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Solving the Lienard equation by differential transform method

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Abstract. In this paper, the differential transform method (DTM) is proposed for solving the Lienard equation. The differential transform method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. Using this method, it is possible to find the exact solution or an approximate solution of the problem.

Keywords: differential transform method, lienard equation

1 Introduction

In this work, we consider the Lienard equation

$$u'' + f(u)u' + g(u) = h(x), \tag{1}$$

which is not only regarded as a generalization of the damped pendulum equation or a damped spring-mass system (where $f(u)u'$ is the damping force, $g(u)$ is the restoring force, and $h(x)$ is the external force), but also used as nonlinear models in many physically significant fields when taking different choices for $f(u), g(u)$ and $h(x)$. For example, the choices $f(u) = \epsilon(u^2 - 1), g(u) = u$, and $h(x) = 0$ lead Eq. (1) to the Van der Pol equation served a a nonlinear model of electronic oscillation[2, 7]. Therefore, studying Eq. (1) is of physical significance. In the general case, it is commonly believed that it is very difficult to find its exact solution by usual ways[3]. Kong studied the following special case of Eq. (1)[1, 4]:

$$u''(x) + lu(x) + mu^3(x) + nu^5(x) = 0,$$

where l, m, n are real coefficients. Finding explicit exact and numerical solutions of nonlinear equations efficiently is of major importance and has widespread applications in numerical methods and applied mathematics. Implement the variational iteration method for Eq. (1) was known in the literature[6]. The differential transform is an analytic method for Solving differential equations. The concept of the differential transform was first introduced by Zhou in [8]. Its main application therein is to solve both linear and non-linear initial value problems in electric circuit analysis. This method constructs and analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders. The differential transform method is an alterative procedure for obtaining analytic Taylor series solution of the differential equations. By using DTM, we get a series solution, in practice a truncated series solution.

2 Differential transform method

Differential transformation of function $u(x)$ is defined as follows[5]

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$$U(k)=\frac{1}{k!}\left[\frac{\mathrm{d}^k u(x)}{\mathrm{d}x^k}\right]_{x=0}. \tag{2}$$

In Eq. (2), $u(x)$ is the original function and $U(k)$ is the transformed function. Differential inverse transform of $U(k)$ is defined as follows

$$u(x)=\sum_{k=0}^{\infty} x^k U(k). \tag{3}$$

In fact, from Eqs. (2) and (3), we obtain

$$u(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=0}. \tag{4}$$

Eq. (4) implies that the concept of differential transformation is derived from the Taylor series expansion. From the definitions Eqs. (2) and (3), it is easy to obtain the following mathematical operations:
(a) If $u(x)=v(x)\pm w(x)$, then $U(k)=V(k)\pm W(k)$.
(b) If $u(x)=av(x), a\in R$, then $U(k)=aV(k)$.
(c) If $u(x)=v(x)w(x)$, then $U(k)=\sum_{r=0}^{k}V(r)W(k-r)$.
(d) If $u(x)=x^n$, then $U(k)=\delta(k-n)$.
(e) If $u(x)=u_1(x)u_2(x)\cdots u_{n-1}(x)u_n(x)$, then
$U(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_1(k_1)U_2(k_2-k_1)\cdots U_{n-1}(k_{n-1}-k_{n-2})U_n(k-k_{n-1})$.

3 Applications and results

In this section we consider the Lienard equation

$$u''(x)+lu(x)+mu^3(x)+nu^5(x)=0, \tag{5}$$

with the initial conditions

$$u(0)=\sqrt{\frac{-2l}{m}},\qquad u'(0)=-\frac{\sqrt{-l}l}{\sqrt{\frac{-2l}{m}}m}, \tag{6}$$

where m and l are arbitrary constants.

Taking differential transform of Eq. (5), we can obtain

$$\begin{aligned}U(k+2)=&\frac{1}{(k+2)(k+1)}\left[-lU(k)-m\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}U(k_1)U(k_2-k_1)U(k-k_2)\right.\\&\left.-n\sum_{k_4=0}^{k}\sum_{k_3=0}^{k_4}\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U(k_1)U(k_2-k_1)U(k_3-k_2)U(k_4-k_3)U(k-k_4)\right].\end{aligned} \tag{7}$$

And the transform of the initial conditions Eq. (6) are

$$U(0)=\sqrt{\frac{-2l}{m}},\qquad U(1)=\frac{-l}{\sqrt{2m}}. \tag{8}$$

Utilizing the recurrence relation Eq. (7) and the transformed initial conditions Eq. (8) and Matlab software, the approximate solution of the Lienard Eq. (5) can be derived as

$$
\begin{aligned}
x_1(t) &= \sqrt{\frac{K}{2+D}} - \frac{1}{2}\sqrt{\frac{K}{2+D}}\left[l + \frac{mK}{2+D} + \frac{nK^2}{(2+D)^2}\right]t^2,\\
u(x) &= \sqrt(\frac{-2l}{m}) + \frac{-1}{\sqrt(2m)}x + \frac{-1}{21(-2l/m)^{1/2}}(-1/2l - 1/2m(-2l/m)^{3/2}\\
&\quad - 1/2n(-2l/m)^{5/2})x^2 + (-5/12l^2 2^{1/2}/m^{(1/2)} + 5/3nl^3 2^{1/2}/m^{5/2})x^3\\
&\quad + (221/30000l - 1/12m(3/2(-2l/m)^{1/2}l^2/m + 663/1250l/m) - 1/12n(5l^2/m(-2l/m)^{3/2}\\
&\quad - 221/125l^2/m^2)x^4 + (-1/20l(-5/12l^2 2^{1/2}/m^{1/2} + 5/3nl^3 2^{1/2}/m^{5/2}) - 1/20m\\
&\quad \times(-2(-5/4l^2 2^{1/2}/m^(1/2) + 5nl^3 2^{1/2}/m^{5/2})l/m + 663/2500l2^{1/2}/m^{1/2}(-2l/m)^{1/2}\\
&\quad - 1/4l^3 2^{1/2}/m^{3/2}) - 1/20n(4(-25/12l^2 2^{1/2}/m^{1/2} + 25/3nl^3 2^{1/2}/m^{5/2})l^2/m^2\\
&\quad + 221/250(-2l/m)^{3/2}l2^(1/2)/m^{1/2} + 5l^4 2^{1/2}/m^{5/2}))x^5\\
&\quad + (-313/300000l - 1/30m\times(-939/5000l/m + 2211/10000l2^{1/2}/m^{1/2}(-2l/m)^{1/2}\\
&\quad + 6757180704780363/288230376151711744(-2l/m)^{1/2} - 663/5000l^2/m) - 1/30n\\
&\quad \times(313/500l^2/m^2 + 737/1000(-2l/m)^{3/2}l2^{1/2}/m^{1/2}\\
&\quad + 2815491960325151/36028797018963968(-2l/m)^{3/2} + 663/250l^3/m^2\\
&\quad + 5/4l^4/m^2(-2l/m)^{1/2}))x^6 + \cdots.
\end{aligned}
$$

Selecting the values of $m = 4$, $n = -3$, $l = -1$, and using the before equation, we can write:

$$u(x) = \frac{7071}{10000} + \frac{221}{625x} - \frac{221}{2500x^2} - \frac{737}{10000x^3} + \frac{313}{10000x^4} + \frac{223}{10000x^5} - \frac{111}{10000x^6} + \cdots.$$

Of course, the exact solution $u(x)$ (see [1]) is

$$u(x) = \sqrt{\frac{-2l(1+\tanh(\sqrt{-l}x))}{m}}. \tag{9}$$

The numerical results with using differential transform method for Lienard equation in comparison with the exact solution of u ($m = 4$, $n = -3$, $l = -1$ and $x = 0.1(0.1)1$) are given in Tab. 1. Also in Fig. 1 we can see the comparison between The exact solution and approximate solution with using DTM for the Lienard equation for the values of $m = 4$, $n = -3$ and $l = -1$.

Table 1. The numerical results for the first example

x	$\lvert U_{\text{Exact}} - U_{\text{DTM}}\rvert$
0.1	$2.2683e-006$
0.2	$2.5222e-006$
0.3	$1.4404e-005$
0.4	$6.1806e-005$
0.5	$2.2099e-004$
0.6	$6.5187e-004$
0.7	$1.6408e-003$
0.8	$3.6469e-003$
0.9	$7.3493e-003$
1.0	$1.3692e-002$

Table 2. The numerical results for the second example

x	$\lvert U_{\text{Exact}} - U_{\text{DTM}}\rvert$
0.1	$4.0668e-006$
0.2	$3.7000e-006$
0.3	$7.1793e-006$
0.4	$4.5002e-005$
0.5	$2.4676e-004$
0.6	$9.9922e-004$
0.7	$3.2168e-003$
0.8	$8.7371e-003$
0.9	$2.0841e-002$
1.0	$4.4891e-002$

In the second example, we will consider the Lienard Eq. (5) with the initial conditions

$$u(0) = \sqrt{\frac{K}{2+D}}, \qquad u'(0) = 0, \tag{10}$$

where with arbitrary constants m and l

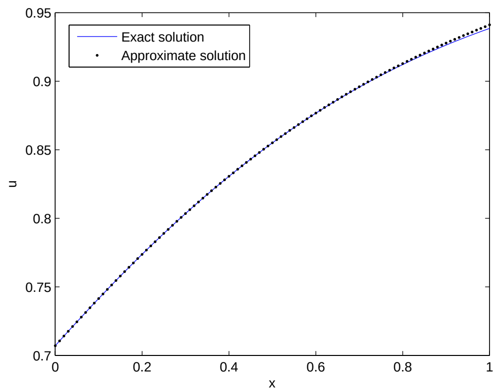

Fig. 1. The exact solution and approximate solution

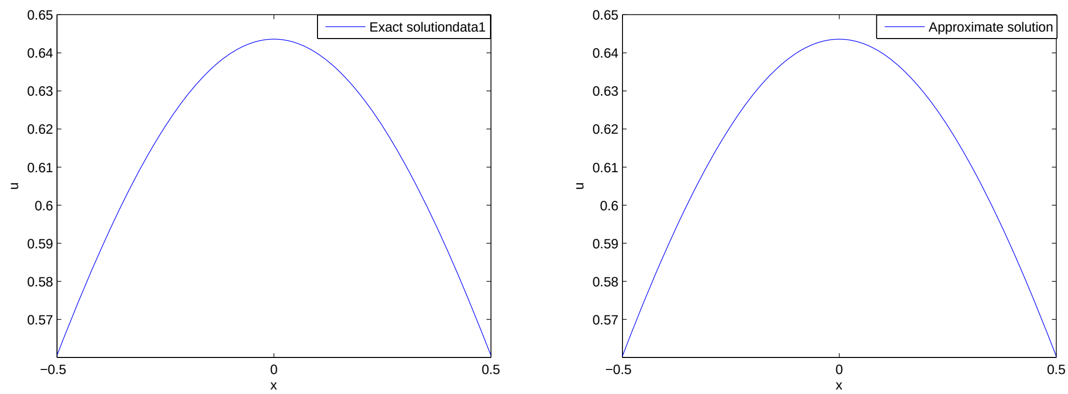

Fig. 2. The exact solution and approximate solution

$$K = 4\sqrt{\frac{3l^2}{3m^2 - 16nl}}, \qquad D = -1 + \frac{\sqrt{3}m}{\sqrt{3m^2 - 16nl}}.$$

Transform of the initial conditions Eq. (10) are

$$U(0) = \sqrt{\frac{K}{2+D}}, \qquad U(1) = 0. \tag{11}$$

Utilizing the recurrence relation Eq. (7) and the transformed initial conditions Eq. (11) and computing with Matlab software, the approximate solution of the Lienard Eq. (5) can be derived as

$$\begin{aligned}
u(x) =& 2(3^{1/2}(l^2/(3m^2 - 16nl))^{1/2}/(1 + 3^{1/2}m/(3m^2 - 16nl)^{1/2}))^{1/2} \\
&+ (-l(3^{1/2}(l^2/(3m^2 - 16nl))^{1/2}/(1 + 3^{1/2}m/(3m^2 - 16nl)^{1/2}))^{1/2} \\
&- 4m(3^{1/2}(l^2/(3m^2 - 16nl))^{1/2}/(1 + 3^{1/2}m/(3m^2 - 16nl)^{1/2}))^{3/2} \\
&- 16n(3^{1/2}(l^2/(3m^2 - 16nl))^{1/2}/(1 + 3^{1/2}m/(3m^2 - 16nl)^{1/2}))^{5/2})x^2 \\
&+ (1/3(112nl^2 - 96m3^{1/2}(-l^2/(-3m^2 + 16nl))^{1/2}nl + l(3m^2 - 16nl)^{1/2}3^{1/2}m + 3lm^2 \\
&+ 18m^2(-l^2/(-3m^2 + 16nl))^{1/2}(3m^2 - 16nl)^{1/2} + 18m^3 3^{1/2}(-l^2/(-3m^2 + 16nl))^{1/2}) \\
&(3lm^2 + 16nl^2 + l(3m^2 - 16nl)^{1/2}3^{1/2}m + 6m^3 3^{1/2}(-l^2/(-3m^2 + 16nl))^{1/2} - 32m3^{1/2} \\
&(-l^2/(-3m^2 + 16nl))^{1/2}nl + 6m^2(-l^2/(-3m^2 + 16nl))^{1/2}(3m^2 - 16nl)^{1/2})3^{1/4}((-l^2/ \\
&(-3m^2 + 16nl))^{1/2}(3m^2 - 16nl)^{1/2}/((3m^2 - 16nl)^{1/2} + 3^{1/2}m))^{1/2}/((3m^2 - 16nl)^{1/2} + 3^{1/2}m)^4) \\
&x^4 + (-32/45 3^{1/4}((-l^2/(-3m^2 + 16nl))^{1/2}(3m^2 - 16nl)^{1/2}/((3m^2 - 16nl)^{1/2} + 3^{1/2}m))^{1/2} \\
&(3lm^2 + 16nl^2 + l(3m^2 - 16nl)^{1/2}3^{1/2}m + 6m^3 3^{1/2}(-l^2/(-3m^2 + 16nl))^{1/2} - 32m3^{1/2}(-l^2/
\end{aligned}$$

$$(-3m^2+16nl))^{1/2}nl+6m^2(-l^2/(-3m^2+16nl))^{1/2}(3m^2-16nl)^{1/2})(284672n^4l^5-614400l^4m3^{1/2} \times(-l^2/(-3m^2+16nl))^{1/2}n^4-269184l^4n^3m^2-57856l^4n^3(3m^2-16nl)^{1/2}3^{1/2}m+1387008l^3 n^3m^33^{1/2}(-l^2/(-3m^2+16nl))^{1/2}+557568l^3n^3m^2(-l^2/(-3m^2+16nl))^{1/2}(3m^2-16nl)^{1/2} +12432l^3n^2(3m^2-16nl)^{1/2}3^{1/2}m^3+71568l^3n^2m^4-473472l^2n^2(-l^2/(-3m^2+16nl))^{1/2} m^53^{1/2}-369792l^2n^2(-l^2/(-3m^2+16nl))^{1/2}(3m^2-16nl)^{1/2}m^4-8208l^2nm^5(3m^2-16nl)^{1/2} 3^{1/2}-48168l^2nm^6+23328ln(-l^2/(-3m^2+16nl))^{1/2}m^73^{1/2}+33696nl(-l^2/(-3m^2+16nl))^{1/2} (3m^2-16nl)^{1/2}m^6+8829lm^8+2943l(3m^2-16nl)^{1/2}m^73^{1/2}+3888(-l^2/(-3m^2+16nl))^{1/2} (3m^2-16nl)^{1/2}m^8+3888(-l^2/(-3m^2+16nl))^{1/2}m^93^{1/2})l/((3m^2-16nl)^{1/2}+3^{1/2}m)^{10})x^6.$$

Selecting the values of $m = 4$, $n = 3$, $l = -1$, and using the above equation, we can write:

$$u(x) = 64359/100000 - 37699/100000x^2 + 20559/100000x^4 - 2983/25000x^6.$$

The exact solution $u(x)$ (see [1]) is

$$u(x) = \sqrt{\frac{K\text{sech}^2(\sqrt{-l}x)}{2 + D\text{sech}^2(\sqrt{-l}x)}},$$

where

$$K = 4\sqrt{\frac{3l^2}{3m^2 - 16nl}}, \qquad D = -1 + \frac{\sqrt{3}m}{\sqrt{3m^2 - 16nl}}.$$

The numerical results with using differential transform method for Lienard equation in comparison with the exact solution of u ($m = 4$, $n = -3$, $l = -1$ and $x = 0.1(0.1)1$) are given in Tab. 2. Also in Fig. 2 we can see the comparison between The exact solution and approximate solution with using DTM for the Lienard equation for the values of $m = 4$, $n = -3$ and $l = -1$.

4 Conclusions

In this paper, differential transform method was employed successfully for solving the Lienard equation. The exact solutions are compared with the numerical solutions in Tabs. 1 and 2. The results show that the differential transform method is a powerful mathematical tool for finding the exact and approximate solutions of nonlinear equations. In our work we use the Matlab to calculate the series obtained from the differential transform method.

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