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New Iterative Method for Time-Fractional Schrödinger Equations

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Abstract. New Iterative Method (NIM) is applied to tackle time- fractional Schrödinger equations. The proposed technique is fully compatible with the complexity of these problems and obtained results are highly encouraging. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the suggested algorithm.

Keywords: new iterative method, fractional Schrödinger partial differential equations, nonlinear problems

1 Introduction

Nonlinear partial differential equations[1–12, 14–19, 24, 25] are of extreme importance in applied and engineering sciences. The through study of literature reveals that most of the physical phenomena are nonlinear in nature and hence there is a dire need to fine their appropriate solutions, see [1–25] and the references therein. Recently, scientists have observed that number of real time problems is modeled by fractional nonlinear differential equations[1–11, 14–17, 19, 24, 25] which are very hard to tackle.

In 2006, Dafardar-Gejji and Jafari [8] have proposed a new technique for solving linear and nonlinear functional equations, namely the New Iterative Method. The Method has proven useful in solving equations such as algebraic equations, integral equations, ordinary and partial differential equations of integer and fractional order see [5–8] and the references therein. In the similar context, we apply New Iterative Method (NIM) to solve time- fractional Schrödinger partial differential equations[15, 17, 25]:

$$D_t^{\alpha} u(x,t) + \iota u(x,t) = 0;\ u(x,0) = f(x),\ i^2 = -1, \tag{1}$$

or

$$iD_t^{\alpha} u(x,t) + u(x,t) - y|u(x,t)|^2 u(x,t) = 0;\ u(x,0) = f(x),\ \iota^2 = -1, \tag{2}$$

where $0 \leq \alpha \leq 1$. The fractional derivatives are considered in the Caputo sense. It is to be highlighted that such equations arise frequently in applied, physical and engineering sciences. The basic motivation of this paper is the extension of a very reliable and efficient technique which is called New Iterative Method (NIM) to find approximate solutions of time-fractional Schrödinger partial differential equations. It is observed that the proposed algorithms is fully synchronized with the complexity of fractional differential equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

2 Definitions

Definition 1. *A real function $f(x)$, $x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C\ [0, \infty)$, and it is said to be in the space C_μ^∞ if $f^m \in C_\mu, \mu \geq 1, m \in N$.*

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Definition 2. *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$, *of a function* $f \in C, \ \geq -1$, *is defined as*
$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$. $\alpha > 0, x > 0$,
$J^0 f(x) = f(x)$.

Properties of the operator j^α *can be found in [1–3, 8–11, 14, 16], we mention only the following For* $f \in C, \ \geq -1, \alpha, \beta \geq 0$ *and* $\gamma > -1$*:*
1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 3. *The fractional derivative of* $f(x)$ *in the Caputo sense is defined as* $D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t)dt$, *for* $m-1 < \alpha \leq m, \ m \in Z, x > 0, \ f \in C_{-1}^m$. *Also, we need here two of its basic properties.*

Lemma 1. *if* $m-1 < \alpha \leq m, \ m \in N$ *and* $f \in C^m, \ \geq -1$, *then* $D_*^\alpha J^\alpha f(x) = f(x)$, *and* $J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+) x^k}{k!}, \ x > 0$.

3 Analysis of New Iterative Method

Consider the following general functional equation

$$u(\bar{x}) = f(\bar{x}) + N(u(\bar{x})), \tag{3}$$

where N is a nonlinear operator from a Banach space $B \rightarrow B$ and f is a known function. $\bar{x} = (x_1, x_2, x_3, \cdots, x_n)$. we are looking for a solution u of Eq. (3) having the series form

$$u(\bar{x}) = \sum_{i=1}^{\infty} u_i(\bar{x}). \tag{4}$$

The nonlinear operator N can b decomposed as

$$N\left(\sum_{i=1}^{\infty} u\right) = N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=1}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \tag{5}$$

From Eqs. (4) and (5), Eq. (3) is equivalent to

$$\left(\sum_{i=1}^{\infty} u_i\right) = f + N(u_0) + \sum_{i=1}^{\infty} \left\{ N(\sum_{j=1}^{i} u_j) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}. \tag{6}$$

We define the recurrence relation

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \\ m = 1, 2, \cdots \end{cases} \tag{7}$$

Then

$$(u_0 + \cdots + u_{m+1}) = N(u_0 + \cdots + u_m), \ m = 1, 2, \cdots \tag{8}$$

And

$$\sum_{i=1}^{\infty} u_i = f + N\left(\sum_{i=1}^{\infty} u_i\right). \tag{9}$$

The k-term approximate solution of Eq. (3) and Eq. (4) is given by $u = u_0 + u_1 + \cdots\cdots + u_{k-1}$ [4, 18].

3.1 Solving PDEs using NIM

Consider the partial differential equation of arbitrary order

$$D_t^\alpha u(x,t) = A(u, \partial u) + B(x,t),\ m-1 < \alpha \leq m,\ m \in N, \tag{10}$$

$$\frac{\partial^k u}{\partial t^k}(x,0) = h_k(x),\ k = 0, 1, \cdots, m-1, \tag{11}$$

where A is a nonlinear function of u and ∂u (partial derivatives of u with respect to x and t), and B is the source function. In view of (2.4), the initial value problems considered in section 4 are equivalent to the following integro-partial differential equation:

$$u(x,t) = \sum_{k=0}^{m-1} h_k(x)\frac{t^k}{k!} + I_t^\alpha B + I_t^\alpha A = f + N(u), \tag{12}$$

where $f = \Sigma_{k=0}^{m-1} h_k(x)\frac{t^k}{k!} + I_t^\alpha B I_t^\alpha$ and $N(u) = I_t^\alpha A$. We get the solution of Eq. (12) by employing the algorithm (7).

4 Numerical examples

In this section, we apply New Iterative Method (NIM) to solve time- fractional Schrödinger equations. Numerical results are very encouraging.

Example 1. We first consider the linear time-fractional Schrödinger equations.

$$D_t^\alpha u + \iota u_{xx} = 0,\ 0 < \alpha \leq 1, \tag{13}$$

with initial conditions $u(x,0) = 1 + \cosh(2x)$.

The problem is equivalent to the following integro-partial differential equation:

$$u = 1 + \cosh(2x) - I_t^\alpha(iu_{xx}) = f + N(u),$$

where $f = 1 + \cosh(2x) - I_t^\alpha(iu_{xx})$ and $N(u) = 0$.

In view of algorithm (7), we get

$u_0(x,t) = 1 + \cosh(2x)$,
$u_1(x,t) = -4\iota\cosh(2x)\frac{t^\alpha}{\Gamma(\alpha+1)}$,
$u_2(x,t) = (4\iota)^2\iota\cosh(2x)\frac{t^2\alpha}{\Gamma(2\alpha+1)}$,
$u_3(x,t) = -(4\iota)^3\iota\cosh(2x)\frac{t^3\alpha}{\Gamma(3\alpha+1)}$,
$\cdots$

The solution in the series form is given by

$$\begin{aligned} u(x,t) =& 1 + \cosh(2x)\Bigg(1 - 4\iota\cosh(2x)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ (4\iota)^2\iota\cosh(2x)\frac{t^2\alpha}{\Gamma(2\alpha+1)} - (4\iota)^3\iota\cosh(2x)\frac{t^3\alpha}{\Gamma(3\alpha+1)} + \cdots\Bigg). \end{aligned} \tag{14}$$

For the special case $\alpha = 1$, we obtain from Eq. (14)

$$u(x,t) = 1 + \cosh(2x)e^{-4\iota t}, \tag{15}$$

which is the exact solution of the Schrödinger equation[15]. The results for the exact solution Eq. (15) and the approximate solution Eq. (14) considering the first four term series solution using New Iterative Method, for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in Fig. 1.

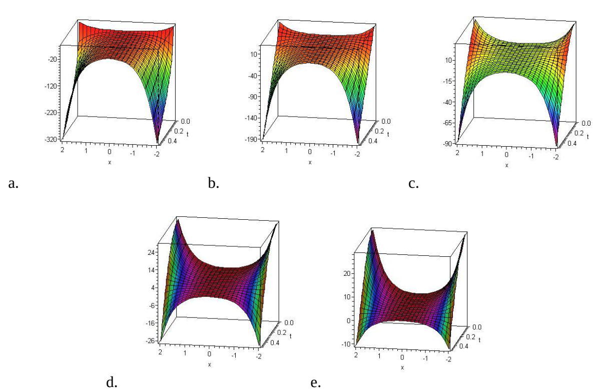

Fig. 1. The surface shows solution $u(x,t)$ for the Eq. (14) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution (15)

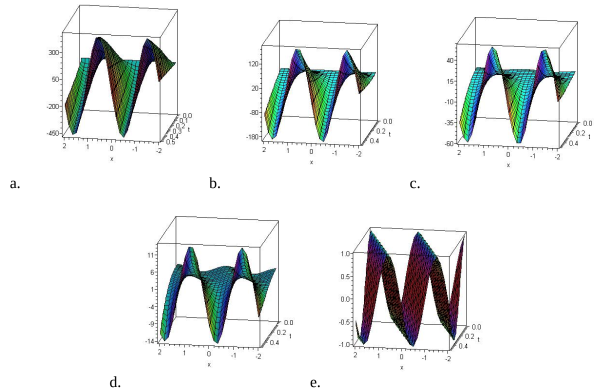

Fig. 2. The surface shows solution $u(x,t)$ for the Eq. (18) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution (17).

Example 2. Consider the following linear time-fractional Schrödinger equation

$$D_t^{\alpha} u + \iota u_{xx} = 0,\ 0 < \alpha \leq 1, \tag{16}$$

with initial conditions $u(x,0) = e^{3\iota x}$.

The given equation is equivalent to the integro-partial differential equation:

$$u = e^{3ix} - I_t^{\alpha}(iu_{xx}) = f + N(u),$$

where $f = e^{3ix} - I_t^{\alpha}(iu_{xx})$ and $N(u) = 0$.

In view of the algorithm (7), we get

$u_0(x,t) = e^{3\iota x}$,
$u_1(x,t) = 9\iota e^{3\iota x}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$,
$u_2(x,t) = (9\iota)^2 e^{3\iota x}\frac{t^2\alpha}{\Gamma(2\alpha+1)}$,
$u_3(x,t) = (9\iota)^3 e^{3\iota x}\frac{t^3\alpha}{\Gamma(3\alpha+1)}$,
...

The solution in the series form is given by

$$u(x,t) = e^{3\iota x}\left(1 + 9\iota\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (9\iota)^2\frac{t^2\alpha}{\Gamma(2\alpha+1)} + (9\iota)^3\frac{t^3\alpha}{\Gamma(3\alpha+1)} + \cdots\right), \tag{17}$$

For the special case $\alpha = 1$, we obtain the form Eq. (17)

$$u(x,t) = e^{3\iota(x+3t)}, \tag{18}$$

which is the exact solution of the Schrödinger equation[15].

The results for the exact solution Eq. (18) and the approximate solution Eq. (19) considering the first four term series solution using New Iterative Method, for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in Fig. 2.

Example 3. Consider the following nonlinear time-fractional Schrödinger equation

$$\iota D_t^{\alpha} u + u_{xx} + 2|u|^2\bar{u} = 0,\ 0 < \alpha \leq 1, \tag{19}$$

with initial conditions $u(x,0) = e^{\iota x}$.

The given equation is equivalent to the integral equation

$$u = e^{\iota x} + I_t^{\alpha}(iu_{xx} + 2iu^2\bar{u}) = f + N,$$

where $f = e^{\iota x} + I_t^{\alpha}(iu_{xx})$ and $N(u) = I_t^{\alpha}(2iu^2\bar{u})$, In view of the algorithm (7), we get

$u_0(x,t) = e^{\iota x}$,
$u_1(x,t) = \iota e^{\iota x}\frac{t^\alpha}{\Gamma(\alpha+1)}$,
$u_2(x,t) = \iota^2 e^{\iota x}\frac{t^2\alpha}{\Gamma(2\alpha+1)}$,
$u_3(x,t) = \iota^3 e^{\iota x}\left[5 - \frac{2\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}\right]\frac{t^3\alpha}{\Gamma(3\alpha+1)}$,
$\cdots$

The solution in the series form is given by

$$u(x,t) = e^{\iota x}\left(1 + \iota\frac{t^\alpha}{\Gamma(\alpha+1)} + \iota^2\frac{t^2\alpha}{\Gamma(2\alpha+1)} + \iota^3\left[5 - \frac{2\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}\right]\frac{t^3\alpha}{\Gamma(3\alpha+1)} + \cdots\right). \tag{20}$$

For the special case $\alpha = 1$, we obtain the form Eq. (20)

$$u(x,t) = e^{\iota(x+t)}, \tag{21}$$

which is the exact solution of the Schrödinger equation[15]. The results for the exact solution Eq. (20) and the approximate solution Eq. (21) considering the first four term series solution using New Iterative Method, for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in Fig. 3.

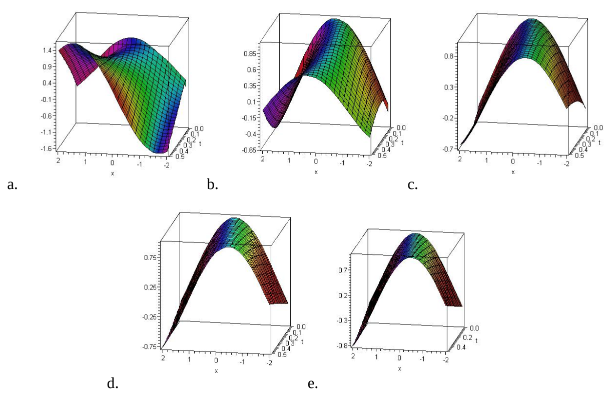

Fig. 3. The surface shows solution $u(x,t)$ for the Eq. (21) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution (20)

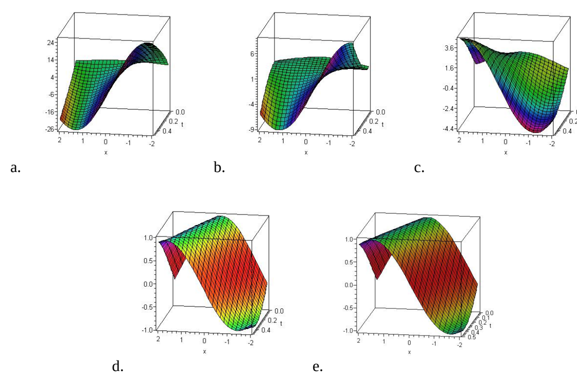

Fig. 4. The surface shows solution $u(x,t)$ for the Eq. (24) when (a) $\alpha = 0.25$, (b) $\alpha = 0.50$, (c) $\alpha = 0.75$, (d) $\alpha = 1$, (e) exact solution (23)

Example 4. Consider the following nonlinear time-fractional Schrödinger equation

$$\iota D_t^\alpha u + u_{xx} - 2|u|^2\bar{u} = 0,\ 0 < \alpha \le 1, \tag{22}$$

with initial conditions $u(x,0) = e^{\iota x}$.

The given equation is equivalent to the integral equation

$$u = e^{\iota x} + I_t^\alpha(iu_{xx} - 2iu^2\bar{u}) = f + N,$$

where $f = e^{\iota x} + I_t^\alpha(iu_{xx})$ and $N(u) = -I_t^\alpha(2iu^2\bar{u})$,

In view of the algorithm (7), we get

$u_0(x,t) = e^{\iota x}$,
$u_1(x,t) = -3\iota e^{\iota x}\frac{t^\alpha}{\Gamma(\alpha+1)}$,
$u_2(x,t) = (3\iota)^2 e^{\iota x}\frac{t^2\alpha}{\Gamma(2\alpha+1)}$,
$u_3(x,t) = -(\iota)^3 e^{\iota x}\left[63 - \frac{18\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}\right]\frac{t^3\alpha}{\Gamma(3\alpha+1)}$,
$\cdots$

The solution in the series form is given by

$$u(x,t) = e^{\iota x}\left(1 + 3\iota\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (3\iota)^2\frac{t^2\alpha}{\Gamma(2\alpha+1)} - (\iota)^3\left[63 - \frac{18\Gamma(1+2\alpha)}{(\Gamma(\alpha+1))^2}\right]\frac{t^3\alpha}{\Gamma(3\alpha+1)} + \cdots\right). \quad (23)$$

For the special case $\alpha = 1$, we obtain the from Eq. (23)

$$u(x,t) = e^{\iota(x+3t)}, \quad (24)$$

which is the exact solution of the Schrödinger equation[15]. The results for the exact solution Eq. (24) and the approximate solution Eq. (23) considering the first four term series solution using New Iterative Method, for $\alpha = 0.25, 0.50, 0.75$ and 1, are shown in Fig. 4.

5 Conclusion

New Iterative Method (NIM) has been implemented to find appropriate solutions of time-fractional non-linear Schrödinger equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the proposed algorithm.

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