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Coupling of laplace transform and correction functional for wave equations

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Abstract. In this paper, we apply Variational Iteration Method coupled with Laplace Transform Method to solve wave equations which arise very frequently in physical problems related to engineering and applied sciences. It is observed that the proposed technique is suitable for such problems and is very user-friendly. Several examples are given to re-confirm the efficiency of the suggested algorithm.

Keywords: variational iteration method, laplace transform method, wave equations, MAPLE 13

1 Introduction

The rapid development of nonlinear sciences[1–14] witnesses a wide range of analytical and numerical techniques by various scientists. Most of the developed schemes have their limitations like limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems[1–14]. The basic motivation of present study is the coupling of correction functional of Variational Iteration Method (VIM) and Laplace transform. It has been observed that the Coupling of Laplace Transform and Correction Functional enhances its efficiency and reduces the computational work to a tangible level. Moreover, this version is more user-friendly and it overcomes some of the basic deficiencies. The suggested algorithm is tested on linear, nonlinear wave equations and wave-like equations in bounded and unbounded domains. Numerical results are very encouraging.

2 Variational Iteration Method (VIM)

To illustrate the basic concept of the He's VIM, we consider the following general differential equation

$$L[u] + N[u] = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method[2–4, 6–11, 14], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\Big(Lu_n(s) + N\tilde{u}_n(s) - g(s)\Big)\mathrm{d}s, \tag{2}$$

where λ is a Lagrange multiplier[2–4, 6–11, 14], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; Eq. (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method

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and its applicability for various kinds of differential equations are given in [2–4, 6–11, 14]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation $u_{n+1},\ n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0 consequently, the solution is given by $u = \lim\limits_{n\to\infty} u_n$.

3 Variational Iteration Method coupled with laplace transform

Consider the general nonlinear, inhomogeneous partial differential equation

$$L_u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t),$$

with the initial condition and in this paper L is operator $\left(\frac{\partial^2}{\partial t^2}\right)$

$$u(x,0) = h(x),\ u_t(x,0) = f(x).$$

Taking the Laplace Transform to the both sides of the given equation

$$\mathfrak{L}Lu(x,t) + \mathfrak{L}Ru(x,t) + \mathfrak{L}Nu(x,t) = \mathfrak{L}f(x,t),$$

with Laplace Transformation

$$s^2\mathfrak{L}(x,s) - su(x,0) - u_t(x,0) = \mathfrak{L}f(x,t) - \mathfrak{L}Ru(x,t) - \mathfrak{L}Nu(x,t).$$

We have

$$\mathfrak{L}u(x,s) = \frac{1}{s}h(x) + \frac{1}{s^2}f(x) + \frac{1}{s^2}\Big[\mathfrak{L}f(x,t) - \mathfrak{L}Ru(x,t) - \mathfrak{L}Nu(x,t)\Big].$$

Taking the inverse Laplace

$$u(x,t) = h(x) + f(x)t + \mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}f(x,t)\right] - \mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}Ru(x,t)\right] - \mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}Nu(x,t)\right].$$

Applying $\frac{\partial}{\partial t}$ on both sides, we have

$$u_t(x,t) + \mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}Ru(x,t)\right] + \mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}Nu(x,t)\right] - \mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}f(x,t)\right] - f(x) = 0.$$

The correction functional of the Variational Iteration Method is given as

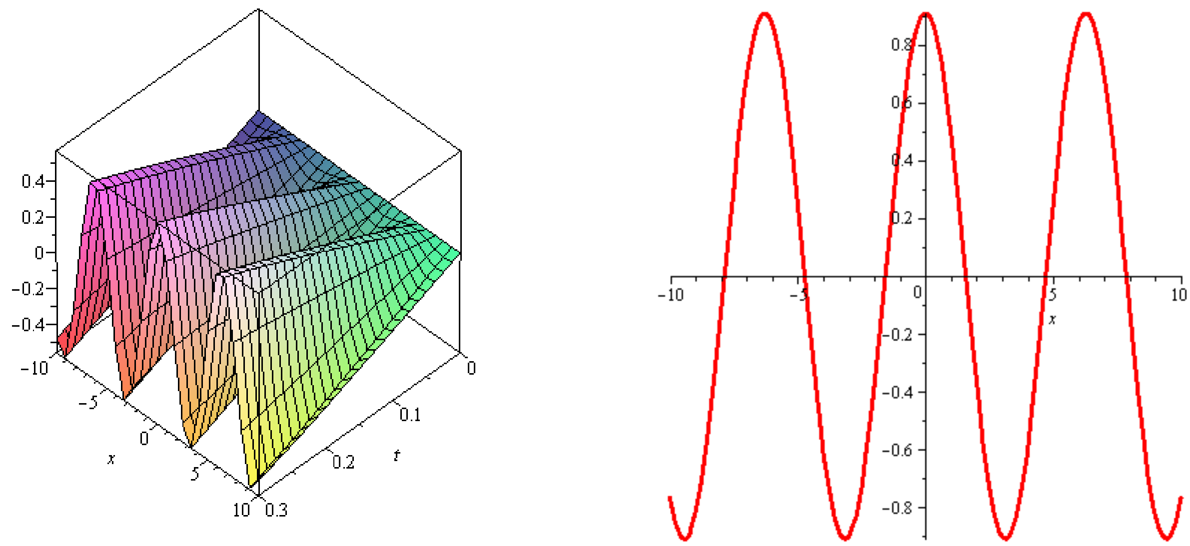
$$u_{n+1}(x,t) = u_n - \int_0^t \left[\frac{\partial u_n}{\partial t} + \mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}Ru(x,t)\right) + \mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}Nu(x,t)\right) - \mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}f(x,t)\right) - f(x)\right](\mathrm{d}\tau).$$

The solution in the series form is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

4 Numerical applications

In this section, we apply Variational Iteration Method (VIM) coupled with Laplace transform to solve linear & nonlinear Wave Equations. Numerical results are very encouraging.



(a) (b)

Fig. 1. (a): 3D plot of $u(x,t)$ for example 1, (b): 2D plot of $u(x,t)$ for example 1

Example 1. Homogeneous wave equation (see Fig. 1)

Consider the homogeneous wave equation

$$u_{tt} = u_{xx} - 3u,\ 0 < x < \pi,\ t > 0,$$

subject to the initial conditions

$$u(x,0) = 0,\ u_t = 2\cos x.$$

Applying Laplace Transformation on both sides of given equation, we get

$$s^2\mathfrak{L}u(x,s) - su(x,0) - u_t(x,0) = \mathfrak{L}u_{xx} - 3\mathfrak{L}u.$$

By inverse Laplace and derivative,

$$u_t(x,t) = 2\cos x + \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}u_{xx}\right] - \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{1}{s^2}3\mathfrak{L}u\right].$$

Applying Variational Iteration Method (VIM), we get

$$u_{n+1}(x,t) = u_n - \int_0^t \left[\frac{\partial u_n}{\partial t} - 2\cos x - \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}u_{nxx}\right) + \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left(\frac{1}{s^2}3\mathfrak{L}u_n\right)\right]\mathrm{d}\tau.$$

Consequently, the following approximations are obtained successively

$$\begin{aligned}
u_0(x,t) &= 2t\cos x,\\
u_1(x,t) &= 2t\cos x - \frac{(2t)^2}{3!}\cos x,\\
u_2(x,t) &= 2t\cos x - \frac{(2t)^2}{3!}\cos x + \frac{(2t)^5}{5!}\cos x,\\
u_3(x,t) &= 2t\cos x - \frac{(2t)^2}{3!}\cos x + \frac{(2t)^5}{5!}\cos x - \frac{(2t)^7}{7!}\cos x,\\
&\ \ \vdots
\end{aligned}$$

Finally, the solution in series form is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),\ u(x,t) = \cos\left[(2t) - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \cdots\right],$$

the closed form solution is given as

$$u(x,t) = \cos x \sin 2t.$$

Example 2. Inhomogeneous wave equation[14] (see Fig. 2)

Consider the following inhomogeneous nonlinear wave equation

$$u_{tt} = u_{xx} + u + u^2 - xt - x^2t^2,\ 0 < x < \pi,\ t > 0,$$

subject to the initial conditions

$$u(x,0) = 0,\ u_t(x,0) = x.$$

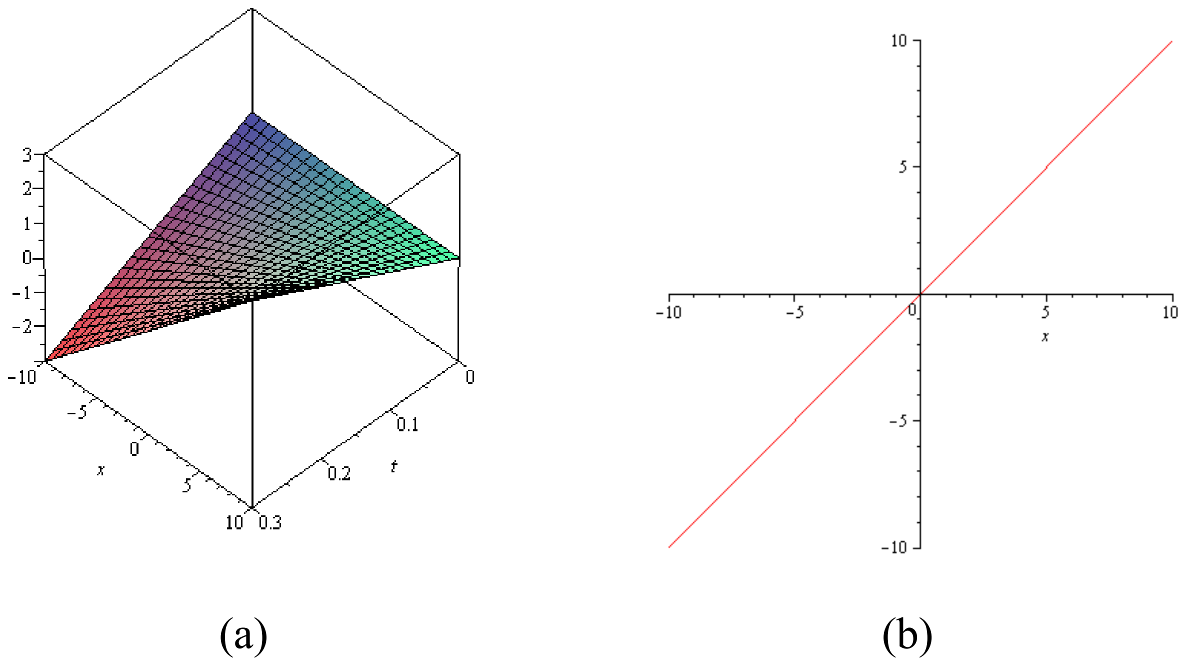


Fig. 2. (a): 3D plot of $u(x,t)$ for example 2, (b): 2D plot of $u(x,t)$ for example 2

Applying Laplace Transformation on both sides of given equation, we get

$$s^2\mathfrak{L}u(x,s) - su(x,0) - u_t(x,0) = \mathfrak{L}u_{xx} + \mathfrak{L}u + \mathfrak{L}u^2 - \mathfrak{L}xt - \mathfrak{L}x^2t^2.$$

By inverse Laplace and derivative, we get

$$u_t(x,t) = x + \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{2}{s^2}\mathfrak{L}u_{xx}\right] + \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}u\right] + \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}u^2\right]$$
$$- \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}xt\right] - \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{1}{s^2}\mathfrak{L}x^2t^2\right].$$

Applying Variational Iteration Method (VIM), we get

$$u_{n+1}(x,t) = u_n - \int_0^t \left[\frac{\partial u_n}{\partial \tau} - x - \frac{\partial}{\partial \tau}\mathfrak{L}^{-1}\left(\frac{1}{s^2}Lu_{nxx}\right) - \frac{\partial}{\partial \tau}\mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}u_n\right) - \frac{\partial}{\partial \tau}\mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}u_n^2\right)\right.$$
$$\left. + \frac{\partial}{\partial \tau}\mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}xt\right) + \frac{\partial}{\partial \tau}\mathfrak{L}^{-1}\left(\frac{1}{s^2}\mathfrak{L}x^2t^2\right)\right]\mathrm{d}\tau.$$

Consequently, following approximants are obtained successively

$$\begin{aligned}
u_0(x,t) &= xt,\\
u_1(x,t) &= xt,\\
u_2(x,t) &= xt,\\
u_3(x,t) &= xt,\\
&\vdots
\end{aligned}$$

Finally, the solution in series form is given by

$$\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t),\\
u(x,t) &= xt.
\end{aligned}$$

The closed form solution is given as

$$u(x,t) = xt.$$

Example 3. Wave-like equation (see Fig. 3)

Consider the following wave like equation

$$u_{tt} = \frac{x^2}{2}u_{xx},\ 0 < x < 1,\ t > 0,$$

subject to the initial conditions

$$u(x,0) = 0,\ u_t(x,0) = x^2.$$

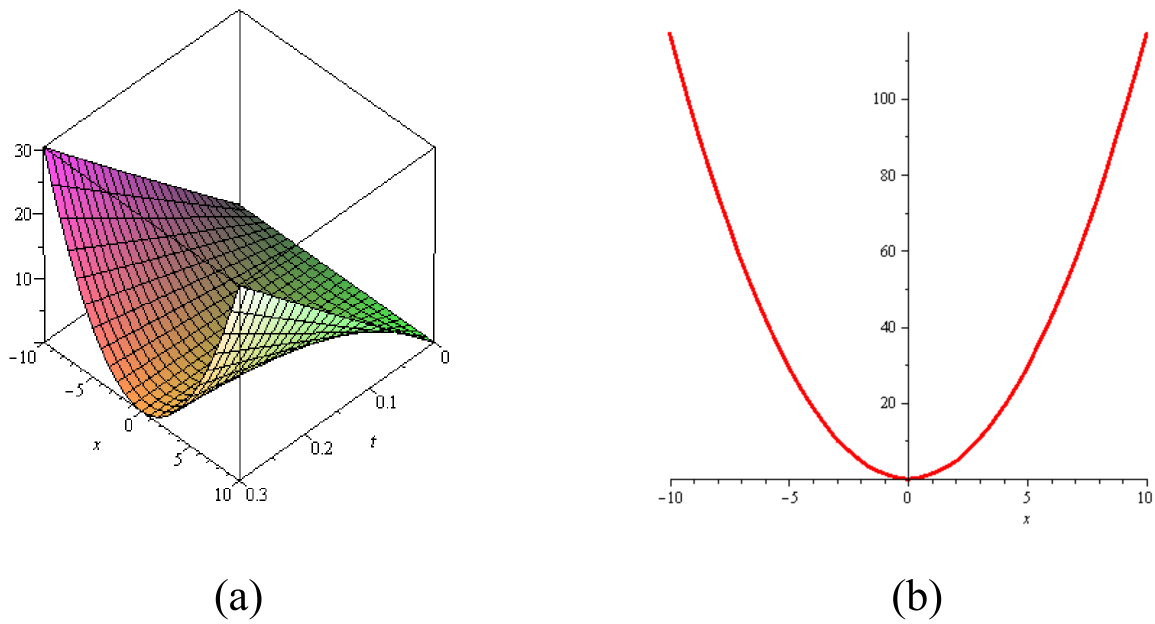


Fig. 3. (a): 3D plot of $u(x,t)$ for example 3, (b): 2D plot of $u(x,t)$ for example 3

Applying Laplace Transform on both sides of given equation, we get

$$s^2\mathcal{L}u(x,s) - su(x,0) - u_t(x,0) = \mathcal{L}\frac{x^2}{2}u_{xx}.$$

By inverse Laplace and derivative, we get

$$u_t(x,t) = x^2 + \frac{\partial}{\partial t}\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{x^2}{2}\mathcal{L}u_{xx}\right].$$

Applying Variational Iteration Method (VIM), we get

$$u_{n+1}(x,t) = u_n - \int_0^t \left[\frac{\partial u_n}{\partial \tau} - x^2 - \frac{\partial}{\partial \tau}\mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{x^2}{2}\mathcal{L}u_{nxx}\right)\right]\mathrm{d}\tau.$$

Consequently, following approximants are obtained successively

$$\begin{aligned}
u_0(x,t) &= x^2 t,\\
u_1(x,t) &= x^2 t + \frac{x^2t^2}{3!},\\
u_2(x,t) &= x^2 t + \frac{x^2t^2}{3!} - \frac{x^2t^5}{5!},\\
u_3(x,t) &= x^2 t + \frac{x^2t^2}{3!} - \frac{x^2t^5}{5!} + \frac{x^2t^7}{7!},\\
&\vdots
\end{aligned}$$

Finally, the solution in series form is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),\ u(x,t) = x^2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots\right).$$

The closed form solution is

$$u(x,t) = x^2 \sinh t.$$

Example 4. Wave-like equation in unbounded domain (see Fig. 4)

We finally study the wave equation in an unbounded domain.

$$u_{tt} = x_{xx},\ -\infty < x < \infty,\ t > 0,$$

subject to the initial conditions

$$u(x,0) = \sin x,\ u_t(x,0) = 0.$$

Applying Laplace Transform on both sides of given equation, we get

$$s^2\mathcal{L}u(x,s) - su(x,0) - u_t(x,0) = \mathcal{L}u_{xx}.$$

By inverse Laplace and derivative, we get

$$u_t(x,t) = \frac{\partial}{\partial t}\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}u_{xx}\right].$$

Applying Variational Iteration Method (VIM), we get

$$u_{n+1}(x,t) = u_n - \int_0^t \left[\frac{\partial u_n}{\partial s} - \frac{\partial}{\partial s}\mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}u_{nxx}\right)\right]\mathrm{d}s.$$

Consequently, following approximants are obtained successively

$$\begin{aligned}
u_0(x,t) &= \sin x,\\
u_1(x,t) &= \sin x - \frac{t^2}{2!}\sin x,\\
u_2(x,t) &= \sin x - \frac{t^2}{2!}\sin x + \frac{t^4}{4!}\sin x,\\
u_3(x,t) &= \sin x - \frac{t^2}{2!}\sin x + \frac{t^4}{4!}\sin x - \frac{t^6}{6!}\sin x,\\
&\vdots
\end{aligned}$$

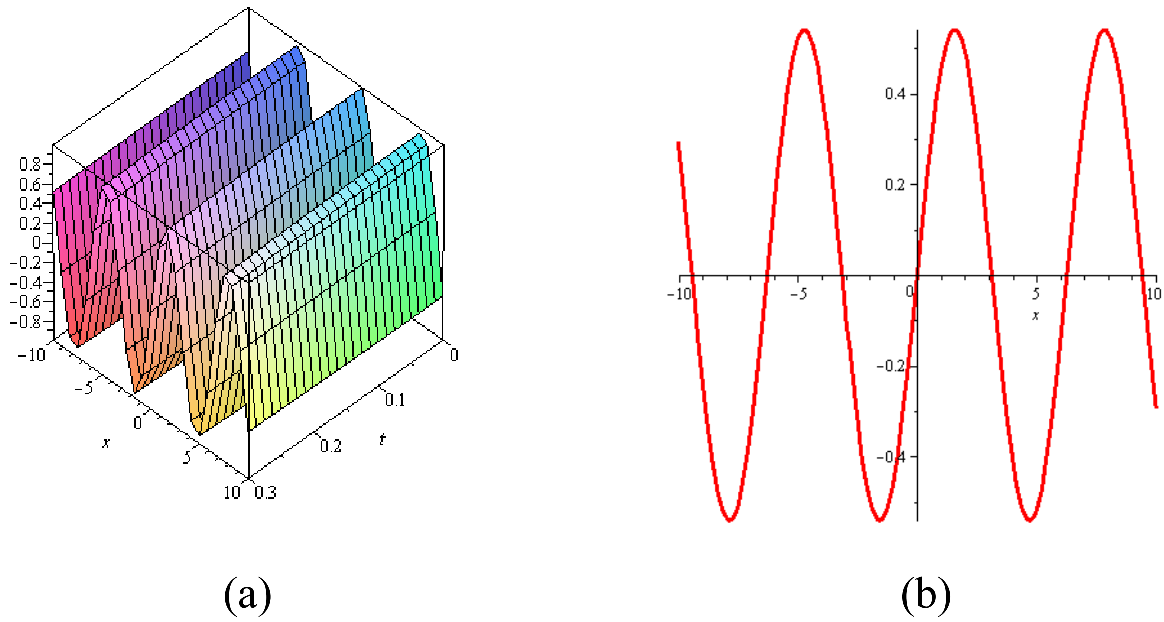


Fig. 4. (a): 3D plot of $u(x,t)$ for example 4, (b): 2D plot of $u(x,t)$ for example 4

Finally, the solution in series form is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t),\ u(x,t) = \sin x\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right).$$

The closed form solution is given as

$$u(x,t) = \sin x \cos t.$$

5 Conclusion

Coupling of Variational Iteration Method (VIM) and Correction Functional proved very effective to solve linear & nonlinear Wave equations. The proposed algorithm is suitable for such problems and is very user-friendly. Computational work and subsequent results are fully supportive of the reliability, efficiency and accuracy of the suggested scheme.

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