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He's semi-inverse method for soliton solutions of Boussinesq system

H. Kheir, A. Jabbari, A. Yildirim*, A. K. Alomari

Department of Mathematics, Faculty of Science, Ege University, Bornova-Izmir 35100, Turkey

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Abstract. In this paper, we apply He's semi-inverse method to establish a variational theory for the Boussinesq system. Based on this formulation, a solitary solution can be easily obtained using Ritz method. Moreover, the results are also compared with He's homotopy perturbation method, Liao's homotopy analysis method and homotopy padé method. The results reveal that the proposed method is very effective and simple.

Keywords: He's semi-inverse method, Boussinesq system, Homotopy analysis method; Homotopy perturbation method; Homotopy padé technique

1 Introduction

There have been various approaches to search for soliton solutions for nonlinear wave equations. These methods include the inverse scattering method[19], Hirota's bilinear method[22], Bäcklund transformation[20], tanh-coth method[4], Jacobi elliptic function method[26], (G'/G)-expansion method[3, 10], homotopy perturbation method (HPM)[12], Luapanov's artificial small parameter method, δ-expansion method, Adomian decomposition method, variational iterative method[27], homotopy analysis method (HAM), homotopy padé method (HPadéM)[1, 2, 5–7, 11, 23–25] and so on.

In the past few decades, qualitative analysis together with ingenious mathematical techniques for handling various nonlinear problems has been studied. Among them, variational approaches, such as the semi-inverse method is a powerful and effective method to search for variational principles for physical problems and provides physical insight into the nature of the solution of problem[16, 17, 29–35].

In this paper we use He's semi-inverse method[16, 17, 29–35] to establish a variational formula for Boussinesq system.

The Boussinesq system[18, 21]

$$u_t + v_x = 0, \quad v_t + a(u^2)_x - bu_{xxx} = 0, \tag{1}$$

is used to model two-way propagation of certain water waves in a uniform horizontal channel filled with an irrotational and inviscid liquid[21, 28]. System (1) was solved using different methods. was widely discussed by many authors using different methods. Wazwaz discussed this system by the tanh method and the sine-cosine method[9].

2 He's semi-inverse method

We suppose that the given nonlinear partial differential equation for $u(x, t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots) = 0, \tag{2}$$

* Corresponding author. *E-mail address*: ahmetyildirim@gmail.com.

where P is a polynomial in its arguments. A. Jabbari et.al. in [3] have been written the He's semi-inverse method in the following steps:
step 1. Seek solitary wave solutions of Eq. (2) by taking $u(x,t) = U(\xi)$, $\xi = x - ct$, and transform Eq. (2) to the ordinary differential equation

$$Q(U, U', U'', \cdots) = 0, \tag{3}$$

where prime denotes the derivative with respect to ξ.
step 2. If possible, integrate Eq. (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.
step 3. According to the He's semi-inverse method, we construct the following trial-functional

$$J(U) = \int L\, d\xi, \tag{4}$$

where L is an unknown function of U and its derivatives. There exist alternative approaches to the construction of the trial-functionals, see Refs. [13–15].
step 4. By Ritz method, we can obtain different forms of solitary wave solutions, such as $U(\xi) = p\ \mathrm{csch}^2(q\xi)$, $U(\xi) = p\ \mathrm{sech}^2(q\xi)$,$U(\xi) = p\ \tanh^2(q\xi)$, $U(\xi) = p\ \coth^2(q\xi)$ and so on. For example in this paper we search a solitary wave solution in the form

$$U(\xi) = p\ \mathrm{sech}^2(q\xi), \tag{5}$$

where p and q are constants to be further determined. substituting Eq. (5) into Eq. (4) and making J stationary with respect to p and q, we have

$$\frac{\partial J}{\partial p} = 0, \tag{6}$$

$$\frac{\partial J}{\partial q} = 0. \tag{7}$$

Solving simultaneously the Eq. (6) and Eq. (7) we obtain p and q. Hence, the solitary wave solution Eq. (5) is well determined.

3 Homotopy analysis method

The HAM, first proposed by Liao in his Ph.D dissertation [23], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations[1, 2]. Liao in his book [24] proved that HAM is a generalization of some previously used techniques such as the δ-expansion method, artificial small parameter method and Adomian decomposition method. Moreover, unlike previous analytic techniques, the HAM provides a convenient way to adjust and control the region and rate of convergence[25]. It should be noted that the HPM is a particular case of the HAM[23]. There exist some techniques to accelerate the convergence of a given series of solutions. Among them, the so-called Padé method is widely applied. For further details, the reader is referred to Liao [24].

For convenience of the readers, we will first present a brief description of the standard HAM. To achieve our goal, let us assume the nonlinear system of differential equations be in the form of

$$N_j[u_1(x,t), u_2(x,t), \cdots, u_m(x,t)] = 0, \qquad j = 1, 2, \cdots, n, \tag{8}$$

where N_j are nonlinear operators, t is an independent variable, $u_i(t)$ are unknown functions. By means of generalizing the standard homotopy method, Liao construct the zeroth-order deformation equation as follows

$$(1-q)L_j[\phi_i(x,t,q) - u_{i,0}(x,t)] = q\hbar H(t) N_j[\phi_1(x,t,q), \phi_2(x,t,q), \cdots, \phi_m(x,t,q)], \tag{9}$$
$$i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n,$$

where $q \in [0,1]$ is an embedding parameter, L_j are linear operators, $u_{i,0}(x,t)$ are initial guesses of $u_i(x,t)$, $\phi_i(x,t;q)$ are unknown functions, $\hbar$ and $H(x,t)$ are auxiliary parameter and auxiliary function respectively. It is important to note that, one has great freedom to choose auxiliary objects such as $\hbar$ and L_j in HAM; This freedom plays an important role in establishing the keystone of validity and flexibility of HAM as shown in this paper. Obviously, when $q=0$ and $q=1$, both

$$\phi_i(x,t,0) = u_{i,0}(x,t) \quad \text{and} \quad \phi_i(x,t,1) = u_i(x,t), \quad i=1,2,\cdots,m, \tag{10}$$

hold. Thus as q increases from 0 to 1, the solutions of $\phi_i(x,t;q)$ change from the initial guesses $u_{i,0}(x,t)$ to the solutions $u_i(x,t)$. Expanding $\phi_i(x,t;q)$ in Taylor series with respect to q, one has

$$\phi_i(x,t,q) = u_{i,0}(x,t) + \sum_{k=1}^{+\infty} u_{i,k}(x,t)q^k, \quad i=1,2,\cdots,m, \tag{11}$$

where

$$u_{i,k}(x,t) = \frac{1}{k!}\frac{\partial^k \phi_i(x,t,q)}{\partial q^k}\Big|_{q=0}, \quad i=1,2,\cdots,m. \tag{12}$$

If the auxiliary linear operators, the initial guesses, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, then the series Eq. (11) converges at $q=1$, then one has

$$\phi_i(x,t,1) = u_{i,0}(x,t) + \sum_{k=1}^{+\infty} u_{i,k}(x,t), \quad i=1,2,\cdots,m, \tag{13}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao. Define the vectors

$$\vec{u}_{i,n}(t) = \{u_{i,0}(x,t), u_{i,1}(x,t), \cdots, u_{i,n}(x,t)\}, \quad i=1,2,\cdots,m. \tag{14}$$

Differentiating Eq. (9), k times with respect to the embedding parameter q and then setting $q=0$ and finally dividing them by $k!$, we have the so-called kth-order deformation equation

$$L_j[u_{i,k}(x,t) - \chi_k u_{i,k-1}(x,t)] = \hbar R_{j,k}(\vec{u}_{i,k-1}(x,t)), \quad i=1,2,\cdots,m; \; j=1,2,\cdots,n, \tag{15}$$

subject to the initial conditions $L_j(0)=0$, where

$$R_{j,k}(\vec{u}_{i,k-1}(x,t)) = \frac{1}{(k-1)!}\frac{\partial^{k-1} N_j[\phi_1(x,t,q), \phi_2(x,t,q), \cdots \phi_m(x,t,q)]}{\partial q^{k-1}}\Big|_{q=0} \tag{16}$$

and

$$\chi_{\mathrm{m}} = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \tag{17}$$

It should be emphasized that $u_{i,k}(x,t)$ is governed by the linear Eq. (15) and Eq. (16) with the linear boundary conditions that come from the original problem. These equations can be easily solved by symbolic computation softwares such as Maple and Mathematica.

4 Homotopy padé method

Traditionally the $[m,n]$ Padé for $u(x,t)$ is in the form

$$\frac{\sum_{k=0}^{m} F_k(x)t^k}{1+\sum_{k=1}^{n} F_{m+1+k}(x)t^k} \quad \text{or} \quad \frac{\sum_{k=0}^{m} G_k(t)x^k}{1+\sum_{k=1}^{n} G_{k+m+1}(t)x^k},$$

where $F_k(r)$ and $G_k(t)$ are functions.

In Homotopy Padé approximation, we employ the traditional Padé technique to the series Eq. (11) for the embedding parameter q to gain the $[m, n]$ Padé approximation in the form of

$$\frac{\sum_{k=0}^{m} w_k(x,t)q^k}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)q^k}, \tag{18}$$

where $w_k(t, x)$ is a function and for $i = 0, 1, \cdots, m, m + 2, \cdots, m + n + 1$, $w_i(x, t)$ is determined by product of the denominator of the above expression in the $\sum_{i=0}^{m+n} u_i(x,t)q^i$ and equating the powers of q^i, $i = 0, 1, \cdots, m+n$. Thus we have $m+n+1$ equations and $m+n+1$ unknowns $w_i(x, t)$, $i = 0, 1, \cdots, m, m+2, \cdots, m + n + 1$. By setting $q = 1$ in Eq. (18) the so-called $[m, n]$ Homotopy Padé approximation in the following form is yield.

$$\frac{\sum_{k=0}^{m} w_k(x,t)}{1+\sum_{k=1}^{n} w_{m+k+1}(x,t)}. \tag{19}$$

It is found that the $[m, n]$ Homotopy Padé approximation often converges faster than the corresponding traditional $[m, n]$ Padé approximation and in many cases the $[m, m]$ Homotopy Padé approximation is independent of the auxiliary parameter $\hbar$. In these cases, even if the corresponding solution series diverge, utilizing the Homotopy-Padé technique will result in a convergent series[8]. However, up to now, It has not seen a mathematical proof about it in general cases in literature [24].

The HPadéM can greatly enlarge the convergence region of the solution series. Besides, the results solutions of HPadéM often converge faster than solutions calculated by HAM.

5 Applications

In this section we apply the proposed methods to solve the Boussinesq system.

5.1 He's semi-inverse method

In order to seek a travelling wave solution of Boussinesq system, we introduce a transformation

$$u(x,t) = u(\xi), \quad v(x,t) = v(\xi), \quad xi = x - ct, \tag{20}$$

where c is arbitrary constant.

Substituting Eq. (20) into Eq. (1) yields

$$-cu' + v' = 0, \quad -cv' + a(u^2)' - bu''' = 0, \quad a, b \neq 0, \tag{21}$$

where the prime expresses the derivative with respect to ξ.

Integrating the resulting system, and neglecting the constant of integration we find

$$v = cu, \quad -cv + au^2 - bu'' = 0. \tag{22}$$

As a result we obtain the ODE

$$c^2u - au^2 + bu'' = 0. \tag{23}$$

By He's semi-inverse method[16], we can obtain at the following variational formulation

$$J = \int_0^{\infty} [\frac{1}{2}c^2u^2 - \frac{a}{3}u^3 - \frac{b}{2}(u')^2]d\xi. \tag{24}$$

By Ritz-like method, we search for a solitary wave solution in the form

$$u(\xi) = p\,\mathrm{sech}^2(q\xi), \tag{25}$$

where p and q are unknown constant to be further determined.

Substituting Eq. (25) into Eq. (24), we have

$$J = \int_0^\infty [\frac{c^2p^2}{2}\mathrm{sech}^4(q\xi) - \frac{ap^3}{3}\mathrm{sech}^6(q\xi) - 2bp^2q^2\mathrm{sech}^4(q\xi)\tanh^2(q\xi)]d\xi = \frac{c^2p^2}{3q} - \frac{8ap^3}{45q} - \frac{4bp^2q}{15}. \tag{26}$$

Making J stationary with p and q results in

$$\frac{\partial J}{\partial p} = \frac{2c^2p}{3q} - \frac{8ap^2}{15q} - \frac{8bpq}{15} = 0, \tag{27}$$

$$\frac{\partial J}{\partial q} = -\frac{c^2p^2}{3q^2} + \frac{8ap^3}{45q^2} - \frac{4bp^2}{15} = 0. \tag{28}$$

From Eq. (27) and Eq. (28), we get

$$p = \frac{3c^2}{2a}, \quad a \neq 0 \tag{29}$$

and

$$q = \frac{c}{2\sqrt{-b}}, \quad b < 0. \tag{30}$$

The solitary solution is, therefore, obtained as follows

$$u(x,t) = \frac{3c^2}{2a}\mathrm{sech}^2(\frac{c}{2\sqrt{-b}}(x-ct)), \quad v(x,t) = \frac{3c^3}{2a}\mathrm{sech}^2(\frac{c}{2\sqrt{-b}}(x-ct)), \tag{31}$$

where $a \neq 0$ and $b < 0$. Similarly, other types of solutions can be found.

5.2 Homotopy analysis method

Let us consider the system Eq. (1) with the initial conditions

$$u_0(x,t) = u(x,0) = \frac{3c^2}{2a}\mathrm{sech}^2(\frac{c}{2\sqrt{-b}}(x-ct)), \quad v_0(x,t) = v(x,0) = \frac{3c^3}{2a}\mathrm{sech}^2(\frac{c}{2\sqrt{-b}}(x-ct)). \tag{32}$$

The system Eq. (1) with the initial conditions Eq. (32) have the exact solutions Eq. (31). To solve system Eq. (1) by means of the HAM, we choose the auxiliary linear operators as follows:

$$L_1[\phi_1(x,t;q)] = \frac{\partial \phi_1(x,t;q)}{\partial t}, \quad L_2[\phi_2(x,t;q)] = \frac{\partial \phi_2(x,t;q)}{\partial t}$$

with the property $L_1[c_1] = 0, L_2[c_2] = 0$, where c_1 and c_2 are constants, and ϕ_1 and ϕ_2 are real functions. For simplicity, using equations in Eq. (1), we define the nonlinear operator as

$$N_1[\phi_1(x,t;q), \phi_2(x,t;q)] = \frac{\partial \phi_1(x,t;q)}{\partial t} + \frac{\partial \phi_2(x,t;q)}{\partial x},$$

$$N_2[\phi_1(x,t;q), \phi_2(x,t;q)] = \frac{\partial \phi_2(x,t;q)}{\partial t} + a\frac{\partial \phi_1^2(x,t;q)}{\partial x} - b\frac{\partial^3 \phi_1(x,t;q)}{\partial x^3}.$$

With the aid of the above definition, we construct the zeroth-order deformation equations

$$(1-q)L_1[\phi_1(x,t;q) - u_0(x,t)] = q\hbar_1 H_1(x,t)N_1[\phi_1(x,t;q), \phi_2(x,t;q)], \tag{33}$$

$$(1-q)L_2[\phi_2(x,t;q) - v_0(x,t)] = q\hbar_2 H_2(x,t)N_2[\phi_1(x,t;q), \phi_2(x,t;q)]. \tag{34}$$

Obviously, in case q = 0 in Eq. (33) and Eq. (34), we have

$$phi_1(x,t;0) = u_0(x,t), \quad phi_2(x,t;0) = v_0(x,t) \tag{35}$$

and with $q = 1$, we obtain

$$\phi_1(x,t;1) = u(x,t), \quad phi_2(x,t;1) = v(x,t). \tag{36}$$

Therefore, as the embedding parameter q increases from 0 to 1, $\phi_1(x,t;q)$ and $\phi_2(x,t;q)$ vary from the initial guess $u_0(x,t)$ and $v_0(x,t)$ to the solution $u(x,t)$ and $v(x,t)$, respectively. For simplicity, we suppose $\hbar_1 = \hbar_2 = \hbar$ and $H_2(x,t) = H_1(x,t) = 1$. Differentiating equations Eq. (33) and Eq. (34)m times with respect to the embedding parameter q, the mth-order deformation equations read

$$L_1[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_{m,1}(u_{m-1}, v_{m-1}), \tag{37}$$
$$L_2[v_m(x,t) - \chi_m v_{m-1}(x,t)] = \hbar R_{m,2}(u_{m-1}, v_{m-1}) \tag{38}$$

subject to initial conditions

$$u_m(x,0) = 0, \quad v_m(x,0) = 0, \tag{39}$$

where

$$R_{m,1}(u_{m-1}, v_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \frac{\partial v_{m-1}}{\partial x}, \tag{40}$$
$$R_{m,2}(u_{m-1}, v_{m-1}) = \frac{\partial v_{m-1}}{\partial t} + 2a\sum_{j=0}^{m-1}(u_j\frac{\partial u_{m-1-j}}{\partial x}) - b\frac{\partial^3 u_{m-1}}{\partial x^3} \tag{41}$$

and χ_m is defined by Eq. (17). The solutions of the mth-order deformation equations Eq. (37) and Eq. (38) for $m \geq 1$ become

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L_1^{-1}(R_{m,1}(u_{m-1}, v_{m-1})), \tag{42}$$
$$v_m(x,t) = \chi_m v_{m-1}(x,t) + \hbar L_2^{-1}(R_{m,2}(u_{m-1}, v_{m-1})). \tag{43}$$

Now, from Eq. (42), Eq. (43) and Eq. (32), we can successively obtain

$$u_1(x,t) = -\frac{3\hbar c^4 t}{2a\sqrt{-b}}\frac{\sinh(\frac{c}{2\sqrt{-b}}x)}{\cosh^3(\frac{c}{2\sqrt{-b}}x)}, \quad v_1(x,t) = -\frac{3\hbar c^5 t}{2a\sqrt{-b}}\frac{\sinh(\frac{c}{2\sqrt{-b}}x)}{\cosh^3(\frac{c}{2\sqrt{-b}}x)},$$

$$u_2(x,t) = \frac{3\hbar c^4 t}{8a\sqrt{(-b)^3}}(\frac{4b\sinh(\frac{c}{2\sqrt{-b}}x)}{\cosh^3(\frac{c}{2\sqrt{-b}}x)}) - \frac{3\hbar c^2\sqrt{-b}t}{\cosh^4(\frac{c}{2\sqrt{-b}}x)} + \frac{2\hbar c^2\sqrt{-b}t}{\cosh^2(\frac{c}{2\sqrt{-b}}x)} + \frac{4\hbar b\sinh(\frac{c}{2\sqrt{-b}}x)}{\cosh^3(\frac{c}{2\sqrt{-b}}x)}),$$
$$v_2(x,t) = \frac{3\hbar c^5 t}{8a\sqrt{(-b)^3}}(\frac{4b\sinh(\frac{c}{2\sqrt{-b}}x)}{\cosh^3(\frac{c}{2\sqrt{-b}}x)}) - \frac{3\hbar c^2\sqrt{-b}t}{\cosh^4(\frac{c}{2\sqrt{-b}}x)} + \frac{2\hbar c^2\sqrt{-b}t}{\cosh^2(\frac{c}{2\sqrt{-b}}x)} + \frac{4\hbar b\sinh(\frac{c}{2\sqrt{-b}}x)}{\cosh^3(\frac{c}{2\sqrt{-b}}x)}),$$

and so on. Then the series solutions obtained by the HAM can be written in the form

$$u(x,t) = \quad = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots, \tag{44}$$

and

$$v(x,t) = \quad = v_0(x,t) + v_1(x,t) + v_2(x,t) + \cdots. \tag{45}$$

6 Convergence of ham solution

Theorem 1. *If the series*

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \text{ and } v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t),$$

converge, where $u_m(x,t)$ and $v_m(x,t)$ are governed by the Eq. (42) and (43) under the definitions Eq. (40) and Eq. (41), then they must be the exact solutions of Eq. (1) with initial conditions Eq. (32).

Proof. If the series Eq. (44) and Eq. (44) are convergent, then we can write

$$S_1 = \sum_{m=0}^{\infty} u_m, \quad S_2 = \sum_{m=0}^{\infty} v_m,$$

Thus,

$$\lim_{n\to+\infty} u_n = 0, \quad \lim_{n\to+\infty} v_n = 0,$$

are hold. Due to Eq. (33), Eq. (37) and Eq. (38), we have

$$\begin{aligned}
\hbar \sum_{m=1}^{\infty}(R_{m,1}(u_{m-1}, v_{m-1})) &= \lim_{n\to+\infty} \sum_{m=1}^{n} L_1[u_m(x,t) - \chi_m u_{m-1}(x,t)] \\
&= L_1(\lim_{n\to+\infty} \sum_{m=1}^{n} [u_m(x,t) - \chi_m u_{m-1}(x,t)]) \\
&= L_1(\lim_{n\to+\infty} u_n(x,t)) = 0,
\end{aligned}$$

and

$$\begin{aligned}
\hbar \sum_{m=1}^{\infty}(R_{m,2}(u_{m-1}, v_{m-1})) &= \lim_{n\to+\infty} \sum_{m=1}^{n} L_2[v_m(x,t) - \chi_m v_{m-1}(x,t)] \\
&= L_2(\lim_{n\to+\infty} \sum_{m=1}^{n} [v_m(x,t) - \chi_m v_{m-1}(x,t)]) \\
&= L_2(\lim_{n\to+\infty} v_n(x,t)) = 0,
\end{aligned}$$

Since $\hbar \neq 0$, we arrive at

$$\sum_{m=1}^{\infty}(R_{m,1}(u_{m-1}, v_{m-1})) = 0,$$

and

$$\sum_{m=1}^{\infty}(R_{m,2}(u_{m-1}, v_{m-1})) = 0,$$

Substituting Eq. (40) and Eq. (41) in the above expressions, we have

$$\sum_{m=1}^{\infty}(R_{m,1}(u_{m-1}, v_{m-1})) = \sum_{m=1}^{\infty}(\frac{\partial u_{m-1}}{\partial t} + \frac{\partial v_{m-1}}{\partial x}) = \sum_{m=1}^{\infty}\frac{\partial u_{m-1}}{\partial t} + \sum_{m=1}^{\infty}\frac{\partial v_{m-1}}{\partial x} = (S_1)_t + (S_2)_x = 0,$$

and

$$\begin{aligned}\sum_{m=1}^{\infty}(R_{m,2}(u_{m-1},v_{m-1})) &= \sum_{m=1}^{\infty}(\frac{\partial v_{m-1}}{\partial t}+2a\sum_{j=0}^{m-1}(u_j\frac{\partial u_{m-1-j}}{\partial x})-b\frac{\partial^3 u_{m-1}}{\partial x^3})\\ &= \sum_{m=1}^{\infty}\frac{\partial v_{m-1}}{\partial t}+2a\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}(u_j\frac{\partial u_{m-1-j}}{\partial x})-b\sum_{m=1}^{\infty}\frac{\partial^3 u_{m-1}}{\partial x^3}\\ &= \sum_{m=1}^{\infty}\frac{\partial v_{m-1}}{\partial t}+2a\sum_{j=0}^{\infty}\sum_{m=j+1}^{\infty}(u_j\frac{\partial u_{m-1-j}}{\partial x})-b\sum_{m=1}^{\infty}\frac{\partial^3 u_{m-1}}{\partial x^3}\\ &= \sum_{m=1}^{\infty}\frac{\partial v_{m-1}}{\partial t}+2a\sum_{j=0}^{\infty}u_j\sum_{i=0}^{\infty}\frac{\partial u_i}{\partial x}-b\sum_{m=1}^{\infty}\frac{\partial^3 u_{m-1}}{\partial x^3}\\ &= (S_2)_t+a(S_1^2)_x-b(S_1)_{xxx}=0.\end{aligned}$$

Moreover, due to initial conditions Eq. (39) and Eq. (32), we get

$$S_1(x,0)=\sum_{m=0}^{\infty}u_m(x,0)=u_0(x,0)=u(x,0),$$

and

$$S_2(x,0)=\sum_{m=0}^{\infty}v_m(x,0)=v_0(x,0)=v(x,0).$$

Therefore, $S_1(x,t)$ and $S_2(x,t)$ satisfy Eq. (1) and Eq. (32), and they are the exact solutions of Eq. (1) with initial conditions Eq. (32).

The auxiliary-parameter $\hbar$ can be regarded as an iteration factor that is widely used in numerical computations. It is well known that a properly chosen iteration factor can ensure the convergence of iteration[24, 25]. Similarly, it is found that the convergence of the homotopy-series like Eq. (11) is dependent upon the value of $\hbar$: one can ensure the convergence of the homotopy-series solution simply by means of choosing a proper value of $\hbar$. In fact, it is the auxiliary-parameter $\hbar$ that provides us, for the first time, a simple way to ensure the convergence of series solution. Due to this reason, it seems reasonable to rename $\hbar$ the convergence-control parameter[25]; more information is like to find in this reference. In general, by means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. To influence of $\hbar$ on the convergence of solution, we plot the so-called $\hbar$-curve of $u(-2.5, 0.7)$ by 8th-order approximation of solution, as shown in Fig. 1. It is easy to discover that $-1.2 < \hbar < -0.6$ is the valid region of $\hbar$. It is easy to see that in order to have a good approximation, $\hbar$ has to be chosen in $-1.2 < \hbar < -0.6$. This means that for these values of $\hbar$ the series Eq. (44) and Eq. (45) converge to the exact solution Eq. (1).

7 Result and discussion

We declare the results for 8th order HAM approximations in Tab. 1 and Tab. 2. The results obtained with $\hbar = -0.962$ and $\hbar = -0.983$ are better than $\hbar = -1$. Hence, the outputs of HAM are better than the HPM. Moreover the absolute error for u is drawn in Fig. 2

The evolution results for the approximate solutions obtained by HPM and HAM, and the exact solutions of Eq. (1) are given in Fig. 7 (a)-(c), respectively. In addition, we give the numerical results for the approximate solutions obtained by HPM and HAM, and the exact solutions of the Boussinesq system Eq. (1) with $t = 0.7$ in Fig. 4. From the numerical results of Fig. 7 and Fig. 4, we can easily conclude that both methods present remarkable accuracy for the approximate solutions of Eq. (1). It is important to note that the accuracy can be further improved by considering more terms of the series Eq. (27). In Tab.3, the absolute errors of approximation results are given with $[4, 4]$ HPadéM. It is shown the HPadéM accelerate the convergence of the related series.

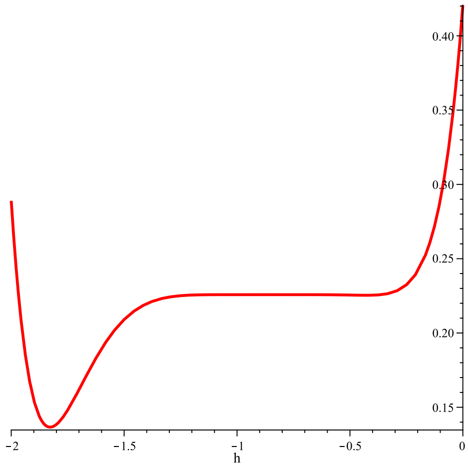

Fig. 1. The $\hbar$ curve of Boussinesq system for u(-2.5,0.7) obtained from the 8th order HAM.

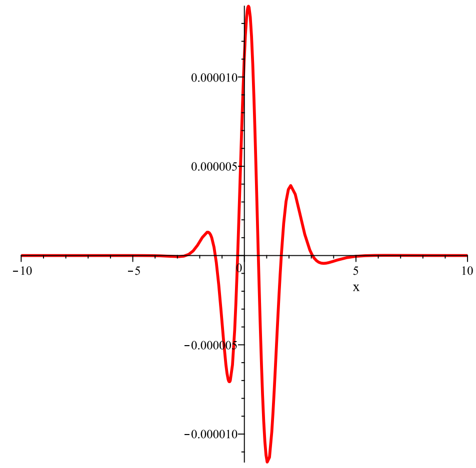

Fig. 2. Absolute error curve of Boussinesq system with 8th order HAM.

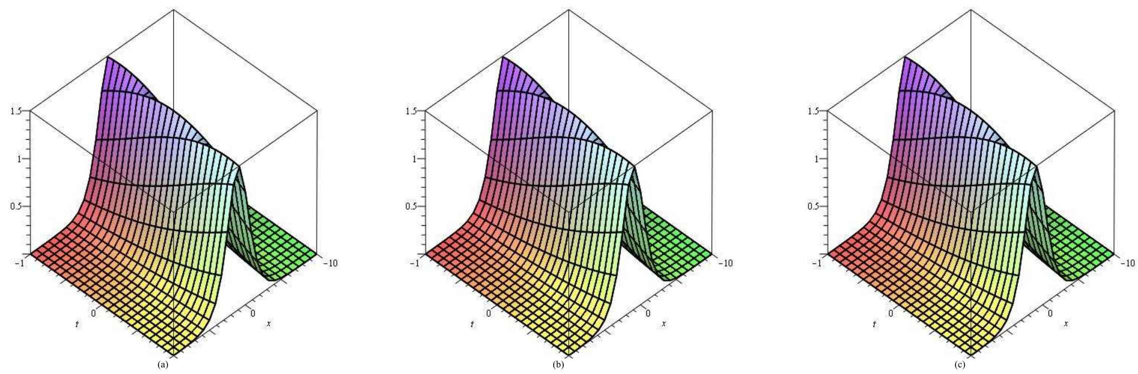

Fig. 3. The evolution results for the Boussinesq system with $a = c = 1$ and $b = -1$: (a) Exact solution (He's semi-inverse solution), (b) HPM, (c) HAM.

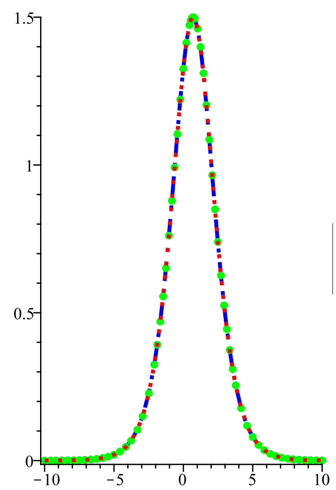

Fig. 4. The numerical results for the Boussinesq system with t = 0.7, $a = c = 1$ and $b = -1$: Exact solution (He's semi-inverse solution)(red), HPM(green), HAM (blue).

8 Conclusion

We established variational formulations for Boussinesq system by He's semi-inverse method. We have also given the comparison of results with homotopy perturbation method, homotopy analysis method and

Table 1. Absolute error of Boussinesq system for 8th order HAM with $\hbar = -0.962$, $a = c = 1$ and $b = -1$.

x		t=0.1	t=0.4	t=0.7	t=1
$\|u_a - u_e\|$	-10	$5.33E-13$	$4.49E-10$	$6.38E-08$	$6.53E-07$
	-7.5	$6.96E-12$	$7.43E-09$	$7.22E-07$	$7.57E-06$
	-5.0	$1.59E-10$	$3.49E-07$	$1.19E-06$	$4.05E-05$
	-2.5	$2.05E-08$	$6.35E-06$	$3.01E-04$	$2.15E-03$
$\|v_a - v_e\|$	-10	$5.33E-13$	$4.49E-10$	$6.38E-08$	$6.53E-07$
	-7.5	$6.96E-12$	$7.43E-09$	$7.22E-07$	$7.57E-06$
	-5.0	$1.59E-10$	$3.49E-07$	$1.19E-06$	$4.05E-05$
	-2.5	$2.05E-08$	$6.35E-06$	$3.01E-04$	$2.15E-03$

Table 2. Absolute error of Boussinesq system for 8th order HPM with $a = c = 1$ and $b = -1$.

x		t=0.1	t=0.4	t=0.7	t=1
$\|u_a - u_e\|$	-10	$2.23E-11$	$2.17E-08$	$3.40E-07$	$1.93E-06$
	-7.5	$2.63E-10$	$2.57E-07$	$4.02E-06$	$2.29E-05$
	-5.0	$2.01E-09$	$2.01E-06$	$3.23E-05$	$1.87E-04$
	-2.5	$4.19E-08$	$4.05E-05$	$6.25E-04$	$3.49E-03$
$\|v_a - v_e\|$	-10	$2.23E-11$	$2.17E-08$	$3.40E-07$	$1.93E-06$
	-7.5	$2.63E-10$	$2.57E-07$	$4.02E-06$	$2.29E-05$
	-5.0	$2.01E-09$	$2.01E-06$	$3.23E-05$	$1.87E-04$
	-2.5	$4.19E-08$	$4.05E-05$	$6.25E-04$	$3.49E-03$

Table 3. Absolute error of Boussinesq system for [4,4] HPad$\acute{e}$M with $a = c = 1$ and $b = -1$.

x		t=0.1	t=0.4	t=0.7	t=1
$\|u_a - u_e\|$	-10	$9.70E-21$	$1.89E-15$	$2.18E-13$	$4.06E-12$
	-7.5	$1.17E-19$	$2.29E-14$	$2.64E-12$	$4.92E-11$
	-5.0	$3.73E-19$	$8.34E-14$	$1.07E-11$	$2.18E-10$
	-2.5	$4.87E-17$	$9.84E-12$	$1.14E-09$	$2.11E-08$
$\|v_a - v_e\|$	-10	$9.70E-21$	$1.89E-15$	$2.21E-13$	$4.06E-12$
	-7.5	$1.17E-19$	$2.29E-14$	$2.64E-12$	$4.92E-11$
	-5.0	$3.73E-19$	$8.34E-14$	$1.07E-11$	$2.18E-10$
	-2.5	$4.87E-17$	$9.84E-12$	$1.14E-09$	$2.11E-08$

homotopy pad$\acute{e}$ Method. It is obvious that the He's semi-inverse method is useful and manageable and remarkably simple to find various kinds of solitary solutions of Boussinesq system.

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