1}\bar{T}_n(\vec{r},t) + \frac{1}{C_p\tau_q}\Big(D_t^{-\beta}(\nabla\cdot(K(\bar{T}_n+\cdots+\bar{T}_0)\cdot\nabla(\bar{T}_n+\cdots+\bar{T}_0)))\Big)\\ &- D_t^{-\beta}\Big(\nabla\cdot(K(\bar{T}_{n-1}+\cdots+\bar{T}_0)\cdot\nabla(\bar{T}_{n-1}+\cdots+\bar{T}_0)))\Big)\\ &+ \frac{\tau_T}{C_p\tau_q}\nabla\cdot\Big(D_t^{-\alpha}(K(\bar{T}_n+\bar{T}_{n-1}+\cdots+\bar{T}_0)\\ &\cdot\nabla(\bar{T}_n+\bar{T}_{n-1}+\cdots+\bar{T}_0)) - (D_t^{-\alpha}(K(\bar{T}_{n-1}+\cdots+\bar{T}_0)\cdot\nabla(\bar{T}_{n-1}+\cdots+\bar{T}_0)))\Big).\end{aligned} \tag{12}$$

Example 1. Let in Eq. (8) and consider initial conditions (as in [11, 17]) $T(\vec{r},0) = e^r$ and $\frac{\partial T(\vec{r},0)}{\partial t} = e^r$. According to [4, 22, 26] we apply MADM on Eq. (9) and using above assumptions in the Eq. (10) and Eq. (11) we get

$$T_0(\vec{r},t) = e^r\left[1 - \frac{1}{\tau_q}\frac{t}{\Gamma(2)}\right] + e^r t, \tag{13}$$

$$T_{n+1}(\vec{r},t) = -\frac{1}{\tau_q}D_t^{-1}T_n(\vec{r},t) + \frac{1}{C_p\,\tau_q}D_t^{-\beta}\Big(\nabla\cdot(T_n\cdot\nabla T_n)\Big) + \frac{\tau_T}{C_p\,\tau_q}\nabla\cdot\Big(D_t^{-\alpha}(T_n\cdot\nabla T_n)\Big), \tag{14}$$

for $n = 0, 1, 2, \cdots$.

We take only three iterations i.e. for $n = 0, 1, 2, \cdots$

$$T(\vec{r},\,t) = T_0(\vec{r},\,t) + T_1(\vec{r},\,t) + T_2(\vec{r},\,t) + \cdots. \tag{15}$$

According to [8, 10, 20] we apply NIM on Eq. (9) and using above assumptions in the Eq. (12) we get

$$\bar{T}_0(\vec{r},\,t) = e^r\left[1 - \frac{1}{\tau_q}\frac{t}{\Gamma(2)}\right] + e^r t,$$

$$\begin{aligned}\bar{T}_{n+1}(\vec{r},t) = &-\frac{1}{\tau_q}D_t^{-1}\bar{T}_n(\vec{r},t) + \frac{1}{C_p\tau_q}\Big(D_t^{-\beta}(\nabla\cdot((\bar{T}_n+\cdots+\bar{T}_0)\cdot\nabla(\bar{T}_n+\cdots+\bar{T}_0)))\Big)\\ &- D_t^{-\beta}\Big(\nabla\cdot((\bar{T}_{n-1}+\cdots+\bar{T}_0)\cdot\nabla(\bar{T}_{n-1}+\cdots+\bar{T}_0))\Big)\\ &+ \frac{\tau_T}{C_p\tau_q}\nabla\cdot\Big(D_t^{-\alpha}((\bar{T}_n+\bar{T}_{n-1}+\cdots+\bar{T}_0)\cdot\nabla(\bar{T}_n+\bar{T}_{n-1}+\cdots+\bar{T}_0))\\ &- (D_t^{-\alpha}((\bar{T}_{n-1}+\cdots+\bar{T}_0)\cdot\nabla(\bar{T}_{n-1}+\cdots+\bar{T}_0)))\Big).\end{aligned} \tag{16}$$

4 Numerical results and discussion

For $\tau_T = 89.286 \times 10^{-12}$, $\tau_q = 0.7838 \times 10_{-12}$ and $C_p = 2.5 \times 10^6$ (as taken in [21]) the plots are drawn in Fig. 2 and Fig. 3, which shows that the slope of the curve i.e. $\frac{\delta T}{\delta t}$ is decreasing very slowly (almost constant in Fermi seconds range). In Fig. 4, $\frac{\delta T}{\delta t}$ changes rapidly within Pico region, i.e. T decreases firstly then increases rapidly and hence there is a point of minima (for consider problem in which we take K = T). In Fig. 5, Fig. 6, Fig. 7 and Fig. 8 and onward we take pico to Micro and Micro to mile-seconds now $\frac{\delta T}{\delta t}$ is increasing only, i.e. T increases rapidly.

The interesting result is that the variation in order of derivative shows great change in T as shown in Fig. 6, Fig. 7 and Fig. 8 in which it is clear that as order increases $\frac{\delta T}{\delta t}$ increases i.e. T increases rapidly for higher order of derivative. It is obvious from Fig. 7 and Fig. 8 that for the value of $\alpha > 1$ we arrived different case of discussion, which is totally different from our case.

In Fig. 1 we compare the results obtain by NIM and MADM and it is found that whatever methods either NIM or MADM we used both gives identical solutions.

Mathematica 5.2 version has been used for all kind of calculations.

5 Conclusion

We proposed Dual-Phase-Lag heat equation of fractional order and successfully we establish it by a suitable example. We believe that this equation is able to cut off all the limitations which are occurred due to structure of medium or material in microscale heat transport.

The Modified Decomposition Method (MDM) and New Iterative Method (NIM) are successfully used to solve this equation. Both MDM and NIM are very powerful in finding the solutions for various physical problems. The main interest is in establishing the solution and constructing a comparative study of finding numerical solutions Fractional Dual-Phase-Lag heat transfer equation with these two methods. It is seen that both methods are efficient for finding solutions to a high degree of accuracy (Fig. 1). The basic difference between the methods is that NIM is direct and straightforward and it avoids the volume of calculations resulting from requiring the Adomian polynomials for finding the solution by modern decomposition methods.

It is observed that the order of the proposed equation affects the transfer of heat significantly with in micro region Fig. 7 and Fig. 8.

The authors' strongly believe that this novel approach will facilitate many young researchers who are working in this area.

6 Acknowledgements

The author would like to thank the anonymous reviewers. Their comments helped improve this manuscript.

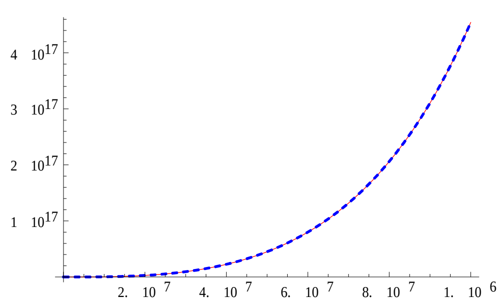

Fig. 1. Comparison between MDM(in Red) and NIM(in dashed Blue) (for $\alpha = 0.9, r = 10^{-6}$)

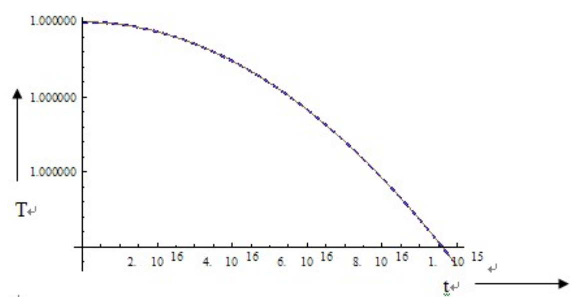

Fig. 2. Plot between Temperature (T) and time (t) for all values of α

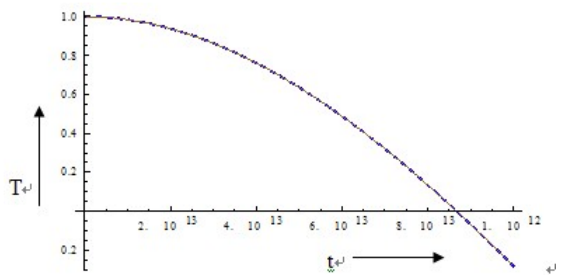

Fig. 3. Plot between Temperature (T) and time (t) for all values of α

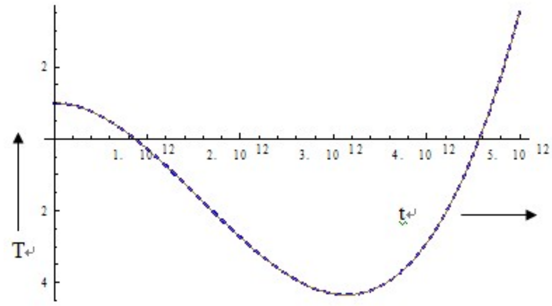

Fig. 4. Plot between Temperature (T) and time (t) for all values of α

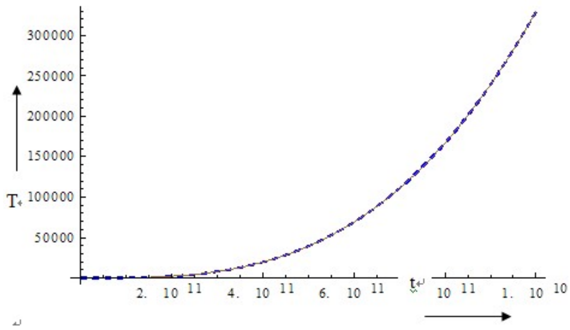

Fig. 5. Plot between Temperature (T) and time (t) for all values of α

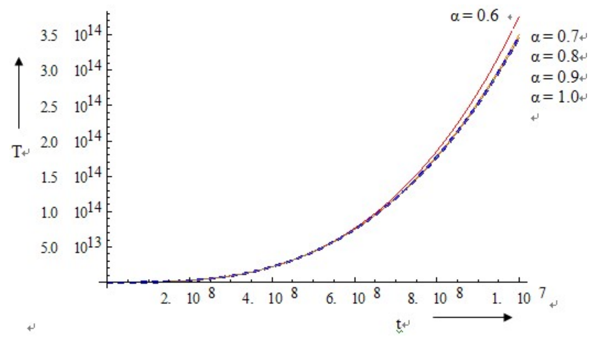

Fig. 6. Plot between Temperature (T) and time (t) for all values of α

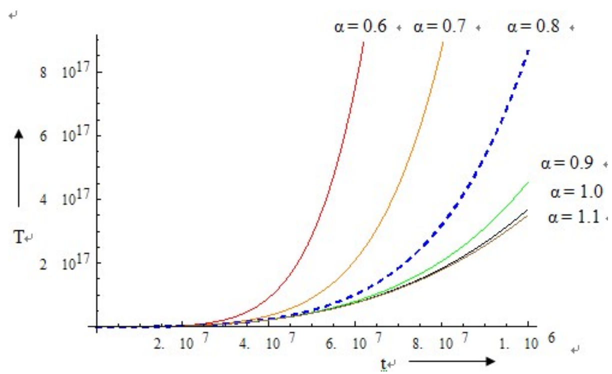

Fig. 7. Plot between Temperature (T) and time (t) for all values of α

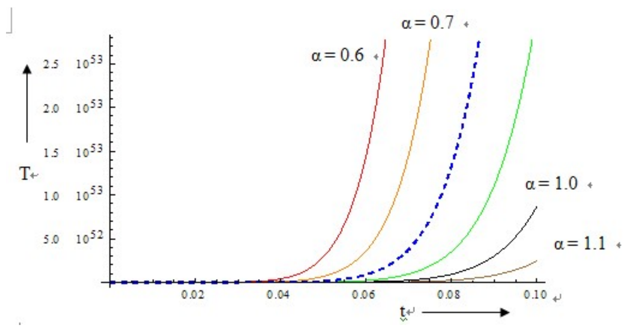

Fig. 8. Plot between Temperature (T) and time (t) for all values of α

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