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Vibration analysis of multi-rotor system through extended Lagrangian formalism*

Chandan Kumar[1] , Vikas Rastogi[2†]

[1] Research Scholar, Department of Mechanical Engineering, SLIET, Longowal, Punjab 148106, India
[2] Associate Professor, Department of Mechanical Engineering, SLIET, Longowal, Punjab 148106, India

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Abstract. This work deals with the applications of extended Lagrangian-Hamiltonian mechanics on the asymmetric multi rotor system, where the symmetries are broken due to rotor mounted on the shaft with different masses and dimensions. The whole dynamics of asymmetries of multi-rotor system is investigated through extended Lagrangian formalism with a case study. In this work, an analytical model through extended Lagrangian- Hamiltonian approach is developed considering a two-rotor system with different masses. The amplitude and natural frequency of the multi-rotor system are determined analytically. The multi-rotor system is modeled through bond graph modeling techniques as it brings the unified modeling approach in several energy domain. Symbol-shakti® software is used for the simulation of the model. The effects of the mass of the rotor system on the amplitudes and frequencies are analyzed using numerical simulations. It is further observed that amplitude of the rotor increases with the mass of the rotors, which are also affirmed theoretically.

Keywords: stiffness, multi rotor, asymmetries, amplitude

1 Introduction

In most of the applications more than one disc are mounted on shafts, accurate prediction of dynamic behaviour of multi rotor system is very important to ensure the efficient and effective working of rotor. It also helps in taking necessary corrective or preventive measures well before critical to the structures and machineries. The purpose of this investigation is to develop and test models for the vibration analysis of asymmetric multi-rotors. These asymmetries may be of geometric in a nature or may be due to the shaft crack. This research is widely studied by many researchers because of possible sudden catastrophic failure of a rotor from fatigue. Sometimes, stress concentration factor, high rotational speeds, inhomogeneous material properties also exaggerate the problems. Dimarogonas [7] provided an earlier literature review of the vibration of the cracked and asymmetric structures and cited more than 250 papers. In his review, several aspects of local flexibility due to cracks, non-linearities introduced into the system and local stiffness matrix descriptions of the asymmetric section. Few researchers have analysed the multi-rotor system using additional external excitation such as active magnetic bearings. Muszynska et al. [21] and Bently et al. [2] studied rotor-coupled lateral and torsional vibrations due to inbalance as well as due to shaft asymmetry under a constant force. More recently, Wu and Meagher [28] has presented a mathematical model of a crack rotor and an asymmetric rotor with two disks representing a turbine and generator to study the vibration due to imbalance and side load. In this study, analytical model is derived from Lagrange's equation taking into considerations of lateral/torsional coupling. However, the amplitude variations due to asymmetries are not considered in their model. Several analytical techniques are used by various researchers to analyze the dynamic behaviour of multi-rotor system. Through

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† Corresponding author. Tel.: +91-1672-253306. *E-mail address*: rastogivikas@yahoo.com.

Lagrangian mechanics, very few works have been done to study the dynamic behaviour of multi-rotor. It is well known phenomenon that asymmetric rotating component produces non potential and dissipative forces, so some additional information of system interior and exterior is needed in generating extended Lagrangian equations[3, 10, 12], which may be applicable to the multi-rotor system.

The symmetries of the system provide useful information to derive the constant of motion of the system as they reduce the complexity of the dynamical systems such as multi-rotor. Noether's theorem [22] played a significant role in determination of invariants of motion. Extended Noether's theorem with Umbra's Hamiltonian provides the insight of dynamics of the asymmetric rotor system. Some significant works in this area were reported by Vujanovic [26, 27] and Djukic [8, 9] in their several papers. Vujanovic [26, 27] has formulated a method for finding the conserved quantities of non-conservative holonomic system, which is based on the differential variational principle of d'Alembert. Some other useful results related with the symmetry aspects of Lagrangian and Hamiltonian formalism are discussed in papers of Katzin and Levine [13], Sarlet and Cantrazin [24] and Simic [25]. Recently, Arizmendi et al. [1] has defined new Lagrangian in extended configuration space comprising of both the original configuration of the systems and all virtual displacements joining any two integral curves. In other papers, Damianou and Sophocleous [5, 6] classified the symmetries (Lie point and Noether) of Hamiltonian systems with two and three degree of freedom.

To enlarge the scope of Lagrangian-Hamiltonian mechanics, a new proposal of additional time like variable 'umbra-time' was made by Mukherjee [14] and new form of equation termed as Umbra Lagrangian is formed. A brief and candid commentary on this idea is given by Brown [4]. This idea was further consolidated by presenting an important issue of invariants of motion for the general class of system by extending Noether's theorem[15]. One of the most important insights gained from the umbra-Lagrangian formalism is that its underlying variational principle[19] is possible, which is based on their cursive minimization of functions. The umbra Lagrangian theory has been used successfully to study invariants of motion for non-conservative mechanical and thermo-mechanical systems[17]. Recently, Mukherjee et. al [18] applied umbra Lagrangian to study dynamics of an electro-mechanical system comprising of an induction motor driving an elastic rotor. Rastogi and Kumar [23] have applied this theory on single rotor system to study dynamic behaviour of asymmetric rotor due to stiffness variation.

The main focus of the present paper is to develop a mathematical model of multi-rotor system through extended Lagrangian Formalism. The main contribution of the paper is to find invariants of motions through umbra-Lagrangian, Umbra-Hamiltonian and extended Noether's theorem. One may obtain amplitudes of both rotors (If two rotors are taken as case study) analytically and this methodology is quite useful, when vibration amplitude increases with mass asymmetry or stiffness asymmetry. This is in line to broaden the scope of umbra-Lagrangian formalism applied to rotor dynamic research.

The paper is organised as follows: Second section present the basic idea of umbra-Lagrangian and umbra-Hamiltonian, which are used to analyze multi-rotor system. In section three, case study of two rotor system is presented alongwith its mathematical model and bond graph model. Section four gives simulation studies conducted for the parameters to elucidate the results.

2 Methodology

Mukherjee [14] introduced a concise and modified form of Lagrange's equation and manifested the use of this new scheme to arrive at system models in the presence of time fluctuating parameters, general dissipation and gyroscopic couplings etc. In this scheme, real and virtual energies (or work) are separated by introduction of an additional time like parameter, which is termed as "umbra-time". The prefix "umbra" was appended to all type of energies, and corresponding Lagrangian was termed as the "umbra Lagrangian". The basic idea presented in [19] leading to umbra-Lagrangian and umbra-Lagrange's equation may be briefly expressed as follows:

(1) Umbra-time is the beholder of d'Alembert's basic idea of allowing displacements, when the real time is frozen.

(2) Umbra-time may be viewed as the interior time of a system.

(3) Potential, kinetic and co-kinetic energies stored in storage elements like symmetric compliant and inertial

fields can be expressed as functions in umbra-time (umbra-displacements and umbra-velocities).
(4) The effort of any external force, resistive element or field, gyroscopic element (treated as anti-symmetric resistive field), transformer or lever element, anti-symmetric compliant field and sensing element depends on displacements and velocities in real time. The potentials associated with them are obtained by evaluation of work-done through umbra-displacements.
(5) The broad principle on which the creation of umbra-Lagrangian and other relevant energies are based can be summarized in [19] and Umbra-Lagrange's equation[17, 18] for a general class of systems may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\lim_{\eta\to t}\frac{\partial L^*}{\partial \dot{q}_i(\eta)}\right\} - \lim_{\eta\to t}\frac{\partial L^*}{\partial q_i(\eta)} = 0,$$

where a dot represents the derivative. The detailed proposal of this Equation is presented in [19]. The umbra-Hamiltonian may be represented as

$$[H^*[q(\eta), p(\eta), q(t), \dot{q}(t), t] = \dot{q}(\eta)p(\eta) - L^*[q(\eta), \dot{q}(\eta), q(t), t],$$
$$H^* = H_i^*\{q(\eta), p(\eta)\} + H_e^*\{q(\eta), p(\eta), q(t), \dot{q}(t), t\}.$$

The umbra-Hamiltonian H^* is composed of two components as H_i^* and H_e^*, H_i^* is the interior Hamiltonian, which does not depend on any function of real displacement, real velocity and real time, and is the rest of the umbra-Hamiltonian, called the exterior Hamiltonian. The two theorems of umbra-Hamiltonian have been used to analyze the dynamic behaviour of multi-rotor system. The detailed proofs of three theorems are presented in [19].

Theorem 1.

$$\lim_{\eta\to t}\left[\frac{dH^*}{d\eta}\right] = 0,$$

Theorem 2.

$$\frac{dH_i^*}{dt} = -\lim_{\eta\to t}\left[\frac{dH_e^*}{d\eta}\right].$$

Corollary of Theorem 2:
If for a system $\lim_{\eta\to t}[\frac{\mathrm{d}H_e^*}{\mathrm{d}\eta}]$, then $H_i^*(q(t), p(t))$ is a constant of motion.

3 Case study of two rotor system

In this case study, a rotor carrying two identical discs mounted on the equal distance as shown in Fig. 1.Rotor is driven by a DC motor and shaft is supported by two bearings. One may approximate the rotor by two discrete pairs of equivalent springs and dashpots in the two orthogonal transverse directions. Such dampers include external damping and a part of the internal damping. The additional circulatory forces due to the internal damping are introduced using modulated gyrator elements in the bond graph model. The twisting of the shaft is assumed to be negligible on account of high torsional stiffness. The bond graph for the complete system is shown in Fig. 2.

The artificial flow sources are incorporated and causalities of the I-elements are reversed in order to follow the procedure of extended Karnopp's algorithm to obtain umbra-Lagrangian of the system. The details of extended Karnopp's Algorithm have been presented in [11, 18].

3.1 The umbra-lagrangian of two-rotor system

The umbra-Lagrangian for the electro-mechanical system as shown in Fig. 1 is obtained following the procedure of Extended Karnopp's algorithm discussed in [11, 18]. The umbra-Lagrangian corresponding to the bond graph model shown in Fig. 2 may be expressed as

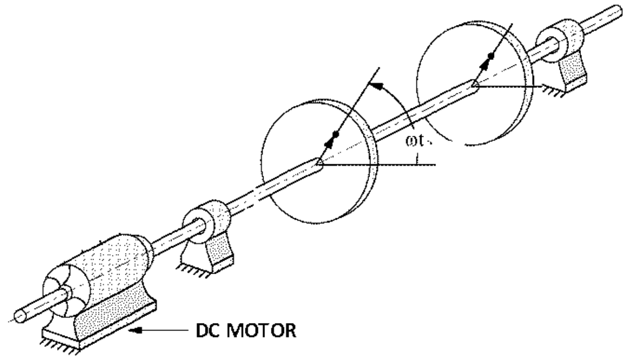

Fig. 1. Multi-rotor system with DC motor

$$\begin{aligned} L^* =& \frac{1}{2}\{(m_1(\dot{x}_1(\eta)+\dot{y}_1(\eta)) - k_1(x_1^2(\eta)+y_1^2(\eta)) + J\dot{\theta}^2(\eta) + m_2(\dot{x}_2^2(\eta)+\dot{y}_2^2(\eta)) - k_2(x_2^2(\eta)+y_2^2(\eta))\} \\ & - [x_1(\eta)y_1(\eta)]\begin{bmatrix} 0 & R_i\dot{\theta}(t) \\ -R_i\dot{\theta}(t) & 0 \end{bmatrix}\begin{Bmatrix} x_1(t) \\ y_1(t) \end{Bmatrix} - [x_2(\eta)y_2(\eta)]\begin{bmatrix} 0 & R_i'\dot{\theta}(t) \\ -R_i'\dot{\theta}(t) & 0 \end{bmatrix}\begin{Bmatrix} x_2(t) \\ y_2(t) \end{Bmatrix} \\ & - (R_a+R_i)(x_1(\eta)\dot{x}_1(t)+y_1(\eta)\dot{y}_1(t)) - (R_a+R_i)(x_2(\eta)\dot{x}_2(t)+y_2(\eta)\dot{y}_2(\eta)). \end{aligned} \tag{1}$$

In Eq. (1), m_1 and m_2 are the mass of two rotors, K is the stiffness of the shaft, J is the moment of inertia of the rotor mass, R_i and R_a are the internal and external damping of the rotor, $x()$ and $y()$ are the displacements in real or umbra time, $\theta()$ is the angular displacements in η or t times, and R_b is the resistance of bearings.

First Noether's rate equation [19] for the first rotor may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\{m_1(\dot{x}_1(t)y_1(t)-\dot{y}_1(t)x_1(t))\} = -\dot{\theta}(t)R_i(x_1^2(t)+y_1^2(t)) + (R_a+R_i)(x_1(t)\dot{y}_1(t)-\dot{x}_1(t)y_1(t)). \tag{2}$$

In Eq. (2), the left hand side term is the classical Noether's term and this momentum may be conserved by making the term on right hand side, i.e., modulatory convection term zero on certain circular trajectories by assuming an orbit for the rotor mass centre as

$$x_1(t) = A\cos(\omega_1 t) \text{ and } y_1(t) = A\sin(\omega_1 t), \tag{3}$$

where ω is the natural frequency of the limiting orbit of the shaft. The right hand side term of Eq. (2) yields

$$-\dot{\theta}(t)R_iA^2 + (R_a+R_i)A^2\omega_1 = 0, \quad \dot{\theta}(t) = \frac{\omega(R_a+R_i)}{R_i}, \quad \dot{\theta}(t) = \omega_1\left(1+\frac{R_a}{R_i}\right). \tag{4}$$

In the similar manner, one may obtain second Noether's rate equation for the rotor two, which may be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\{m_2(\dot{x}_2(t)y_2(t)-\dot{y}_2(t)x_2(t))\} = -\dot{\theta}(t)R_i'(x_2^2(t)+y_2^2(t)) + (R_a+R_i)(x_2(t)\dot{y}_2(t)-\dot{x}_2(t)y_2(t)). \tag{5}$$

Again assumed the right hand side, i.e. modulatory convection term zero on certain circular trajectories by assuming an orbit for the rotor mass center as

$$x_2(t) = A'\cos(\omega_2 t) \text{ and } y_2(t) = A'\sin(\omega_2 t), \tag{6}$$

The right hand side of Eq. (5) yields

$$\dot{\theta}'(t) = \frac{\omega_2(R_a'+R_i')}{R_i'}, \quad \dot{\theta}'(t) = \omega_2\left(1+\frac{R_a'}{R_i'}\right). \tag{7}$$

In Eqs. (4) and (7), $\dot{\theta}(t)$ and $\dot{\theta}'(t)$ are the shaft spinning speeds of rotor one and rotor two.

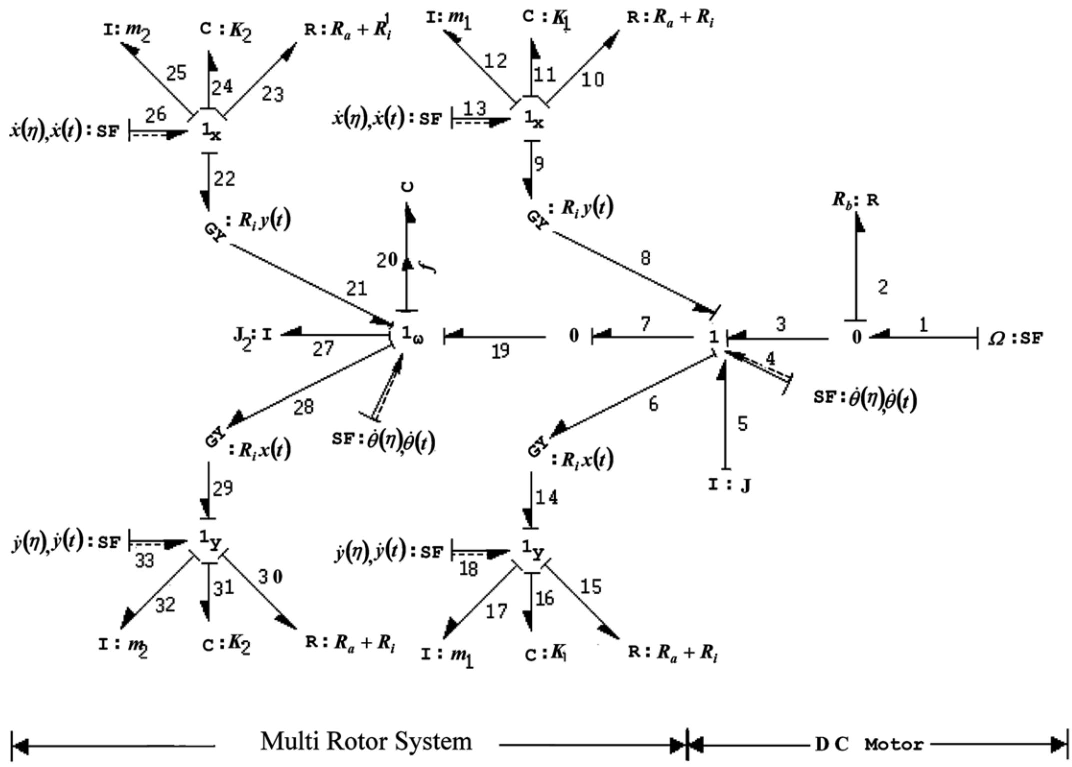

Fig. 2. Bond graph model of multi-rotor with DC motor with artificial flow sources to obtain umbra-Lagrangian

3.2 Umbra hamiltonian of two-rotor system

The umbra-Hamiltonian for the case of electro-mechanical system may be expressed as

$$\begin{aligned} H^* =&\frac{1}{2}\left\{\frac{1}{m_1}\left(p_{x_1}^2(\eta)+p_{y_1}^2(\eta)\right)+k(x_1^2(\eta)+y_1^2(\eta))+\frac{1}{m_2}\left(p_{x_2}^2(\eta)+p_{y_2}^2(\eta)\right)+k(x_2^2(\eta)+y_2^2(\eta))+\frac{1}{2J}p^2\theta(\eta)\right\} \\ &+[x_1(\eta)y_1(\eta)]\begin{bmatrix} 0 & R_i\dot{\theta}(t) \\ -R_i\dot{\theta}(t) & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}+[x_2(\eta)+y_2(\eta)]\begin{bmatrix} 0 & R_i'\dot{\theta}(t) \\ -R_i'\dot{\theta}(t) & 0 \end{bmatrix}\begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} \\ &+(R_a+R_i)(\dot{x}_1(t)x_1(\eta)+\dot{y}_1(t)y_1(\eta))+(R_a'+R_i')(\dot{x}_2(t)x_2(\eta)+\dot{y}_2(t)y_2(\eta)) \\ &+\left\{R_i(x_1(t)\dot{y}_1(t)-y_1(t)\dot{x}_1(t))+R_i'(x_2(t)\dot{y}_2(t)-y_2(t)\dot{x}_2(t))+R_c(\dot{\theta}(t)-\Omega)\right\}. \end{aligned} \tag{8}$$

Now, finding $\lim_{\mathrm{d}H_e^*/\mathrm{d}\eta}$ for Eq. (8) and assuming certain circular trajectories as per Eqs. (3) and (6) , one may obtain the amplitude equation of the rotor one and rotor two, which may be written as

$$A=\frac{R_c(\Omega-\dot{\theta}(t))}{R_i\omega_1}, \tag{9}$$

and

$$A'=\frac{R_c(\Omega-\dot{\theta}'(t))}{R_i\omega_2}, \tag{10}$$

where $\dot{\theta}(t)$ and $\dot{\theta}'(t)$ are the threshold speed of rotation. ω_1 and ω_1 are the natural frequencies of the rotor 1 and rotor 2. If internal damping and natural frequency are $R_1=R_i'=R_i$, $\omega=\omega_1+\omega_2$, one may obtain $A=A'$

$$A=\frac{R_c(\Omega-\dot{\theta}'(t))}{2R_i\omega}. \tag{11}$$

4 Simulation study

The purpose of the simulation study is to gain an insight view into the dynamic behaviour of a multi-rotor system. Moreover, it provides validation of the theoretical results obtained in Section 3. The bondgraph model of a motor with two-rotor with internal and external damping was simulated on the software SYMBOLS-Shakti [16, 20], with parameters shown in Tab. 1.

Table 1. Simulation parameters

Parameter	Parameter	Value
Stiffness of the rotor 1 and rotor 2	$K_1 = K_2$	100 N/m
Mass of the rotor 1 and rotor 2	m1=m2	1,1.2,1.4,1.6 Kg
Mass momentum of inertia for the rotor 1 and rotor 2	J1=j2	1kg-m2
Internal damping coefficient of rotor 1 and 2	Ri	5 N-s/m
External damping coefficient of rotor 1 and rotor 2	Ra	5 N-s/m
Damping coefficient of flexible couplings	Rc	0.2 N-s/m
Excitation frequency		22 rad/sec

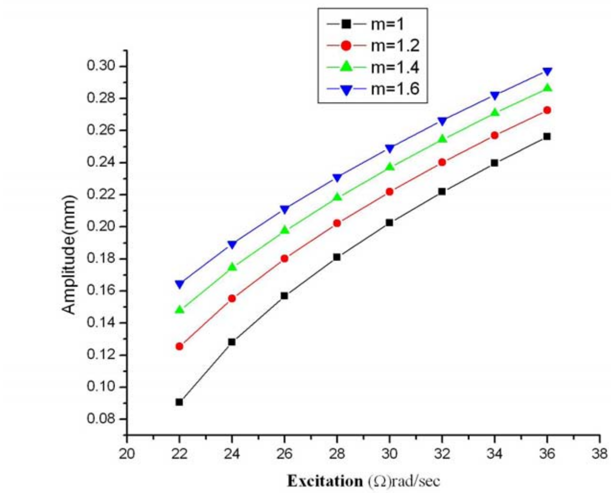

Fig. 3. Amplitude of the rotor at different excitation frequencies and masses

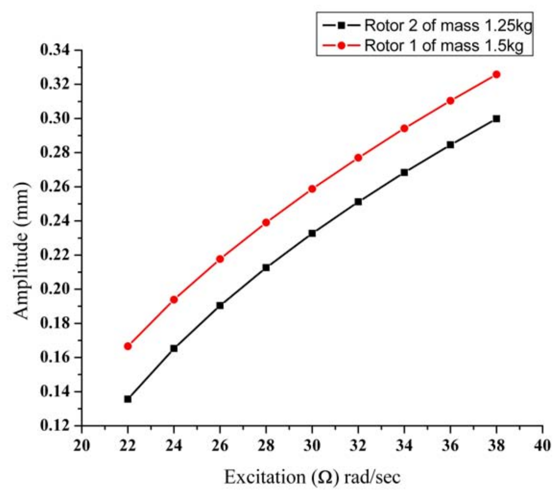

Fig. 4. Excitation Frequency Vs Amplitude for multi-rotor system with non-identical masses

Fig. 3 shows the amplitudes at different excitation frequencies with different masses for the two-rotor system, where two discs mounted on shaft are identical. Amplitude of the rotor increase with the excitation frequency. Amplitude of the rotor also increases with mass. It is well known fact that the natural frequency of the rotor decrease with increase in mass, and simultaneously the amplitude of the rotor increases as per Eq. (11), where amplitude is inversely proportional to the natural frequency of the rotor.

Fig. 4 shows the amplitude of rotor with non-identical mass. The amplitude of vibration of rotor 1 and rotor 2 are plotted at different excitation frequency. The other parameters of the simulation are same. In this case, amplitude is different for each of the rotor. It is also apparent that that amplitude increases with the excitation frequency and with mass.

Fig. 5 shows the amplitude of the rotor at various natural frequencies are shown with different excitation frequency. It is shown that the amplitude decrease with the natural frequency. However, it increases with increase in excitation frequencies. One may easily diagnose any faults in the rotor system by knowing the amplitude of the rotor by using Eq. (11), which is obtained through extended Lagrangian Formalism.

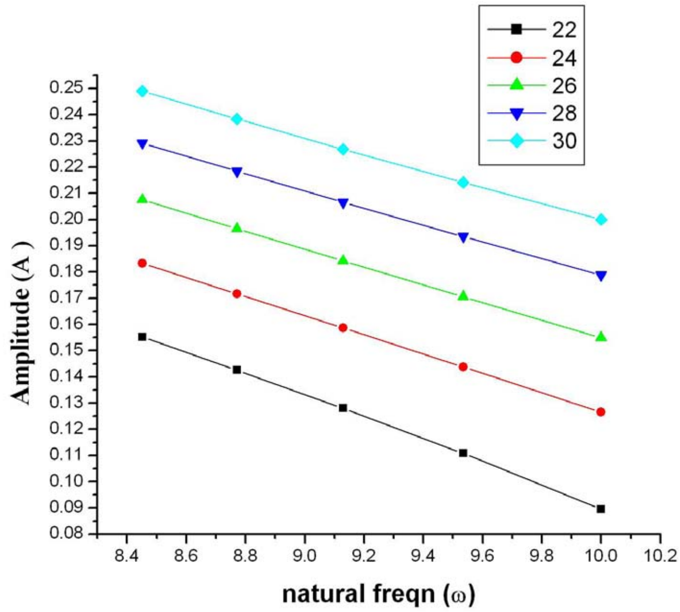

Fig. 5. Amplitude Vs natural frequency

5 Conclusions

The investigation of dynamics of multi-rotor system (two rotor case) has been carried out through extended Lagrangian-Hamiltonian approach. The umbra-Lagrangian for such a system was generated using its bondgraph model. The amplitude equation of rotor 1 and rotor 2 has been achieved through this approach, which are many times very difficult to find out. Analytical results were validated through simulations. Amplitude of the vibration was determined for the two rotor system, which may or may not be identical and results shows considerable agreement with the simulation studies.

Moreover, this study can be applied to three rotors, four rotors or more rotors mounted on the same shaft. This study can further be applied to investigate crack on any one of the rotor, which may be significantly useful in rotor dynamic and power plant industries.

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