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Free flexural vibration analysis of stiffened plates with general elastic boundary supports

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Abstract. The free flexural vibration of a stiffened rectangular plate is investigated by using the finite element method (FEM). In the present model, the plate can have arbitrary elastic boundary conditions and arbitrarily located stiffeners. Numerical studies are conducted to analyze the natural frequencies of concentrically/eccentrically stiffened plates with different boundary conditions and the results show good agreement with earlier published results. The present model is also applied to parametric studies examining the effects of the stiffener on the natural frequencies of a glass window. The results show that the natural frequencies of the window are notably influenced by the stiffener and the stiffener's location. The results also demonstrate the feasibility and effectiveness of the present model.

Keywords: vibration, FEM, stiffener, boundary conditions

1 Introduction

Thin plate is widely used as primary structural components for variety of applications. The characteristics of thin plate vibration, sound radiation and transmission have been extensively studied[1, 3, 16, 27]. It is known from these studies that the properties of the boundary supports have significant effects on the vibration (or sound radiation) response of the plate structure. In order to further understand the effects of different boundary supports, in recent years numerical prediction methods[7, 9, 17, 18, 21, 22] have been constantly developed that are not only limited to the application to the classical boundary conditions but also to the non-classical boundary conditions.

Thin plates stiffened by stiffeners (like ribs) have also been widely used in various engineering areas and its applications can be found in buildings, aircraft, ships and many other industries. The stiffeners can be employed to reinforce the plate and enhance the plate's load-carrying capacities. On the other hand, adding stiffeners to a structure can influence structure dynamic characteristics. Therefore, several studies have been conducted to search for better stiffener design in order to improve the plate's vibration isolation (or sound insulation) property[12, 13, 15].

In the structural dynamic analysis, FEM has several advantages over most other numerical analysis methods[24], such as: (1) it has no geometric restriction; (2) boundary conditions and loading are not restricted; (3) material properties are not restricted; (4) different types of components can be combined; (5) the approximation can be easily improved by grading the mesh. Two recent studies by Ou et al. [21, 22] showed that the FEM method was suitable for the vibration analysis of the plate structures with arbitrary elastic boundary conditions. Meanwhile, various types of stiffened-plate elements have also been developed that can be used to analyze plates with arbitrary shapes and disposition of stiffeners[4, 5, 11].

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Both the boundary conditions and the stiffeners have important effects and cannot be neglected in the vibration analysis of a practical stiffened plate structure. The purpose of the current study is to present a simple but effective method for the free flexural vibration analysis of plate structures with arbitrary elastic boundary conditions and arbitrarily located stiffeners. A four-node stiffened-plate bending element is presented for this purpose. A brief summary of the FEM formulation by using this element is given in Section 2. Section 3 reports the numerical results for the natural frequencies of various stiffened plates and compares them with earlier published results. It also reports the parametric studies on a glass window with non-classical boundary condition to check the effects of the stiffener on the window's natural frequencies. Finally, Section 4 presents the conclusions.

2 Analysis and basic functions

2.1 Description of the model

Consider a thin rectangular plate (length Lx, width Ly and thickness h) stiffened by a stiffener (or stiffeners) at arbitrary position on the plate, as shown in Fig. 1. The boundary of the plate is elastically restrained and is idealized by combining translational (k_{tb}) and rotational (k_{rb}) springs[6, 17, 21, 22].

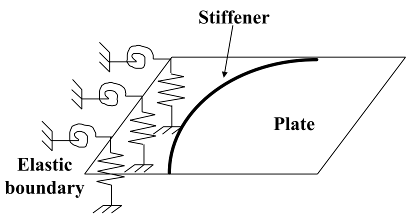

Fig. 1. Schematic illustration of a stiffened rectangular plate system. (for simplicity and clarity, only the supports along the left edge are shown)

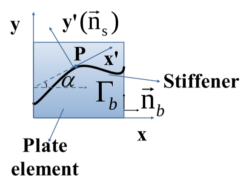

Fig. 2. The stiffened plate element. A local (x', y') axis is set along the tangent to the stiffener at the integration point P making an angle α with the global (x, y) axis

2.2 Free vibration analysis

The equilibrium equation of undamped free vibration of the plate system, as shown in Fig. 1, undergoing small displacements can be expressed in matrix form as

$$[M]\ddot{\delta} + [K]\{\delta\} = \{F\}, \tag{1}$$

where $[M]$, and $[K]$ are the global mass and stiffness matrices of the plate system, $\{\delta\}$ is the global nodal displacement vector, $\{F\}$ is the generalized force vector. To compute the natural frequencies of the system by considering the eigenvalue problem, Eq. (1) can be simplified as

$$(-\omega^2[M] + [K])\{\varphi\} = \{0\}, \tag{2}$$

where ω is the natural angular frequency and $\{\varphi\}$ is the corresponding modal vector. The eigenvalue ω^2 and the eigenvector can now be obtained from Eq. (2). The simultaneous iteration algorithm of Corr and Jennings [8] can be used to solve this eigenproblem.

The four-node rectangle Kirchoff plate element [24] is used in the FEM model. The mesh size of the element is determined by considering both the solution accuracy and computational cost. A suggestion proposed by Kim et al. [14] is a mesh size equal to one quarter of the wavelength of the highest frequency of interest. The formulation of the global mass matrix $[M]$ and stiffness matrix $[K]$ for the stiffened plate system with elastic boundary supports are discussed below.

2.3 Stiffness and mass matrices of the stiffened plate

The stiffness matrix $\{K\}$, in Eq. (2), of the whole plate system is decomposed into plate, boundary supports and stiffeners contributions, and can be expressed as $\{K\} = \{K_p\} + \{K_b\} + \{K_s\}$, where $\{K_p\}$, $\{K_b\}$, $\{K_s\}$ are the stiffness matrices for the plate, boundary supports and stiffeners, respectively. The mass matrix $\{M\}$, in Eq. (2), of the whole plate system is decomposed into plate and stiffeners contributions, and can be expressed as $\{M\} = \{M_p\} + \{M_s\}$, where $\{M_p\}$ and $\{M_s\}$ are the mass matrices for the plate and stiffeners, respectively. In general, the mass effect of the elastic boundary supports can be neglected and supposed to be a null mass component[6, 17, 21, 22].

The total strain energy Π of the stiffened plate element, as shown in Fig. 2, is given by

$$\Pi_e = \Pi_{pe} + \Pi_{be} + \Pi_{se}, \tag{3}$$

where the stain energy of the plate element Π_{pe}, the stain energy of the boundary support in the plate element Π_{be}, and the stain energy of the stiffener in the plate element Π_{se} can be expressed by,

$$\Pi_{pe} = \frac{1}{2}\{\delta\}_e^T\{K_p\}_e\{\delta\}_e, \quad \Pi_{be} = \frac{1}{2}\{\delta\}_e^T\{K_b\}_e\{\delta\}_e, \quad \Pi_{se} = \frac{1}{2}\{\delta\}_e^T\{K_s\}_e\{\delta\}_e,$$

where $\{\delta\}_e$ is the nodal displacement vector of the plate element. $\{K_p\}_e$, $\{K_b\}_e$ and $\{K_s\}_e$ are the element stiffness matrices of $\{K_p\}$, $\{K_b\}$ and $\{K_s\}$, which can be expressed by,

$$\{K_p\}_e = \int \{B_p\}^T\{D_p\}\{B_p\}\mathrm{d}x\mathrm{d}y, \quad \{K_b\}_e = \int \left(k_{tb}\{N\}^T\{N\} + k_{rb}\left\{\frac{\partial N}{\partial \vec{n}_b}\right\}\right)\mathrm{d}\Gamma_b,$$

and

$$\{K_s\}_e = \int \{B_s\}^T\{D_s\}\{B_s\}\mathrm{d}l_s, \tag{4}$$

where the transformation matrix $\{B_p\}$, the elasticity matrix $\{D_p\}$ and the shape function vector $\{N\}$ for the four-node Kirchhoff plate element can be found in [24]. $\vec{n}_b$ is the normal unit vector of the element boundary contour Γ_b, and l_s is taken along the stiffener axis in the x-y plane. Similar to the stiffened plate element in [4] but neglecting the axial force of the stiffener, the matrices $\{B_s\}$ and D_s are given as,

$$\{D_s\} = \begin{bmatrix} E_sI_s & 0 \\ 0 & G_sJ_s \end{bmatrix}, \quad \{B_s\} = \{T_s\}\{B_p\}.$$

where E_s is the elasticity modulus of the stiffener, G_s is the rigidity modulus of the stiffener, I_s is the second moment of area of the stiffener, and J_s is the torsional constant of the stiffener. $\{T_s\}$ is the transformation matrix which takes care of the arbitrary position of the stiffener in the plate element and is given by

$$\{T_s\} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & \frac{1}{2}\sin 2\alpha \\ -\frac{1}{2}\sin 2\alpha & \frac{1}{2}\sin 2\alpha & \frac{1}{2}\cos 2\alpha \end{bmatrix},$$

where α is the angle between the global axis and the local axis of the stiffener (see Fig. 2). The advantage of using Eq. (4) is that the orientation and the position of the stiffener inside the plate element are without any restriction.

The total kinetic energy T_e of the stiffened plate element, as shown in Fig. 2, is given by:

$$T_e = T_{pe} + T_{se},$$

where the kinetic energy of the plate element T_{pe}, the kinetic energy of the stiffener in the plate element T_{se} can be expressed by:

$$T_(pe) = \frac{1}{2}\{\dot{\delta}\}_e^T\{M_p\}_e\{\dot{\delta}\}_e, \quad T_(se) = \frac{1}{2}\{\dot{\delta}\}_e^T\{M_s\}_e\{\dot{\delta}\}_e,$$

where $\{M_p\}_e$ and $\{M_s\}_e$ are the element mass matrices of $\{M_p\}$ and $\{M_s\}$, which can be expressed by,

$$\{M_p\}_e = \rho_p h \int \{N\}^T \{N\} \mathrm{d}x\mathrm{d}y, \quad \{M_s\}_e = \rho_s A_s \int \{N\}^T \{N\} \mathrm{d}l_s,$$

where P_p and P_s are the density of the plate and the stiffener, respectively, and A_s is the cross-sectional area of the stiffener.

Once the element matrices $\{K_p\}_e, \{K_b\}_e, \{K_s\}_e, \{M_p\}_e$ and $\{M_s\}_e$ are solved, the global stiffness and mass matrix $\{K\}, \{M\}$, of the whole plate system can be obtained according to the finite element assembly procedure[24].

3 Numerical studies and discussion

Numerical studies were conducted in this section by using the present model. In order to get more accurate results, the element numbers used in the following numerical studies were set large enough to ensure the element size was at least smaller than one quarter of the wavelength of the highest frequency of interest.

3.1 Concentrically stiffened plate

A square plate clamped in all edges having a centrally placed concentric stiffener has been analyzed by Nair and Rao [20] and several other investigators[4, 25]. The plate was 0.6 m long, 0.6 m wide, and 1 mm thick, with a central stiffener (3.11 mm×20.25 mm) lying in the width direction. The plate and the stiffener were made of the same material, with Young's modulus 68.7 GPa, density 2780 kg/m^3, and Poisson's ratio 0.34. The first six natural frequencies of this stiffened plate were calculated using the present method, with k_{tb} and k_{rb} set to ∞ to represent the clamped boundary condition. Tab. 1 shows the calculation results compared with those of the previous investigators; good agreement can be seen among these results.

Table 1. Natural frequencies of the clamped plate with a concentric stiffener

Method	Mesh size	Natural frequency (Hz)					
		1	2	3	4	5	6
Present	16×16	50.13	63.42	74.08	84.26	111.94	118.37
Barik et al. [4]	16×16	50.15	63.41	74.13	84.24	111.99	118.34
Nair et al. [20]		50.45	63.71	75.16	85.5	113.69	120.89
Sheikh et al. [25]		50.43	63.72	75.07	85.46	113.96	120.82

3.2 Eccentrically stiffened plate

A square plate clamped in all edges having a centrally placed eccentric stiffener has been analyzed by Aksu [2] and several other investigators[10, 19]. The plate was 0.6 m long, 0.41 m wide, and 6.33 mm thick, with a central stiffener (12.7 mm ×22.2 mm) lying in the width direction. The plate and the stiffener were made of the same material, with Young's modulus 211 GPa, density 7830 kg/m^3, and Poisson's ratio 0.3. The first three natural frequencies of this stiffened plate were calculated using the present method, with k_{tb}, k_{rb}, k_{nb} and k_{tb} set to set to ∞ and k_{rb} set to 0, to represent the simply supported boundary condition. Tab. 2 shows the calculation results compared with those of the previous investigators; again, good agreement can be seen among these results.

3.3 Eccentrically stiffened windows

A window is a typical thin plate structure that forms a major noise transmission path in residential buildings. The boundary condition of a practical window in buildings is usually neither simply supported nor

Table 2. Natural frequencies of the simply supported plate with an eccentric stiffener

Method	Mesh size	Natural frequency (Hz)		
		1	2	3
Present	12×8	258.79	273.89	527.29
Mukherjee et al. [19]		257.05	272.1	524.7
Aksu [2]		254.94	269.46	511.64
Harik et al. [10]		253.59	282.02	513.5

clamped, but is a more general condition (such as the one between simply supported and clamped)[23, 26]. The feature of the present method is that it can be used to examine the effects of arbitrary elastic boundary conditions and arbitrarily located stiffeners. Therefore, the present method is suitable for the analysis of a practical window with stiffeners.

The natural frequencies of a structure are important to both its vibration and its sound radiation performance. Numerical studies were conducted to analyze the natural frequencies of a window with different stiffeners. The window was 0.65m long, 0.65 m wide, and 5 mm thick. Young's modulus, density and Poisson's ratio were 65 GPa, 2500 kg/m3 and 0.25, respectively. A single eccentric stiffener of Ls length was used to stiffen the window. The cross-section dimensions of the stiffener cross section had a width of 5 mm and a thickness of 20 mm. The left point of the stiffener was located at the lower left comer of the window, and the whole stiffener was crossed through the window, as shown in Fig. 3. The location of the stiffener and its length Ls were dependent only on the angle θ. The dimensionless boundary parameters $\bar{k}_{tb}$ and $\bar{k}_{rb}$ were set to ∞ and 20, which indicate the boundary condition is between the clamped and simply supported. The dimensionless treatments were the same as those used in [6, 17, 21, 22]. In real applications, one can use the boundary condition identification (BCI) method[22] to determine the approximate boundary condition of a practical window. Considering the window's transparency, the stiffeners were made of the same material as the window.

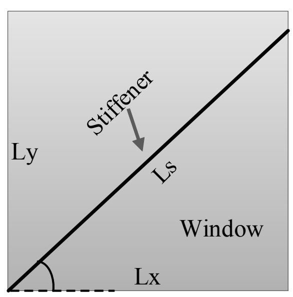

Fig. 3. Schematic diagram of the stiffened window

Tab. 3 shows the calculated natural frequencies (the first six modes) of the stiffened window. The element number used in the calculation was 64 (8×8). For comparison, the predictions of the same window without stiffeners (unstiffened window) are also included in the table.

Table 3. Natural frequencies of the simply supported plate with an eccentric stiffener

Method	Natural frequency (Hz)					
	1	2	3	4	5	6
Unstiffened	88.2	181.96	181.96	266.8	331.03	332.15
	93.14	187.31	194.98	296.1	333.15	344.91
	102.22	190.77	215.42	296.12	356.4	377.4
	113.26	182.82	272.34	282.31	333.83	408.95

The results in Tab. 3 show that even a single stiffener has a significant effect on the natural frequencies of a window. Due to the additional stiffness the stiffened window has higher natural frequencies than the unstiffened window. Moreover, the natural frequencies are influenced by the location of the stiffener. Stiffening the window at proper positions could be an effective way to shift the natural frequencies (especially the fundamental natural frequency) away from unwanted ranges and could thus be a potential way to improve the vibration isolation (or sound insulation) performance of the window in the frequency range of interest.

4 Conclusions

A simple but effective finite element model has been introduced, which can be used for the analysis of the free vibrations of rectangular stiffened plates. This technique allows the plate having arbitrary elastic boundary conditions and arbitrarily located stiffeners. Numerical studies are conducted to analyze the natural frequencies of concentrically/eccentrically stiffened plates with different boundary conditions and the results show good agreement with earlier published results. Parametric studies are also carried out on a stiffened glass window. The results show that the natural frequencies of the window are notably influenced by the stiffener and the stiffener's location.

Although only one stiffener is used in the parametric studies, the current method can be applied to the plates (or windows) with the stiffeners in arbitrary number. Also, the current method can be applied to examine the effects of different boundary conditions on the free vibration of a stiffened plate.

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