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Bifurcation and Stability Analysis of a Prey-predator System with a Reserved Area

Debasis Mukherjee*

Department of Mathematics, Vivekananda College, Thakurpukur, Kolkata 700063, India

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Abstract. This paper analyzes a prey-predator system with a reserved area. The predator functional response is taken to be of Holling type II. It is assumed that the habitat is divided into two zones, namely free zone and the other is reserved zone where predation is prohibited. The local and global stability analysis have been carried out. When the carrying capacity of the environment crosses a critical value, we determine that strictly positive equilibrium enters into Hopf bifurcation. We obtain conditions which influence the persistence of all the populations. Numerical simulation with a hypothetical set of data has been done to support the analytical findings.

Keywords: Prey-predator, reserve zone, stability, Hopf bifurcation, persistence

1 Introduction

Dynamics of interacting biological species has been studied in the past few decades from various angles. Many species become extinct and many others are at the verge of extinction due to several reasons like, over exploitation, over predation, environmental pollution, mismanagement of natural resources etc. To save these species, suitable measures such as restriction on harvesting, creating reserve zones/refuges etc. should be taken so that the species grow in these regions without any external disturbances. The role of reserve zones/refuges in prey-predator dynamics creates a major interest to researchers[2–4, 6, 8–11, 13, 15, 16]. Yakubu [15] shows that the presence of refuge can influences stable coexistence of all species. Krivan [10] investigated the dynamics of predator-prey ecosystem in presence of refuge using Lotka-Volterra time continuous models and remarked that low refuge carrying capacity leads to stability of predator-prey dynamics and stability is lost when the carrying capacity is high. Kar [9] analyzed a predator -prey model incorporating a prey refuge and independent harvesting on either species. He observed that harvesting can destroy cyclic behavior of the system. Gao and Liang [7] studied permanence on a competition system with a refuge. Recently, Dubey [3] investigated the role of reserve zone on the dynamics of predator-prey system with Holling type I predator functional response. He concluded that the positive equilibrium, whenever exists, is always globally stable which indicates that reserve zone has a stabilizing effect on the predator-prey dynamics. Zhang et al. [16] studied local and global stability of a prey-predator fishery model with prey reserve and discussed optimal harvesting policy.

The main thrust of this paper is to study the dynamical behavior of a prey-predator system with a reserved area where the predator functional response is assumed to be of Holling type II.

The paper is structured in the following manner. In the next section, we present the model. In Section 3, the local stability analysis of boundary and interior equilibrium point is given.Global stability condition is derived in the same Section. The criterion for existence of Hopf bifurcation about interior equilibrium point is discussed in Section 4. Persistence condition is developed in Section 5. The numerical examples are given in Section 6. Finally, Section 7 concludes the paper with a brief discussion.

* *E-mail address*: debasis_mukherjee2000@yahoo.co.in.

2 The mathematical model

We consider a habitat where prey and predator species are living together. There are two zones in the habitat, namely reserved and unreserved zones. The predator species cannot enter into the reserved zone. But the prey species can move from reserved to unreserved zone and vice-versa.

Now we state our model equations :

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= r_1x\left(1-\frac{x}{k_1}\right)-m_1x+m_2y-\frac{c_1xz}{1+x},\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= r_2y\left(1-\frac{y}{k_2}\right)-m_1x+m_2y,\\
\frac{\mathrm{d}z}{\mathrm{d}t} &= z\left(\frac{c_2x}{1+x}-d\right),\\
x(0)&\geq 0, y(0)\geq 0, z(0)\geq 0.
\end{aligned} \tag{1}$$

Here $x(t)$ is the density of prey species inside the unreserved area at time t. $y(t)$ is the density of prey species in the reserved area at time t where predation is not permitted. $z(t)$ denotes the predator density at time t. All the parameters are assumed to be positive. r_1 and r_2 are the intrinsic growth rates of prey species inside the unreserved and reserved area, respectively. k_1 and k_2 are their respective carrying capacities. d, c_1 and c_2 are the mortality rate, capturing rate and conversion rate of predators, respectively. m_1 and m_2 are migration rates from the unreserved area to the reserved area and the reserved area to the unreserved area, respectively.

3 Boundedness and dynamical behavior

In this section, we shall show that , all the solutions of system Eq. (1) are bounded in a positive orthant R^3_+.The boundedness of system Eq. (1) is given by the following lemma.

Lemma 1. *All the solutions of system Eq. (1) will be in the region* $B=\{(x,y,z)\in R^3_+ : 0\leq x+y+z\leq r/\lambda\}$ *as* $t\to\infty$ *for all positive initial values* $(x(0),y(0),z(0))\in R^3_+$*, where* $r=\frac{k_1}{4r_1}(r_1+\lambda)^2+\frac{k_2}{4r_2}(r_2+\lambda)^2$ *and* $0<\lambda<d$.

Proof. Proof is trivial and can be deleted.

We discuss the stability of boundary equilibrium points. The system Eq. (1) has the following boundary equilibrium points: $E_1(0,0,0)$, $E_2(\overline{x},\overline{y},0)$. *Clearly* E_1 *always exists and for* E_2,

$\overline{y}=\frac{1}{m_2}\{\frac{r_1\overline{x}^2}{k_1}-(r_1-m_1)\overline{x}\}$ *and* $\overline{x}$ *is the positive root of the following equation*

$$a_0x^3+a_1x^2+a_2x+a_3=0, \tag{2}$$

where

$$\begin{aligned}
a_0&=\frac{r_1^2r_2}{m_2^2k_1^2k_2}, \quad a_1=-\frac{2r_1r_2(r_1-m_1)}{k_1k_2m_2^2},\\
a_2&=\frac{r_2(r_1-m_1)}{k_2m_2^2}-\frac{r_1(r_2-m_2)}{m_2k_1}, \quad a_3=\frac{(r_1-m_1)(r_2-m_2)}{m_2}-m_1,
\end{aligned}$$

Eq. (2) has a positive root if $(r_1-m_1)(r_2-m_2)<m_1m_2$. *For* $\overline{y}$ *to be positive we must have* $\overline{x}>\frac{k_1}{r_1}(r_1-m_1)$.

Theorem 1. *(i) If* $r_1+r_2<m_1+m_2$ *then* E_1 *is stable and unstable otherwise.*

(ii) E_2 *is stable or unstable according as* $\frac{c_2\overline{x}}{1+\overline{x}}<>d$.

Next we are interested about the existence of the interior equilibrium point of system Eq. (1) which is given by $E(x^*,y^*,z^*)$, *where*

$$x^* = \frac{d}{c_2 - d}, \quad y^* = \frac{k_2(r_2 - m_2) + \sqrt{k_2^2(r_2 - m_2)^2 + 4k_2 m_1 x^* r_2}}{2r_2},$$
$$z^* = \frac{1 + x^*}{c_1 x^*}[r_1 x^*(1 - \frac{x^*}{k_1}) - m_1 x^* + m_2 y^*].$$

Clearly, the interior equilibrium E is feasible

$$if\ c_2 > d\ and\ m_2 y^* + \frac{d}{c_2 - d}(r_1 - m_1) > \frac{r_1 d^2}{k_1(c_2 - d)^2}. \tag{3}$$

Eq. (3) gives a threshold value of the carrying capacity of the free access zone for survival of predators.

Theorem 2. *Suppose Eq. (3) holds. Then E is locally asymptotically stable provided the following conditions are satisfied: $p_1 p_2 - p_3 > 0$ and $p_1, p_2 > 0$. where*

$$p_1 = -r_1 + \frac{2r_1 x^*}{k_1} + m_1 + \frac{c_1 z^*}{(1 + x^*)^2} + (m_1 x^*)/y^* + \frac{r_2 y^*}{k_2},$$
$$p_2 = -\left(r_1 - \frac{2r_1 x^*}{k_1} - m_1 - \frac{c_1 z^*}{(1 + x^*)^2}\right)\left(\frac{m_1 x^*}{y^*} + \frac{r_2 y^*}{k_2}\right) - m_1 m_2 + \frac{c_2 c_1 z^* x^*}{(1 + x^*)^3},$$
$$p_3 = \left(\frac{m_1 x^*}{y^*} + \frac{r_2 y^*}{k_2}\right)\frac{c_2 c_1 z^* x^*}{(1 + x^*)^3}. \tag{4}$$

Proof. The characteristic equation about E is given by

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0. \tag{5}$$

The result follows by the application of Routh Hurwitz criterion. We now give another result on local stability in the following theorem.

Theorem 3. *Suppose Eq. (3) holds. Further suppose that $\frac{r_1}{k_1} \geq \frac{c_1 z^*}{(1+x^*)^2}$. Then E is locally asymptotically stable.*

Proof. Consider the following positive definite function about E

$$V(t) = p_1\left(x - x^* - x^* \ln \frac{x}{x^*}\right) + p_2\left(y - y^* - y^* \ln \frac{y}{y^*}\right) + p_3\left(z - z^* - z^* \ln \frac{z}{z^*}\right), \tag{6}$$

where p_1, p_2 and p_3 are positive constants to be chosen later. Differentiating V with respect to t along the solutions of system Eq. (1), we get

$$\frac{dV}{dt} = p_1(x - x^*)\left\{r_1\left(1 - \frac{x}{k_1}\right) - m_1 + \frac{m_2 y}{x} - \frac{c_1 z}{1 + x}\right\} + p_2(y - y^*)\left\{r_2\left(1 - \frac{y}{k_2}\right) + \frac{m_1 x}{y} - m_2\right\}$$
$$+ p_3(z - z^*)\left(\frac{c_2 x}{1 + x} - d\right).$$

We expand $\dot{V}$ about E and obtain

$$\dot{V} = -(x - x^*)^2 p_1\left\{\frac{r_1}{k_1} + \frac{m_2 y^*}{x^{*2}} - \frac{c_1 z^*}{(1 + x^*)^2}\right\}$$
$$+ (x - x^*)(y - y^*)\left(\frac{m_1 p_1}{y^*} + \frac{m_2 p_2}{x^*}\right) - (y - y^*)^2 p_2\left(\frac{r_2}{k_2} + \frac{m_1 x^*}{y^{*2}}\right)$$
$$+ \frac{(x - x^*)(z - z^*)}{1 + x^*}\left\{-p_1 c_1 + \frac{c_2 p_3}{1 + x^*}\right\} + H.O.T,$$

where $H.O.T.$ stands for terms that are cubic or higher order.

Choosing $p_3 = \frac{p_1 c_1(1+x^*)}{c_2}$ we note that $\dot{V}$ is negative definite if

$$4p_1p_2\left\{\frac{r_1}{k_1}+\frac{m_2y^*}{x^{*2}}-\frac{c_1z^*}{(1+x^*)^2}\right\}\left(\frac{r_2}{k_2}+\frac{m_1x^*}{y^{*2}}\right)>\left(\frac{m_1p_1}{y^*}+\frac{m_2p_2}{x^*}\right)^2.$$

It may be noted that if we choose $p_1=\frac{m_2}{x^*}$, $p_2=\frac{m_1}{y^*}$. *Then the above condition becomes*

$$\left\{\frac{r_1}{k_1}+\frac{m_2y^*}{x^{*2}}-\frac{c_1z^*}{(1+x^*)^2}\right\}\left(\frac{r_2}{k_2}+\frac{m_1x^*}{y^{*2}}\right)>\frac{m_1m_2}{x^*y^*}. \tag{7}$$

If $\frac{r_1}{k_1}\geq\frac{c_1z^*}{(1+x^*)^2}$ *then Eq. (7) holds. This shows that* V *is a Lyapunov function and hence the theorem follows.*

In the following theorem, we state condition for which E is globally asymptotically stable.

Theorem 4. *If* $\frac{r_1}{k_1}\geq\frac{c_1z^*}{1+x^*}$ *then the interior equilibrium* E *is globally asymptotically stable with respect to all solutions initiating in the interior of the positive orthant.*

Proof. Consider the following positive definite function about E

$$V(t)=\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+q_1\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+q_2\left(z-z^*-z^*\ln\frac{z}{z^*}\right),$$

where q_1 *and* q_2 *are positive constants to be chosen later. Differentiating* V *with respect to* t *along the solutions of system Eq. (1), we get*

$$\begin{aligned}\frac{dV}{dt}&=(x-x^*)\frac{\dot{x}}{x}+q_1(y-y^*)\frac{\dot{y}}{y}+q_2(z-z^*)\frac{\dot{z}}{z}\\&=(x-x^*)\left\{-\frac{r_1}{k_1}(x-x^*)+\frac{m_2(yx^*-xy^*)}{xx^*}+\frac{c_1(1+x^*)(z^*-z)+c_1z^*(x-x^*)}{(1+x)(1+x^*)}\right\}\\&\quad+q_1(y-y^*)\left\{-\frac{r_2}{k_2}(y-y^*)+\frac{m_1(xy^*-yx^*)}{yy^*}\right\}+\frac{q_2c_2(z-z^*)(x-x^*)}{(1+x)(1+x^*)}.\end{aligned}$$

Choosing $q_1=\frac{y^*m_2}{x^*m_1}$, $q_2=\frac{c_1(1+x^*)}{c_2}$, $\frac{dV}{dt}$ *can further be written as*

$$\frac{dV}{dt}=-\left(\frac{r_1}{k_1}-\frac{c_1z^*}{(1+x^*)}\right)(x-x^*)^2-\frac{y^*m_2r_2}{x^*m_1k_2}(y-y^*)^2-\frac{m_2}{xyx^*}(x^*y-xy^*)^2,$$

which is negative definite by the assumption of the theorem. Hence V *is a Lyapunov function [11] with respect to* E *whose domain of attraction is* B*, proving the theorem.*

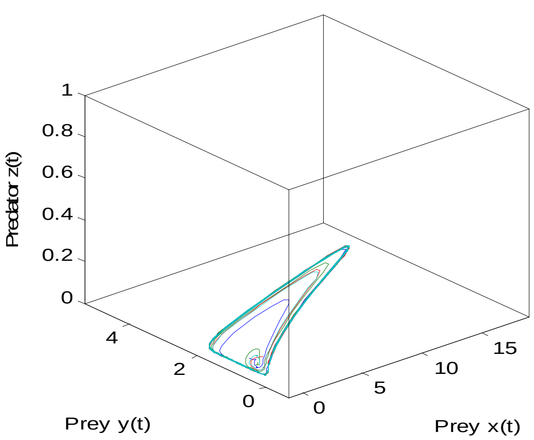

Fig. 1. Here $r_1=4, k_1=30, c_1=146/15, r_2=2, k_2=2, c_2=2, d=1, m_1=1, m_2=2$

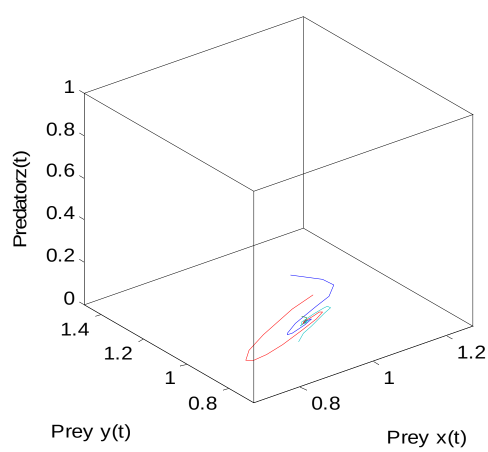

Fig. 2. Here $r_1=1, k_1=2, c_1=3, r_2=2, k_2=2, c_2=2, d=1, m_1=1, m_2=2$

4 Hopf bifurcation around positive equilibrium

In this section we present Hopf bifurcation of nontrivial periodic solution for system Eq. (1).

Now we enquire whether Hopf bifurcation occurs or not for system Eq. (1). Considering k_1 as a bifurcation parameter, in the following we will show Hopf bifurcation occurs for system Eq. (1) at a critical value $k_1 = k_{1c}$.

Theorem 5. *Suppose that Eq. (3) holds. Then system Eq. (1) undergoes a Hopf bifurcation when k_1 passes through k_{1c}*

Proof. Hopf bifurcation will occur if and only if there exists $k_1 = k_{1c}$ such that (i) $p_1(k_1)p_2(k_1) = p_3(k_1)$ with $p_i(k_1) > 0$, $i = 1, 2, 3$ and $\frac{dRe\lambda(k_1)}{dk_1} \neq 0$ at $k_1 = k_{1c}$[13]. Now when $k_1 = k_{1c}$, $p_1p_2 = p_3 =$ with $p_i > 0$, the characteristic equation becomes $(\lambda^2 + p_2)(\lambda + p_1) = 0$ with roots $\lambda_1 = i\sqrt{p_2}$, $\lambda_2 = -i\sqrt{p_2}$, and $\lambda_3 = -p_1$, so there are purely imaginary eigenvalues and a strictly negative eigenvalue... For k_1 in a neighbourhood of k_{1c} the roots have the form $\lambda_1(k_1) = b_1(k_1) + ib_2(k_1)$, $\lambda_2(k_1) = b_1(k_1) - ib_2(k_1)$, $\lambda_3(k_1) = -b_3(k_1)$ where $b_i(k_1), i = 1, 2, 3$ are real. In view of the above roots, the corresponding characteristic equation will be

$$\lambda^3 + (b_3 - 2b_1)\lambda^2 + (b_1^2 + b_2^2 - 2b_1b_3)\lambda + b_3(b_1^2 + b_2^2) = 0. \tag{8}$$

Now comparing this with the equation in Eq. (5), we get

$$p_1 = b_3 - 2b_1, \quad p_2 = b_1^2 + b_2^2 - 2b_1b_3, \quad p_3 = b_3. \quad (b_1^2 + b_2^2). \tag{9}$$

We know that $b_1 = b_1(k_1)$ and $b_1(k_1c) = 0$. From Eq. (9) we find that

$$(p_1 + 2b_1)p_2 = p_3 - 2b_1(p_1 + 2b_1)^2. \tag{10}$$

We can differentiate this with respect to k_1, using the Eq. (5) for k_1 dependence of p_1, p_2 and p_3, and calculate $\frac{db_1}{dk_1}$ at $k_1 = k_1c$.

$$\left(\frac{dp_1}{dk_1} + \frac{2db_1}{dk_1}\right)p_2 + (p_1 + 2b_1)\frac{dp_2}{dk_1} = \frac{dp_3}{dk_1} - 2\frac{db_1}{dk_1}(p_1 + 2b_1)^2 - 2b_1\frac{d(p_1 + 2b_1)^2}{dk_1}.$$

Considering $k_1 = k_1c$, we get

$$\begin{aligned}\frac{db_1}{dk_1} = &\left\{\frac{r_1x^*(2 + x^*)p_2}{k_1^2(1 + x^*)} + p_1\frac{r_1x^*(2 + x^*)}{k_1^2(1 + x^*)}\left(\frac{m_1x^*}{y^*} + \frac{r_2y^*}{k_2}\right)\right.\\ &\left.+\left(r_1 - \frac{2r_1x^*}{k_1} - m_1 - \frac{c_1z^*}{(1 + x^*)^2}\right)\left(\frac{c_2{x^*}^2r_1}{(1 + x^*)^2k_1^2}\right)\right\}\frac{1}{2(p_1^2 + p_2}) > 0.\end{aligned}$$

at $k_1 = k_1c$ since $p_1, p_2 > 0$ and $2p_1^2 + p_2 > 0$. So $\frac{d}{dk_1}(p_1p_2 - p_3) < 0$ at $k_1 = k_1c$. Hence interior equilibrium point is stable when $k_1 < k_1c$ and unstable when $k_1 > k_1c$. Thus Hopf bifurcation occurs.

Condition of Theorem 5 indicates that exchange of stability is possible when carrying capacity of the prey species in the unreserved zone crosses a critical value.

5 Persistence

Biologically, persistence means the long term survival of all populations no matter what the initial populations are. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of the non-negative cone. Now we state a result guaranteeing the uniform persistence of system Eq. (1).

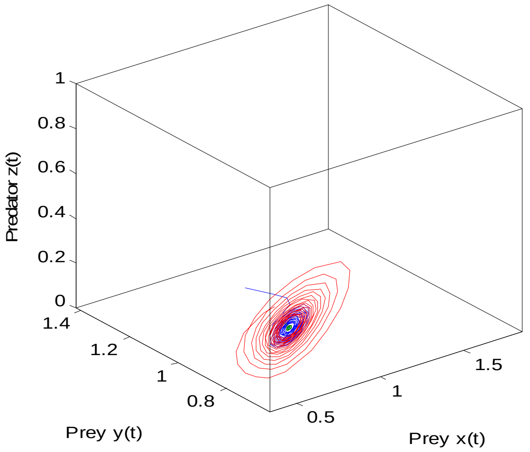

Fig. 3. Here $r_1 = 85/22, c_1 = 90/11, r_2 = 2, k_2 = 2, c_2 = 2, d = 1, m_1 = 1, m_2 = 2, k_1 = 5$

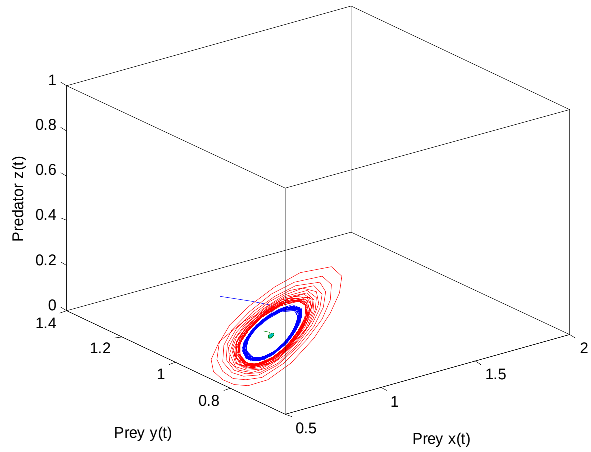

Fig. 4. Limit cycle behavior of system Eq. (1) observed with $k_1 = 5.2$ and other parameters values same as in Fig. 3

Theorem 6. *Suppose E_2 exist. Further assume that $d < \frac{c_2\overline{x}}{1+\overline{x}}$. Then system Eq. (1) is uniformly persistent.*

Proof. *Suppose that α is a point in the positive octant and $o(\alpha)$ is the orbit through α and Ω is the omega limit set of the orbit through α. Note that $\Omega(\alpha)$ is bounded. We claim that E_1 is not a member of $\Omega(\alpha)$. If $E_1 \in \Omega(\alpha)$, then by Butler-McGehee Lemma (Freedman and Waltman [5]), there exists a point P in $\Omega(\alpha) \cap W^s(E_1)$)(which denotes the stable manifold of E_1). Since $o(P)$ lies in $\Omega(\alpha)$ and $W^s(E_1)$ is the z axis and hence unbounded orbit lies in $\Omega(\alpha)$, a contradiction.*

Next we show that E_2 is not a member of $\Omega(\alpha)$, the condition $d < \frac{c_2\overline{x}}{1+\overline{x}}$ implies that E_2 is a saddle point. $W^s(E_2)$ is the $x - y$ plane and hence orbits in the plane emanate either from E_1 or unbounded orbit lies in $\Omega(\alpha)$, once more a contradiction.

There does not exist any equilibrium point in the $x - z$ plane. The orbits in this plane are unbounded $W^s(\Omega(x - z) \bigcap \Omega(\alpha)) = \Phi$. Thus, $\Omega(\alpha)$ does not intersect any of the coordinate planes and hence system Eq. (1) is persistent. Since Eq. (1) is bounded, by main theorem in Butler et al. [1], this implies that the system is uniformly persistent (permanent).

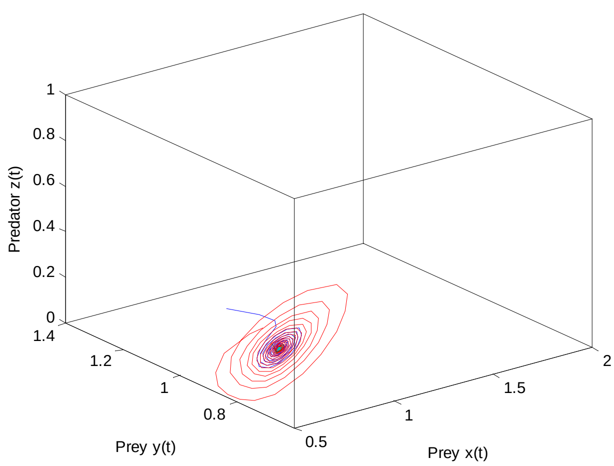

Fig. 5. Stable behavior of system Eq. (1) is observed with $k_1 = 4.8$ and other parameter values are same as in Fig. 3

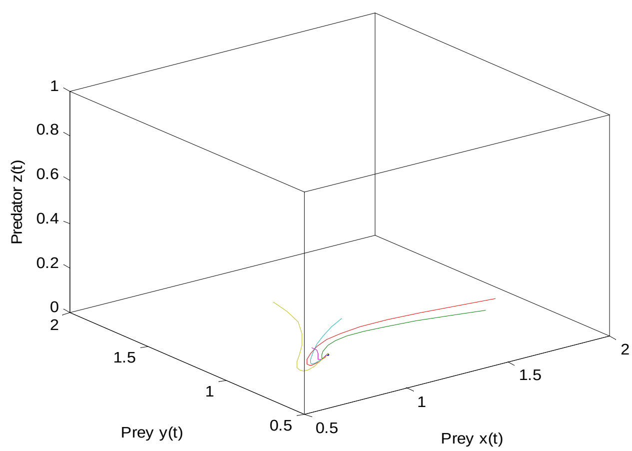

Fig. 6. Here $r_1 = 1.5, k_1 = 1, c_1 = 2, r_2 = 2, k_2 = 2, c_2 = 2, d = 1, m_1 = 1, m_2 = 2$

The condition of Theorem 6 means that if the death of predator population remains below a certain threshold value then all populations will survive in future time.

6 Numerical examples

In the following section we will present four examples to verify our results obtained earlier.

Example 1. Suppose $r_1 = 4, k_1 = 30, c_1 = 146/15, r_2 = 2, k_2 = 2, c_2 = 2, d = 1, m_1 = 1, m_2 = 2$. It is easy to see that system Eq. (1) has a equilibrium point $(1, 1, 1)$. Condition of Theorem 2 is not satisfied. Thus equilibrium $(1, 1, 1)$ is unstable which indicates that all the populations remain in oscillatory manner. We apply Matlab to simulate system Eq. (1) and obtain Fig. 1. In Fig. 1, phase portrait of system Eq. (1) showing that $E(1, 1, 1)$ is unstable.

Example 2. Suppose $r_1 = 1, k_1 = 2, c_1 = 3, r_2 = 2, k_2 = 2, c_2 = 2, d = 1, m_1 = 1, m_2 = 2$. Condition of Theorem 2 is satisfied. Thus we obtain a stable stable equilibrium $(1, 1, 1)$ (see Fig. 2). Phase portrait of system Eq. (1) showing that $E(1, 1, 1)$ is locally asymptotically stable in Fig. 2. This means that all the populations are in coexistence.

Example 3. Suppose $r_1 = 85/22, c_1 = 90/11, r_2 = 2, k_2 = 2, c_2 = 2, d = 1, m_1 = 1, m_2 = 2$. If we choose $k_1 = k_1c = 5$, it is easy to see that system Eq. (1) has a equilibrium point $(1, 1, 1)$. Then it follows from Theorem 5 that Hopf bifurcation of periodic solution occurs at $k_1 = k_1c = 5$ (Hopf bifurcation at $E(1, 1, 1)$, see Fig. 3). Choose k_1 to be 5.2, 4.8 respectively. When $k_1 = 5.2$ the positive equilibrium $(1, 1, 2.9166)$ is unstable (see Fig. 4). When $k_1 = 4.8$, the positive equilibrium point $(1, 1, 2.1875)$is stable (see Fig. 5). We note that bifurcation occurs for certain value of the carrying capacity of prey species in the unreserved area. If we decrease its value then stability is regained.

Example 4. Suppose $r_1 = 1.5, k_1 = 1, c_1 = 2, r_2 = 2, k_2 = 2, c_2 = 2, d = 1, m_1 = 1, m_2 = 2$. Condition of Theorem 4 is satisfied. We observe global convergence of all the solutions of system Eq. (1) (see Fig. 6). In Fig. 6, global behavior of the system Eq. (1) is observed.This means long term survival of the populations in stable manner.

All the above examples show that the stability of the system can be controlled if we decrease the value of r_1 and, k_1 respectively.

7 Discussion

In this paper, we have proposed and analyzed a prey-predator system with a reserved area. We considered Holling type II predator functional response. In [3], Dubey concluded that reserved zone has a stabilizing effect on prey-predator dynamics. But this is not true for Holling type II predator functional response. We have investigated condition for limit cycle to arise by Hopf bifurcation. We have observed that when predator's death rate remains below a threshold value, all the populations can survive. We have discussed local and global stability of the system. If the carrying capacity of prey species of the environment in the unreserved area remains below a threshold value then reserve zone can influence on stability of the prey-predator system. It is to be noted that, whether in the absence or in the presence of predator, prey population may be sustained at an appropriate equilibrium level.

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