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Comparison and coupling of polynomials for singular fourth-order parabolic PDES

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Abstract. In this paper, we apply two modified versions of variational iteration method (VIM) to solve singular fourth-order parabolic partial differential equations. The proposed modifications are made by introducing the He's polynomials and Adomian's polynomials in the correction functional of VIM. The developed algorithms are quite efficient and are practically well suited for use in these problems. The proposed iterative schemes find the solution without any discritization, linearization or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the suggested algorithms. It is observed that the modified version which is based upon He's polynomials (VIMHP) is more user friendly and is easier to implement as compare to the one (VIDM) where Adomian's polynomials along with their complexities are applied.

Keywords: modified variational iteration methods, He's polynomials, Adomian's polynomials, singular fourth-order parabolic PDEs, boundary value problems

1 Introduction

Singular fourth-order parabolic partial differential equations[3–8, 10, 25, 28, 34, 36, 38] with variable coefficient are known to arise in various physical phenomenon including transverse vibrations of a homogeneous beam, viscoelastic and inelastic flows, deformation of beams, plate deflection theory, engineering and applied sciences. The singular fourth-order parabolic partial differential equation with variable co-efficient is of the form:

$$\frac{\partial^2 u}{\partial t^2} + \mu(x,y,z)\frac{\partial^4 u}{\partial x^4} + \frac{1}{y}\lambda(x,y,z)\frac{\partial^4 u}{\partial y^4} + \frac{1}{z}\eta(x,y,z)\frac{\partial u}{\partial z^4} = g(x,y,z,t), \quad z < y, z < b, t > 0,$$

(where $\mu(x,y,z)$, $\lambda(x,y,z)$ and $\eta(x,y,z)$ are positive) subject to the initial conditions:

$$u(x,y,z,t) = g_0(x,y,z), \quad \frac{\partial u}{\partial t} = f_0(x,y,z),$$

and boundary conditions:

$$\begin{aligned} u(x,y,z,t) =& g_0(x,y,z), \quad \frac{\partial u}{\partial t} = f_0(x,y,z), \\ u(a,y,z,t) =& g_0(y,z,y), \quad u(b,y,z,t) = g_1(y,z,t), \\ u(a,y,z,t) =& k_0(y,z,y), \quad u(b,y,z,t) = k_1(y,z,t), \\ u(a,y,z,t) =& h_0(y,z,y), \quad u(b,y,z,t) = h_1(y,z,t), \end{aligned}$$

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$$\frac{\partial^2 u}{\partial x^2}(a,y,z,t)=\bar{g}_0(y,z,y),\quad \frac{\partial^2 u}{\partial x^2}(b,y,z,t)=\bar{g}_1(y,z,t),\quad \frac{\partial^2 u}{\partial y^2}(a,y,z,t)=\bar{k}_0(y,z,y),$$
$$\frac{\partial^2 u}{\partial y^2}(\partial x^2 b,y,z,t)=\bar{k}_1(y,z,t),\quad \frac{\partial^2 u}{\partial z^2}(a,y,z,t)=\bar{h}_0(y,z,y),\quad \frac{\partial^2 u}{\partial z^2}(b,y,z,t)=\bar{h}_1(y,z,t),$$

where the functions $f_i, g_i, k_i, h_i, \bar{g}_i, \bar{k}_i, \bar{h}_i, i = 0, 1$ are continuous, $\mu(x, y, z) > 0$ is the ratio of flexural rigidity[2] of the beam to its mass per unit length, see [3–8, 10, 25, 27, 32, 38, 39] and the references therein. The functions $f_0(x), f_1(x), g_0(t), g_1(t), h_0(t)$ and $h_1(t)$ are continuous functions. Numerical computations of transverse vibrations have been carried out by a number of authors. The main focus was to obtain numerical solutions by using several techniques including, explicit and implicit finite difference schemes, lines approach, separation of variables and Adomian's decomposition, see [1–8, 10, 25, 27, 32, 38, 39] and the references therein. Inspired and motivated by the ongoing research in this area, we apply two modified versions of variational iteration technique[27, 34, 35, 37] which are based on the coupling of He's polynomials[9] and Adomian's polynomials in the correction functional of VIM[11, 13, 15, 19, 21, 22, 24] by converting them into systems of partial differential equations by introducing a suitable transformation. The proposed modified versions (VIMHP and VIDM) are applied to the resultant systems of integro partial differential equations. It is worth mentioning that the modification based on He's polynomials is more user friendly as compare to VIDM where Adomian's polynomials are coupled with the correction functional of VIM. It is worth mentioning that the basic idea of variational iteration method using Adomian's polynomials (VIDM) was given by Abbasbandy [1, 2] to solve Klein Gordon and Ricatti differential equations. In a subsequent work, Noor and Mohyud-Din [26, 27, 32, 34, 35] exploited this concept to solve a wide class of initial and boundary value problems. The variational iteration method using He's polynomials (VIMHP) was introduced by Noor and Mohyud-Din [26, 27, 32, 34, 35] where He's polynomials are inserted in the correction functional of He's VIM. It is to be highlighted that He's polynomials are evaluated from He's homotopy perturbation method[12, 14, 16–18, 20, 26–37]. Moreover, Ghorbani et al. [9] also proved that He's polynomials are fully compatible with He's polynomials but are easier to calculate and hence are more user friendly. Several examples are given which clearly reveal the complete reliability and efficiency of the proposed algorithms and also certify the claim that VIMHP is more user friendly as compare to VIDMs.

2 Homotopy perturbation method (HPM) and He's polynomials

To explain the homotopy perturbation method, we consider a general equation of the type:

$$L(u) = 0, \tag{1}$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by:

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{3}$$

we have:

$$H(u, 0) = F(u), \quad H(u, 1) = L(u).$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(f, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter[9, 12, 14, 16–18, 20, 21, 23, 26–28, 31–37]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter[9, 12, 14, 16–18, 20, 21, 23, 26–28, 31–37]. to obtain:

$$u=\sum_{i=0}^{\infty}p^iu_i=u_0+pu_1+p^2u_2+p^3u_3+\cdots, \tag{4}$$

if $p\to 1$, then Eq. (4) corresponds to Eq. (2) and becomes the approximate solution of the form:

$$f=\lim_{p\to 1}u=\sum_{i=0}^{\infty}u_i. \tag{5}$$

It is well known that series Eq. (5) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; see [4–6, 11, 13, 15, 19, 23]. We assume that Eq. (5) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to [9], He's HPM considers the solution, $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x)=\sum_{i=0}^{\infty}p^iu_i=u_0+pu_1+p^2u_2+\cdots,$$

and the method considers the nonlinear term $N(u)$ as:

$$N(u)=\sum_{i=0}^{\infty}p^iu_i=u_0+pu_1+p^2u_2+\cdots,$$

where H_n's are the so-called He's polynomials[9], which can be calculated by using the formula:

$$H_n(u_0,\cdots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}(N(\sum_{i=0}^{\infty}p^iu_i))_{p=0},\quad n=0,1,2,\cdots.$$

3 Adomian's decomposition method and Adomian's polynomials

Consider the differential equation[9, 27, 38, 39]:

$$Lu+Ru+Nu=g, \tag{6}$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser order than L, Nu represents the nonlinear terms and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq. (4) and using the given conditions, we obtain:

$$u=f-L^{-1}(Ru)-L^{-1}(Nu), \tag{7}$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [9, 27, 38, 39] defines the solution $u(x)$ by the series:

$$u(x)=\sum_{i=0}^{\infty}u_n(x), \tag{8}$$

where the components $u_n(x)$ are usually determined recurrently by using the relation:

$$u_0=f,\quad u_{k+1}=L^{-1}(Ru_k)-L^{-1}(Nu_k),\quad k\geq 0.$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by:

$$F(u)=\sum_{i=0}^{\infty}A_n,$$

where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [9, 27, 38, 39] which yields.

4 Variational iteration method using He's polynomials (VIMHP)

To illustrate the basic concept of the variational iteration method using He's polynomials (VIMHP), we consider the following general differential equation[11, 13, 15, 19, 21–24]:

$$Lu + Nu = g(x), \tag{9}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method[11, 13, 15, 19, 21, 22, 24], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)(Lu_n(\xi) + N\tilde{u}_n)(\xi) - g(\xi))\mathrm{d}\xi, \tag{10}$$

where λ is a Lagrange multiplier[11, 13, 15, 19, 21, 22, 24], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0;(10)$ is called a correction functional. It is well-known that He's homotopy perturbation method (HPM) provides the solution as a series; whereas He's variational iteration method provides the solution as a sequence. Now, we apply the homotopy perturbation method:

$$\sum_{n=0}^{\infty} u_n = u_0(x) + p\int_0^x \lambda(\xi)(\sum_{n=0}^{\infty} p^n L(u_n) + \sum_{n=0}^{\infty} p^n N(\tilde{u}_n))\mathrm{d}\xi - \int_0^x \lambda(\xi)g(\xi)\mathrm{d}\xi, \tag{11}$$

which is the coupling of variational iteration method and He's polynomials[11, 13, 15, 19, 21, 22, 24] and comparison of like powers of p gives solutions of various orders.

5 Variational iteration method using Adomian's polynomials (VIDM)

To illustrate the basic concept of the VIDM, we consider the following general differential Eq. (12)[1, 2, 27, 32]:

$$Lu + Nu = g(x). \tag{12}$$

The correction functional is given as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + N\tilde{u}_n(s) - g(s))\mathrm{d}s. \tag{13}$$

where λ is a Lagrange multiplier[1, 2, 27, 32], $\tilde{u}_n$ is considered as a restricted variation i.e. $\delta\tilde{u}_n = 0$; We define the solution $u(x)$ by the series $u(x) = \sum_{i=0}^{\infty} u_i(x)$ and the nonlinear term $N(u) = \sum_{i=0}^{\infty} A_n(u_0, u_1, u_2, \cdots, u_i)$ where are the so-called Adomian's polynomials and can be generated for all type of nonlinearities according to the algorithm developed, in [1, 2, 27, 32, 38, 39]. Hence, we obtain the following iterative scheme:

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda(Lu_n(x) + \sum_{n=0}^{\infty} A_n - g(x))\mathrm{d}x, \tag{14}$$

which is called variational iteration method using Adomian's polynomials (VIDM) [1, 2, 26, 27].

6 Numerical applications

In this section, we apply VIMHP and VIDM to solve the fourth-order parabolic partial differential equations by converting them into systems of integral equations. The proposed modified versions are applied on the resultant systems of integral equations.

Example 1. Consider the following singular fourth-order parabolic equation:

$$\frac{\partial^2 u}{\partial t^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}=0,\quad \frac{1}{2}<x<1,\quad t>0, \tag{15}$$

subject to the initial conditions:

$$u(x,0)=0,\quad \frac{\partial u}{\partial t}(x,0)=1+\frac{x^5}{120},\quad \frac{1}{2}<x<1, \tag{16}$$

and the boundary conditions:

$$u\left(\frac{1}{2},t\right)=\left(1+\frac{(1/2)^5}{120}\right)\sin t,\quad u(1,t)=\frac{121}{120}\sin t,\quad t>0,$$
$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},t\right)=\frac{1}{6}(\frac{1}{2})^3\sin t,\quad \frac{\partial^2 u}{\partial x^2}(1,t)=\frac{1}{6}\sin t,\quad t>0.$$

Using the transformation $\frac{\partial u}{\partial t}=q(t)$ the above problem can be converted to the following system of partial differential equations:

$$\begin{cases}\frac{\partial u}{\partial t}=q(t)\\ \frac{\partial q}{\partial t}=-\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}\end{cases} \tag{17}$$

with initial conditions:

$$u(x,0)=0,\quad q(x,0)=1+\frac{x^5}{120}.$$

The correction functional for the above system is given by:

$$\begin{cases}u_{n+1}(x,t)=\int_0^t\lambda_1\left(\frac{\partial u_n}{\partial t}-\tilde{q}_n(t)\right)\mathrm{d}t\\ q_{n+1}(x,t)=\left(1+\frac{x^5}{120}\right)+\int_0^t\lambda_2\left(\frac{\partial q_n}{\partial t}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4\tilde{u}_n}{\partial x^4}\right)\mathrm{d}t\end{cases} \tag{18}$$

Applying the variational iteration method using He's polynomials (VIMHP), we get:

$$\begin{cases}u_0+pu_1+p^2u_2+\cdots= & p\int_0^t\lambda_1\left(\left(\frac{\partial u_0}{\partial t}+p\frac{\partial u_1}{\partial t}+p^2\frac{\partial u_2}{\partial t}+\cdots\right)-(\tilde{q}_0+p\tilde{q}_1+p^2\tilde{q}_2+\cdots)\right)\mathrm{d}t\\ q_0+pq_1+p^2q_2+\cdots= & \left(1+\frac{x^5}{120}\right)+p\int_0^t\lambda_2\left(\left(\frac{\partial q_0}{\partial t}+p\frac{\partial q_1}{\partial t}+p^2\frac{\partial q_2}{\partial t}+\cdots\right)\right)\mathrm{d}t\\ & +p\int_0^t\lambda_2\left(\frac{1}{x}+\frac{x^4}{120}\right)\left(\frac{\partial^4\tilde{u}_0}{\partial x^4}+p\frac{\partial^4\tilde{u}_1}{\partial x^4}+p^2\frac{\partial^4\tilde{u}_2}{\partial x^4}+\cdots\right)\mathrm{d}t\end{cases} \tag{19}$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as:

$$\lambda_1=-1,\quad \lambda_2=-1,$$
$$\begin{cases}u_0+pu_1+p^2u_2+\cdots= & -p\int_0^t\left(\left(\frac{\partial u_0}{\partial t}+p\frac{\partial u_1}{\partial t}+p^2\frac{\partial u_2}{\partial t}+\cdots\right)-(\tilde{q}_0+pq_1+p^2p_2+\cdots)\right)\mathrm{d}t\\ q_0+pq_1+p^2q_2+\cdots= & \left(1+\frac{x^5}{120}\right)-p\int_0^t\left(\left(\frac{\partial q_0}{\partial t}+p\frac{\partial q_1}{\partial t}+p^2\frac{\partial q_2}{\partial t}+\cdots\right)\right)\mathrm{d}t-p\int_0^t\left(\frac{1}{x}+\frac{x^4}{120}\right)\\ & \left(\frac{\partial^4\tilde{u}_0}{\partial x^4}+p\frac{\partial^4\tilde{u}_1}{\partial x^4}+p^2\frac{\partial^4\tilde{u}_2}{\partial x^4}+\cdots\right)\mathrm{d}t\end{cases} \tag{20}$$

Comparing the co-efficient of like powers of p, following approximants are obtained:

$$p^{(0)}:\begin{cases} u_0(x,t)=0 \\ q_0(x,t)=1+\frac{x^5}{120} \end{cases} \tag{21}$$

$$p^{(1)}:\begin{cases} u_1(x,t)=\left(1+\frac{x^5}{120}\right)t \\ q_1(x,t)=1+\frac{x^5}{120} \end{cases} \tag{22}$$

$$p^{(2)}:\begin{cases} u_2(x,t)=\left(1+\frac{x^5}{120}\right)t \\ q_2(x,t)=\left(1+\frac{x^5}{120}\right)-\left(1+\frac{x^5}{120}\right)\frac{t^2}{2!} \end{cases} \tag{23}$$

$$p^{(3)}:\begin{cases} u_3(x,t)=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!} \\ q_3(x,t)=\left(1+\frac{x^5}{120}\right)-\left(1+\frac{x^5}{120}\right)\frac{t^2}{2!} \end{cases} \tag{24}$$

$$p^{(4)}:\begin{cases} u_4(x,t)=\left(1+\frac{x^5}{120}\right)t-\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!} \\ q_4(x,t)=\left(1+\frac{x^5}{120}\right)-\left(1+\frac{x^5}{120}\right)\frac{t^2}{2!}-\left(1+\frac{x^5}{120}\right)\frac{t^4}{4!} \end{cases} \tag{25}$$

$$\vdots$$

The solution in a series form is:

$$u(x,0)=\left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right)=\left(1+\frac{x^5}{120}\right)\sin t, \tag{26}$$

which is the exact solution.

Now, we apply variational iteration method using Adomian's polynomials (VIDM) on the reformulated system of integral Eq. (18), we get:

$$\begin{cases} u_{n+1}(x,t)=-\int_0^t\left(\frac{\partial u_n}{\partial t}-\sum\limits_{n=0}^{\infty}q_n(t)\right)\mathrm{d}t \\ q_{n+1}(x,t)=\left(1+\frac{x^5}{120}\right)-\int_0^t\left(\frac{\partial q_n}{\partial t}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\sum\limits_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial x^4}\right)\mathrm{d}t \end{cases}$$

consequently, we get the same approximants Eqs. (21) $\sim$ (25) as that of VIMHP and hence our series solution is fully compatible with the one obtained in Eq. (26).

Example 2. Consider the following singular fourth order parabolic partial differential equation in two space variables:

$$\frac{\partial^2 u}{\partial t^2}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}=0, \tag{27}$$

with initial conditions:

$$u(x,y,0)=0,\quad \frac{\partial u}{\partial t}(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!}, \tag{28}$$

and the boundary conditions:

$$\begin{aligned} &u\left(\frac{1}{2},y,t\right)=\left(2+\frac{(0.5)^6}{6!}+\frac{y^6}{6!}\right)\sin t,\quad u(1,y,t)=\left(2+\frac{1}{6!}+\frac{y^6}{6!}\right)\sin t,\\ &\frac{\partial^2 u}{\partial x}\left(\frac{1}{2},y,t\right)=\frac{(0.5)^4}{24}\sin t,\quad \frac{\partial^2 u}{\partial x}(1,y,t)=\frac{1}{24}\sin t,\\ &\frac{\partial^2 u}{\partial y^2}\left(x,\frac{1}{2},t\right)=\frac{(0.5)^4}{24}\sin t,\quad \frac{\partial^2 u}{\partial y^2}(x,1,t)=\frac{1}{24}\sin t. \end{aligned}$$

Using the transformation $\frac{\partial u}{\partial t}=q(t)$ the above problem can be converted to the following system of partial differential equations:

$$\begin{cases} \frac{\partial u}{\partial t} = q(t) \\ \frac{\partial q}{\partial t} = -2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} - 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4} = 0 \end{cases} \tag{29}$$

with initial conditions:

$$u(x, y, 0) = 0, \quad q(x, y, 0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}.$$

The above system of differential equations can be converted to the following system of integral equations:

$$\begin{cases} \frac{\partial u}{\partial t} = \int_0^t q(t)\mathrm{d}t \\ q(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) - 2\left(\int_0^t \left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} + \left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}\right)\mathrm{d}t \end{cases} \tag{30}$$

The correction functional is given as:

$$\begin{cases} u_{n+1}(x, t) = \int_0^t \lambda_1(u_n(x, t) - \tilde{q}_n(t))\mathrm{d}t \\ q_{n+1}(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) + \int_0^t \lambda_2 \left(q_n + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 \tilde{u}}{\partial x^4} + \left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 \tilde{u}_n}{\partial y^4}\right)\mathrm{d}t \end{cases} \tag{31}$$

Applying the variational iteration method using He's polynomials (VIMHP), we get:

$$\begin{cases} u_0 + pu_1 + p^2u_2 + \cdots = & p\int_0^t((u_0 + pu_1 + p^2u_2 + \cdots) - (\tilde{q}_0 + p\tilde{q}_1 + p^2\tilde{q}_2 + \cdots))\mathrm{d}t \\ q_0 + pq_1 + p^2q_2 + \cdots = & \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) + p\int_0^t \lambda_2\left((q_0 + pq_1 + p^2q_2 + \cdots) + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\right. \\ & \left.\left(\frac{\partial^4 \tilde{u}_0}{\partial x^4} + p\frac{\partial^4 \tilde{u}_1}{\partial x^4} + p^2\frac{\partial^4 \tilde{u}_2}{\partial x^4} + \cdots\right)\right)\mathrm{d}t + 2p\int_0^t \lambda_2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right) \\ & \left(\frac{\partial^4 \tilde{u}_0}{\partial y^4} + p\frac{\partial^4 \tilde{u}_1}{\partial y^4} + p^2\frac{\partial^4 \tilde{u}_2}{\partial y^4} + \cdots\right)\mathrm{d}t \end{cases}$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as:

$$\lambda_1 = -1, \quad \lambda_2 = -1,$$

$$\begin{cases} u_0 + pu_1 + p^2u_2 + \cdots = & -p\int_0^t((u_0 + pu_1 + p^2u_2 + \cdots) - (q_0 + pq_1 + p^2q_2 + \cdots))\mathrm{d}t \\ q_0 + pq_1 + p^2q_2 + \cdots = & \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right) - p\int_0^t \left((q_0 + pq_1 + p^2q_2 + \cdots) + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\right. \\ & \left.\left(\frac{\partial^4 u_0}{\partial x^4} + p\frac{\partial^4 u_1}{\partial x^4} + p^2\frac{\partial^4 u_2}{\partial x^4} + \cdots\right)\right)\mathrm{d}t - 2p\int_0^t \left(\frac{1}{y^2} + \frac{y^4}{6!}\right) \\ & \left(\frac{\partial^4 u_0}{\partial y^4} + p\frac{\partial^4 u_1}{\partial y^4} + p^2\frac{\partial^4 u_2}{\partial y^4} + \cdots\right)\mathrm{d}t \end{cases}$$

Comparing the co-efficient of like powers of p, following approximants are obtained:

$$p^{(0)}: \begin{cases} u_0(x, t) = 0 \\ q_0(x, t) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \end{cases} \tag{32}$$

$$p^{(1)}: \begin{cases} u_1(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)t \\ q_1(x, t) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \end{cases} \tag{33}$$

$$p^{(2)}: \begin{cases} u_2(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)t \\ q_2(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(1 - \frac{t^2}{2!}\right) \end{cases} \tag{34}$$

$$p^{(3)}: \begin{cases} u_3(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(t - \frac{t^3}{3!}\right) \\ q_3(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!}\right) \end{cases} \tag{35}$$

$$p^{(4)}: \begin{cases} u_4(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!}\right) \\ q_4(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right) \end{cases} \tag{36}$$

$$p^{(5)}: \begin{cases} u_5(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{y^7}{7!}\right) \\ q_5(x, t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{y^8}{8!}\right) \end{cases} \tag{37}$$

$$p^{(6)}:\begin{cases} u_6(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{y^7}{7!}+\frac{y^9}{9!}\right)\\ q_6(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\frac{y^8}{8!}-\frac{y^{10}}{10!}\right)\end{cases} \tag{38}$$

$$\vdots$$

The exact solution is recognized easily:

$$u(x,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{y^7}{7!}+\frac{y^9}{9!}=\cdots\right)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin t. \tag{39}$$

Now, we apply variational iteration method using Adomian's polynomials (VIDM) on the reformulated system of integral Eq. (23), we get:

$$\begin{cases} u_{n+1}(x+t)=-\int_0^t(u_n(x,t)-\sum\limits_{n=0}^{\infty}q_n(t))\mathrm{d}t\\ q_{n+1}(x,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)-\int_0^t\left(q_n+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\sum\limits_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial x^4}+\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\sum\limits_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial y^4}\right)\mathrm{d}t\end{cases}$$

consequently, we get the same approximants Eq. (32) $\sim$ (38) as that of VIMHP and hence our series solution is fully compatible with the one obtained in Eq. (39).

Example 3. Consider the following three dimensional nonhomogeneous singular parabolic partial differential equation:

$$\frac{\partial^2 u}{\partial t^2}+\frac{1}{4!z}\frac{\partial^4 u}{\partial y^4}+\frac{1}{4!z}\frac{\partial^4 u}{\partial z^4}=-\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right]\cos t, \tag{40}$$

with initial conditions:

$$u(x,y,z,0)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x},\quad \frac{\partial u}{\partial t}(x,y,z,0)=0, \tag{41}$$

and the boundary conditions:

$$\begin{aligned} &u\left(\frac{1}{2},y,z,t\right)=\left(\frac{1}{2y}+\frac{y}{z}+2z\right)\cos t,\quad u(1,y,z,t)=\left(\frac{1}{y}+\frac{y}{z}+z\right)\cos t,\\ &u\left(x,\frac{1}{2},z,t\right)=\left(2x+\frac{1}{2z}+\frac{z}{x}+\frac{z}{x}\right)\cos t,\quad u(x,1,z,t)=\left(x+\frac{1}{z}+\frac{y}{x}\right)\cos t,\\ &u\left(x,y,\frac{1}{2},t\right)=\left(2y+\frac{x}{y}+\frac{1}{2x}\right)\cos t,\quad u(x,y,1,t)=\left(y+\frac{x}{y}+\frac{1}{x}\right)\cos t,\\ &\frac{\partial u}{\partial x}\left(\frac{1}{2},y,z,t\right)=\left(\frac{1}{y}-4z\right)\cos t,\quad \frac{\partial u}{\partial x}(1,y,z,t)=\left(\frac{1}{y}-z\right)\cos t.\end{aligned}$$

Using the transformation $\frac{\partial u}{\partial t}=q(t)$ the above problem can be converted to the following system of partial differential equations:

$$\begin{cases} \frac{\partial u}{\partial t}=q(t)\\ \frac{\partial q}{\partial t}=\left(\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)\cos t+\frac{1}{4!z}\frac{\partial^4 u}{\partial x^4}+\frac{1}{4!x}\frac{\partial^4 u}{\partial y^4}+\frac{1}{4!y}\frac{\partial^4 u}{\partial z^4}\right)\end{cases} \tag{42}$$

with initial conditions:

$$x(x,y,z,0)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x},\quad q(x,y,z,0)=0.$$

The above system of differential equations can be converted to the following system of integral equations

$$\begin{cases} u(x,y,z,t) = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) + \int_0^t q(t)\mathrm{d}t \\ q(x,y,z,t) = \int_0^t \left(\frac{1}{4!z}\frac{\partial^4 u}{\partial x^4} + \frac{1}{4!x}\frac{\partial^4 u}{\partial y^4} + \frac{1}{4!y}\frac{\partial^4 u}{\partial z^4}\right)\mathrm{d}t \end{cases} \tag{43}$$

The correction functional is given as:

$$\begin{cases} u_{n+1}(x,y,z,t) = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) + \int_0^t \lambda_1(u_n - \tilde{q}(t))\mathrm{d}t \\ q_{n+1}(x,y,z,t) = \int_0^t \lambda_2\left(q_n - \left(\frac{1}{4!z}\frac{\partial^4 \tilde{u}_n}{\partial x^4} + \frac{1}{4!x}\frac{\partial^4 \tilde{u}_n}{\partial y^4} + \frac{1}{4!y}\frac{\partial^4 \tilde{u}_n}{\partial z^4}\right)\right)\mathrm{d}t \end{cases} \tag{44}$$

Applying the variational iteration method using He's polynomials (VIMHP), we get:

$$\begin{cases} u_0+pu_1+p^2u_2+\cdots = & \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)+p\int_0^t \lambda_1((u_0+pu_1+p^2u_2+\cdots)-(\tilde{q}_0+p\tilde{q}_1+p^2\tilde{q}_2+\cdots))\mathrm{d}t \\ q_0+pq_1+p^2q_2+\cdots = & p\int_0^t \lambda_2\left((q_0+pq_1+p^2q_2+\cdots)-\left(\frac{1}{4!z}\frac{\partial^4\tilde{u}_0}{\partial x^4}+p\frac{\partial^4\tilde{u}_1}{\partial x^4}+p^2p\frac{\partial^4\tilde{u}_1}{\partial x^4}+\cdots\right)\right)\mathrm{d}t \\ & -p\int_0^t \lambda_2\left(\frac{1}{4!x}\frac{\partial^4\tilde{u}_0}{\partial y^4}+p\frac{\partial^4\tilde{u}_1}{\partial y^4}+p^2p\frac{\partial^4\tilde{u}_1}{\partial y^4}+\cdots\right)\mathrm{d}t \\ & -p\int_0^t \lambda_2\left(\frac{1}{4!y}\frac{\partial^4\tilde{u}_0}{\partial z^4}+p\frac{\partial^4\tilde{u}_1}{\partial z^4}+p^2p\frac{\partial^4\tilde{u}_1}{\partial z^4}+\cdots\right)\mathrm{d}t \end{cases}$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as:

$\lambda_1 = -1, \quad \lambda_2 = -1,$

$$\begin{cases} u_0+pu_1+p^2u_2+\cdots = & \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)-p\int_0^t((u_0+pu_1+p^2u_2+\cdots)-(q_0+pq_1+p^2q_2+\cdots))\mathrm{d}t \\ q_0+pq_1+p^2q_2+\cdots = & -p\int_0^t\left((q_0+pq_1+p^2q_2+\cdots)-\left(\frac{1}{4!z}\frac{\partial^4u_0}{\partial x^4}+p\frac{\partial^4u_1}{\partial x^4}+p^2p\frac{\partial^4u_1}{\partial x^4}+\cdots\right)\right)\mathrm{d}t \\ & +p\int_0^t\left(\frac{1}{4!x}\frac{\partial^4u_0}{\partial y^4}+p\frac{\partial^4u_1}{\partial y^4}+p^2p\frac{\partial^4u_1}{\partial y^4}+\cdots\right)\mathrm{d}t \\ & +p\int_0^t\left(\frac{1}{4!y}\frac{\partial^4u_0}{\partial z^4}+p\frac{\partial^4u_1}{\partial z^4}+p^2p\frac{\partial^4u_1}{\partial z^4}+\cdots\right)\mathrm{d}t \end{cases}$$

Comparing the co-efficient of like powers of p, following approximants are obtained:

$$p^{(0)}:\begin{cases} u_0(x,y,z,t) = \frac{x}{y}+\frac{y}{z}+\frac{z}{x} \\ q_0(x,y,z,t) = 0 \end{cases} \tag{45}$$

$$p^{(1)}:\begin{cases} u_1(x,y,z,t) = \frac{x}{y}+\frac{y}{z}+\frac{z}{x} \\ q_1(x,y,z,t) = -\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\sin t+\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)(\sin t+t) \end{cases} \tag{46}$$

$$p^{(2)}:\begin{cases} u_2(x,y,z,t) = \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t+\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)(\cos t+\frac{t^2}{2!}-1) \\ q_2(x,y,z,t) = -\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\sin t+\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^3}\right)(\sin t+t) \end{cases} \tag{47}$$

$$p^{(3)}:\begin{cases} u_3(x,y,z,t) = & \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t+\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)\left(\cos t+\frac{t^2}{2!}-1\right) \\ q_3(x,y,z,t) = & -\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\sin t+\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^3}\right)(\sin t+t)-\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)t \\ & \cos+70\left(\frac{1}{zx^9}+\frac{1}{xy^9}+\frac{1}{yz^9}\right)\left(\sin t-t+\frac{t^3}{3!}\right) \end{cases} \tag{48}$$

$$p^{(4)}:\begin{cases} u_4(x,y,z,t) = & \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t+\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)\left(\cos t+\frac{t^2}{2!}-1\right) \\ & +\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t+70\left(\frac{1}{zx^9}+\frac{1}{xy^9}+\frac{1}{yz^9}\right)\left(1-\cos t-\frac{t^2}{2!}-\frac{t^4}{4!}\right) \\ & -\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\sin t+133650\left(\frac{1}{z^2x^{13}}+\frac{1}{x^2y^{13}}+\frac{1}{y^2z^{13}}\right)\left(\sin t-t-\frac{t^3}{3!}+\frac{t^5}{5!}\right) \end{cases} \tag{49}$$

The sequences tends to $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t$, as $n\to\infty$ therefore, the exact solution is given as:

$$u(x,y,z,t) = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t. \tag{50}$$

Now, we apply variational iteration method using Adomian's polynomials (VIDM) on the reformulated system of integral equations Eq. (36), we get:

$$\begin{cases} u_{n+1}(x,y,z,t) = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) - \int_0^t \lambda_1(u_n - \sum\limits_{n=0}^{\infty} q_n(t))\mathrm{d}t \\ q_{n+1}(x,y,z,t) = -\int_0^t \left(q_n - \left(\frac{1}{4!z}\sum\limits_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial x^4} + \frac{1}{4!x}\sum\limits_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial y^4} + \frac{1}{4!y}\sum\limits_{n=0}^{\infty}\frac{\partial^4 u_n}{\partial z^4}\right)\right)\mathrm{d}t \end{cases}$$

Consequently, we get the same approximants Eq. (45) $\sim$ Eq. (49) as that of VIMHP and hence our series solution is fully compatible with the one obtained in Eq. (50).

Example 4. Consider the following fourth-order singular parabolic partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u}{\partial x^4} = 0, \quad 0 < x < 1, \tag{51}$$

with initial conditions:

$$u(x,0) = x - \sin x, \quad \frac{\partial u}{\partial t}(x,0) = -(x - \sin x), \quad 0 < x < 1, \tag{52}$$

and the boundary conditions:

$$\begin{aligned} &u(0,t) = 0, \quad u(1,t) = e^{-t}(1 - \sin 1), \quad t > 0, \\ &\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1,t) = e^{-t}\sin 1, \quad t > 1. \end{aligned}$$

Using the transformation $\frac{\partial u}{\partial t} = q(t)$ the above problem can be converted to the following system of partial differential equations:

$$\begin{cases} \frac{\partial u}{\partial t} = q(t) \\ \frac{\partial q}{\partial t} = -\left(\frac{x}{\sin x} - 1\right)\frac{\partial^4 u}{\partial x^4} \end{cases} \tag{53}$$

with initial conditions:

$$u(x,0) = x - \sin x, \quad q(x,0) = -(x - \sin x).$$

The above system of differential equations can be converted to the following system of integral equations:

$$\begin{cases} u(x,t) = (x - \sin x) + \int_0^t q(t)\mathrm{d}t \\ q(x,t) = -(x - \sin x) + \int_0^t \left(-\frac{x}{\sin x} - 1\right)\frac{\partial^4 u}{\partial x^4}\mathrm{d}t \end{cases} \tag{54}$$

The correction functional is given as:

$$\begin{cases} u_{n+1}(x,t) = (x - \sin x) + \int_0^t \lambda_1(u_n - \tilde{q}(t))\mathrm{d}t \\ q(x,t) = -(x - \sin x) + \int_0^t \lambda_2\left(q_n - \left(-\frac{x}{\sin x} - 1\right)\frac{\partial^4 \tilde{u}_n}{\partial x^4}\right)\mathrm{d}t \end{cases} \tag{55}$$

Applying the variational iteration method using He's polynomials (VIMHP), we get:

$$\begin{cases} u_0 + pu_1 + p^2u_2 + \cdots = & (x - \sin x) + p\int_0^t \lambda_1((u_0 + pu_1 + p^2u_2 + \cdots) - (\tilde{q}_0 + p\tilde{q}_1 + p^2\tilde{q}_2 + \cdots))\mathrm{d}t \\ q_0 + pq_1 + p^2q_2 + \cdots = & -(x - \sin x) + \int_0^t \lambda_2\left((q_0 + pq_1 + p^2q_2 + \cdots) - \left(-\frac{x}{\sin x} - 1\right)\right. \\ & \left.\left(\frac{\partial^4 \tilde{u}_0}{\partial x^4} + p\frac{\partial^4 \tilde{u}_1}{\partial x^4} + \cdots\right)\right)\mathrm{d}t \end{cases}$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as:

$$\lambda_1 = -1, \quad \lambda_2 = -1,$$

$$\begin{cases} u_0 + pu_1 + p^2u_2 + \cdots = (x - \sin x) - p\int_0^t((u_0 + pu_1 + p^2u_2 + \cdots) - (q_0 + pq_1 + p^2q_2 + \cdots))\mathrm{d}t \\ q_0 + pq_1 + p^2q_2 + \cdots = -(x-\sin x)-\int_0^t\Big((q_0+pq_1+p^2q_2+\cdots)-(-\frac{x}{\sin x}-1)\left(\frac{\partial^4 u_0}{\partial x^4}+p\frac{\partial^4 u_1}{\partial x^4}+\cdots\right)\Big)\mathrm{d}t \end{cases}$$

Comparing the co-efficient of like powers of p, following approximants are obtained:

$$p^{(0)}: \begin{cases} u_0(x,t) = x - \sin x \\ q_0(x,t) = -(x - \sin x) \end{cases} \tag{56}$$

$$p^{(1)}: \begin{cases} u_1(x,t) = x - \sin x - (x - \sin x)t \\ q_1(x,t) = -(x - \sin x) - (x - \sin x)\left(t - \frac{t^2}{2!}\right) \end{cases} \tag{57}$$

$$p^{(2)}: \begin{cases} u_2(x,t) = (x - \sin x) - (x - \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) \\ q_2(x,y,z,t) = -(x - \sin x) - (x - \sin x)\left(t - \frac{t^2}{2!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) \end{cases} \tag{58}$$

$$p^{(3)}: \begin{cases} u_3(x,t) = & (x - \sin x) - (x - \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) \\ q_3(x,t) = & -(x-\sin x) - (x - \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) \\ & +(x - \sin x)\left(\frac{t^5}{5!} - \frac{t^6}{6!}\right) \end{cases} \tag{59}$$

$$p^{(4)}: \begin{cases} u_4(x,t) = & (x - \sin x) - (x - \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) \\ & +(x - \sin x)\left(\frac{t^6}{6!} - \frac{t^7}{7!}\right) \\ q_4(x,t) = & (x - \sin x) - (x - \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) \\ & +(x - \sin x)\left(\frac{t^5}{5!} - \frac{t^6}{6!}\right) + (x - \sin x)\left(\frac{t^7}{7!} - \frac{t^8}{8!}\right) \end{cases} \tag{60}$$

$$p^{(5)}: \begin{cases} u_5(x,t) = & (x - \sin x) - (x - \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) \\ & +(x - \sin x)\left(\frac{t^6}{6!} - \frac{t^7}{7!}\right) + (x - \sin x)\left(\frac{t^8}{8!} - \frac{t^9}{9!}\right) \\ q_5(x,t) = & (x - \sin x) - (x - \sin x)t + (x - \sin x)\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right) + (x - \sin x)\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right) \\ & +(x - \sin x)\left(\frac{t^5}{5!} - \frac{t^6}{6!}\right) + (x - \sin x)\left(\frac{t^7}{7!} - \frac{t^8}{8!}\right) + (x - \sin x)\left(\frac{t^9}{9!} - \frac{t^{10}}{10!}\right) \end{cases} \tag{61}$$

$\vdots$

The solution is given as:

$$u(x,t) = (x - \sin x)\left(1 - t + \frac{t^2}{2!} - \frac{t^2}{2!} + \frac{t^2}{2!} - \frac{t^2}{2!} + \cdots\right) = (x - \sin x)e^{-t}, \tag{62}$$

which is the exact solution. Now, we apply variational iteration method using Adomian's polynomials (VIDM) on the reformulated system of integral Eq. (47), we get:

$$\begin{cases} u_{n+1}(x,t) = (x - \sin x) - \int_0^t \left(u_n - \sum_{n=0}^{\infty} q_n(t)\right)\mathrm{d}t \\ q(x,t) = -(x - \sin x) - \int_0^t \left(q_n - \left(-\frac{x}{\sin x} - 1\right)\sum_{n=0}^{\infty} \frac{\partial u_n}{\partial x^4}\right)\mathrm{d}t \end{cases} \tag{63}$$

consequently, we get the same approximants Eqs. (56) $\sim$ (61) as that of VIMHP and hence our series solution is fully compatible with the one obtained in Eq. (62).

7 Conclusion

In this paper, we applied variational iteration method using He's polynomials (VIMHP) and variational iteration method using Adomian's polynomials (VIDM) to solve the singular fourth-order parabolic partial differential equations with variable co-efficient by converting them into systems of integro partial differential equations. The proposed methods are successfully implemented by using the initial conditions only. It is concluded that VIMHP is more user friendly and is easier to implement as compare to VIDM.

References

[1] S. Abbasbandy. A new application of He's variational iteration method for quadratic riccati differential equation by using adomian's polynomials. *Journal of Computational and Applied Mathematics*, 2007, **207**: 59–63.

[2] S. Abbasbandy. Numerical solutions of nonlinear klein-gordon equation by variational iteration method. *International Journal of Mechanical Engineering Education*, 2007, **70**: 876–881.

[3] M. Abdou, A. Soliman. New applications of variational iteration method. *Journal of Physics D: Applied Physics*, 2005, **211**(1-2): 1–8.

[4] C. Andrade, S. McKee. High frequency A.D.I. methods for fourth-order parabolic equations with variable coefficients. *International Journal of Computational and Applied Mathematics*, 1977, **3**: 11–14.

[5] S. Conte. A stable implicit difference approximation to a fourth-order parabolic equation. *Journal of Association for Computing Machinery*, 1957, **4**: 210–212.

[6] S. Conte, W. Royster. Convergence of finite difference solution to a solution of the equation of a vibration beam. **in:** *Proceedings of the American Mathematical Society*, vol. 7, 1956, 742–749.

[7] D. Evans. A stable explicit method for the finite difference solution of fourth-order parabolic partial differential equations. *The Computer Journal*, 1965, **8**: 280–287.

[8] D. Evans, W. Yousef. A note on solving the fourth-order parabolic equation by a age method. *International Journal of Computer Mathematics*, 1991, **40**: 93–97.

[9] A. Ghorbani, J. Nadjfi. He's homotopy perturbation method for calculating adomian's polynomials. *The International Journal of Nonlinear Sciences and Numerical Simulation*, 2007, **8**(2): 229–332.

[10] D. Gorman. *Free vibrations analysis of beams and shafts*. Wiley, New York, 1975.

[11] J. He. A new approach to nonlinear partial differential equations. *Communications in Nonlinear Science and Numerical Simulation*, 1997, **2**(4): 203–205.

[12] J. He. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, 1999, 178–257.

[13] J. He. Variational iteration method—a kind of non-linear analytical technique: some examples. *International Journal of Non-Linear Mechanics*, 1999, **34**(4): 699–708.

[14] J. He. A coupling method of homotopy technique and perturbation technique for nonlinear problems. *International Journal of Non-Linear Mechanics*, 2000, **35**(1): 115–123.

[15] J. He. Variational iteration method for autonomous ordinary differential systems. *Applied Mathematics and Computation*, 2000, **2-3**: 115–123.

[16] J. He. Comparison of homotopy perturbation method and homotopy analysis method. *Applied Mathematics and Computation*, 2004, **156**: 527–539.

[17] J. He. The homotopy perturbation method for nonlinear oscillators with discontinuities. *Applied Mathematics and Computation*, 2004, **151**: 287–292.

[18] J. He. Homotopy perturbation method for solving boundary value problems. *Physics Letters A*, 2004, **156**: 87–88.

[19] J. He. A generalized variational principle in micro phonic thermo elasticity. *Mechanics Research Communications*, 2005, **32**: 93–98.

[20] J. He. Homotopy perturbation method for bifurcation of nonlinear problems. *Communications in Nonlinear Science and Numerical Simulation*, 2005, **6**(2): 207–208.

[21] J. He. Some asymptotic methods for strongly nonlinear equation. *International Journal of Modern Physics*, 2006, **20**(10): 1144–1199.

[22] J. He. Variational iteration method—some recent results and new interpretations. *The Journal of Computational and Applied Mathematics*, 2007, **207**: 3–17.

[23] J. He, X. Wu. Construction of solitary solution and compacton-like solution by variational iteration method. *Chaos, Solitons & Fractals*, 2006, **29**(1): 108–113.

[24] J. He, X. Wu. Variational iteration method: new development and applications. *Applied Mathematics and Computation*, 2007, **54**: 881–894.

[25] A. Khaliq, E. Twizell. A family of second-order methods for variable coefficient fourth-order parabolic partial differential equations. *International Journal of Computer Mathematics*, 1987, **23**: 63–76.

[26] S. Mohyud-Din. Solving heat and wave like equations using He's polynomials. *Mathematical Problems in Engineering, Hindawi*, 2009. Article ID 427516.

[27] S. Mohyud-Din. *Variational iteration techniques for boundary value problems*. VDM Verlag, 2010. ISBN 978-3-639-27664-0.

[28] S. Mohyud-Din, M. Noor. Homotopy perturbation method for solving fourth-order boundary value problems. *Mathematical Problems in Engineering, Hindawi*, 2007, 1–15. Doi:10.1155/2007/98602.

[29] S. Mohyud-Din, M. Noor, et al. Solution of singular equations by He's variational iteration method. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2010, **11**(2): 81–86.

[30] S. Mohyud-Din, M. Noor, et al. Variational iteration method for re-formulated partial differential equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2010, **11**(2): 87–92.
[31] S. Mohyud-Din, M. Noor, K. Noor. Parameter-expansion techniques for strongly nonlinear oscillators. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2009, **10**(5): 581–583.
[32] S. Mohyud-Din, M. Noor, K. Noor. Some relatively new techniques for nonlinear problems. *Mathematical Problems in Engineering, Hindawi*, 2009, 25. DOI:10.1155/2009/234849.
[33] S. Mohyud-Din, M. Noor, K. Noor. Travelling wave solutions of seventh-order generalized KDV equations using He's polynomials. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2009, **10**(2): 223–229.
[34] S. Mohyud-Din, M. Noor, K. Noor. Variational iteration method for flieral-petviashivili equations using He's polynomials and pade approximants. *World Applied Sciences Journal*, 2009, **6**(8): 1298–1303.
[35] S. Mohyud-Din, A. Yildirim, et al. Modified variational iteration method for free-convective boundary-layer equation using pade approximation. *Mathematical Problems in Engineering, Hindawi*, 2010, 11. Doi:10.1155/2010/318298.
[36] M. Noor, S. Mohyud-Din. Homotopy perturbation method for nonlinear higher-order boundary value problems. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2008, **9**(4): 395–408.
[37] M. Noor, S. Mohyud-Din. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials. *International Journal of Nonlinear Sciences and Numerical Simulation*, 2008, **9**(2): 141–157.
[38] A. Wazwaz. On the solution of the fourth-order parabolic equations by the decomposition method. *International Journal of Computer Mathematics*, 1995, **57**: 213–217.
[39] A. Wazwaz. Analytic treatment for variable coefficient fourth-order parabolic partial differential equations. *Applied Mathematics and Computation*, 2001, **123**: 219–227.