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Permanence and extinction for a nonautonomous SVIR epidemic model with distributed time delay*

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Abstract. In this paper we have considered a nonautonomous $SVIR$ epidemic model with varying total population size and distributed time delay to become infectious. Instead of assuming that vaccinees gain immunity immediately, we have assumed that they are different from susceptible and recovered persons and it takes some time for them to gain immunity and then enter into the recovered class. Here, we have established some sufficient conditions on the permanence and extinction of the disease by using inequality analytical technique. We have obtained the explicit formula of the eventual lower bound of infected persons. We have introduced some new threshold values R_0 and R^* and further obtained that the disease will be permanent when $R_0 > 1$ and the disease will be going to extinct when $R^* < 1$. By Lyapunov functional method, we have also obtained some sufficient conditions for global asymptotic stability of this model. In special case, our model reduces to the standard $SIRS$ model without vaccination with the classical basic reproduction number. Computer simulations are carried out to explain the analytical findings. The aim of the analysis of this model is to identify the parameters of interest for further study, with a view to informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness.

Keywords: vaccination, time delay, permanence, Lyapunov functional, global stability

1 Introduction

It is well known that the spectrum of infectious disease is changing rapidly in conjunction with dramatic social and environmental changes. Worldwide, explosive population growth with increasing poverty and urban migration is going on, international travel and commerce are expanding, technology is changing rapidly, all of which are affecting the risk of exposure to infectious agents. Vaccination or immunization is a commonly used method for controlling diseases such as polio, measles, diphtheria, tetanus, influenza, etc. It is a central factor in improving the standards of living and the standards of health[4, 9]. Routine vaccination is now given in all developing countries against all these diseases. The eradication of smallpox, which was last observed in a natural case in 1977, has been recognized as the most dramatic success of vaccination[34]. Vaccinations work by stimulating the immune system, the natural disease-fighting system of the body, against the germ, thereby preventing disease. The healthy immune system is able to recognize invading bacteria and viruses, and produce antibodies to destroy or disable them. Immunizations or vaccinations prepare the immune system to prevent a disease. To immunize against viral diseases, the virus used in the vaccine has been weakened or killed. To only immunize against bacterial diseases, it is generally possible to use a small portion of the

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dead bacteria to stimulate the formation of antibodies against the whole bacteria. Most vaccines are given orally or by intramuscular or subcutaneous injection. Hence one may obtain an idea about how vaccines work. Firstly, it provides the immune system with harmless copies of an antigen; once it is detected by the immune system, white blood cells (B-lymphocytes) create antibody that is precisely designed to attack that antigen. Since immune systems are designed to 'remember', once exposed to a particular bacterium or virus, they retain immunity against it for years, decades, or even a lifetime, and are prepared to overcome a later infection[19, 34].

In addition to the initial immunization process, it has been found that the effectiveness of immunizations can be improved by periodic repeat injections or "boosters", usually there are different schedules for different diseases and vaccines. For each schedule, some doses should be taken by vaccinees several times and there should be some fixed time intervals between two successive doses. For example, Gabbuti et al.[11] recommended three doses ($20\mu g$/dose) recombinant hepatitis B vaccine (Engerix B, Smith Kline Beecham Biologicals, Rixensart, Belgium) given at 0, 1 and 6 months for vaccination against hepatitis B. They observed that one month after the third dose of vaccine, 99.8% of vaccinees have gained anti-HBs antibody and eleven years after vaccination, 91.2% of vaccinees still have a protective level of anti-HBs.

Recently, in some developed countries, we have observed several instances of declines in vaccination coverage for several diseases; in some cases this is a consequences of rumours and adverse publicity against vaccination. The decline in coverage of the Measles-Mumps-Rubella vaccine (MMR) was observed in the U. K.[29–32] and it has been explained by the role of adverse publicity about possible links between the vaccine, autism, and Crohn's disease[35]. Similar events was found in Scotland[10]. Another example is the decline in HBV coverage due to the 'Thimerosal' case[20]. In the future negative effects on vaccination coverage could be derived from the arguments, often raised by anti-vaccination movements, that vaccines could favour the onset of allergic reactions, a point that is still debated by the scientific literatures[5, 7, 16, 24].

According to the mathematical theory of epidemics[15], the spread of infectious diseases usually can be described by compartmental models such as SIR or $SIRS$ models with each letter referring as a compartment in which an individual can be placed. As such vaccination can also be considered by adding some compartment into the basic epidemic models for certain diseases. In their literatures, Kribs-Zaleta and Velasco-Hernandez[18] included a compartment V into an SIS model and discussed the vaccination of diseases; it has been generalized by Arino[3], considering individuals recovering from the disease to enter into a temporarily immune class rather than directly back into the susceptible class. Kribs-Zaleta and Martcheva[17] analyzed the effects of a vaccination campaign upon the spread of nonfatal diseases such as Hepatitis A, B and feline calici virus (FCV), which features both acute and chronic infective stages, as well as variable infectivity and recovery rates in the chronic stage. Alexander et al.[1] and Shim[25] studied the transmission dynamics of influenza with vaccination by using $SVIR$ models, d'Onofrio et al.[9] suggested a family of models for information-related vaccinating behaviour, and Liu et al.[19] analyzed continuous vaccination strategy and pulse vaccination strategy by utilizing $SVIR$ models.

In fact, as soon as susceptible persons start the vaccination process, they are different from susceptible individuals. Also, they should be distinguished from recovered persons, who has gained immunity against the disease. For the 0, 1 and 6 month schedule of vaccination dose against hepatitis B vaccine, usually $30\sim50\%$ persons gain anti-HBs antibody after the first dose, $80\sim90\%$ will gain after the second dose, and almost all the persons will have high anti-HBs concentrations one month after the last dose. The anti-HBs concentrations may decline slowly, but still be in an effective level for protection and may last for more than ten years[11, 13, 19, 33]. Hence when the vaccinees gain immunity, they may be treated as recovered persons.

Nonautonomous phenomenon often occurs in many realistic epidemic models. The nonautonomous phenomenon occurs mainly due to the seasonal variety, which makes the population to behave periodically. Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically varying environment are considered as important selective forces on systems in a fluctuating environment. To investigate this kind of phenomenon, in the model, the coefficients should be periodic functions, then the system is called periodic system. The nonautonomous epidemic models can be regarded as an extension of the periodic epidemic models. To the best of our knowledge, the research works on the nonautonomous epidemic dynamical models are very few[14, 23, 27, 28, 36, 37]. Therefore, the research on the nonautonomous epidemic dynamical models is also very important.

Research on epidemic models that incorporates time dependent biological and environmental parameters, disease related death, varying total population, and time delay is becoming one of the important areas in the mathematical theory of epidemiology. Motivated by the above facts, in this paper we have considered a nonautonomous $SVIR$ epidemic model with varying total population size and distributed time delay to become infectious. Instead of assuming that vaccinees gain immunity immediately, we have assumed that they are different from susceptible and recovered persons and it takes some time for them to gain immunity and then enter into the recovered class. Here, we have established some sufficient conditions on the permanence and extinction of the disease by using inequality analytical technique. We have obtained the explicit formula of the eventual lower bound of infected persons. We have introduced some new threshold values R_0 and R^* and further obtained that the disease will be permanent when $R_0 > 1$ and the disease will be going to extinct when $R^* < 1$. By Lyapunov functional method, we have also obtained some sufficient conditions for global asymptotic stability of this model. Our analytical results are validated through numerical simulations. The aim of the analysis of this model is to identify the parameters of interest for further study, with a view to informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness[26].

2 Nonautonomous svir epidemic model with distributed time delay

The model is formulated as the following system of nonautonomous delay differential equations:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda(t) - \beta(t)S(t)\int_0^h I(t-s)\mathrm{d}\eta(s) - \{\mu(t)+\delta(t)\}S(t), \tag{1}$$

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \delta(t)S(t) - \beta_1(t)V(t)\int_0^h I(t-s)\mathrm{d}\eta(s) - \{\mu(t)+\gamma_1(t)\}V(t), \tag{2}$$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta(t)S(t)\int_0^h I(t-s)\mathrm{d}\eta(s) + \beta_1(t)V(t)\int_0^h I(t-s)\mathrm{d}\eta(s) - \{\mu_1(t)+\gamma(t)\}I(t), \tag{3}$$

$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma_1(t)V(t) + \gamma(t)I(t) - \mu(t)R(t), \tag{4}$$

where $N(t) = S(t) + V(t) + I(t) + R(t)$ denotes the total number of high-risk human population at time t; $S(t), V(t), I(t), R(t)$ are the densities (or fractions) of susceptible, vaccinees (who have divided from susceptible and begun vaccination process), infected and recovered individuals, respectively at time t. The recovered individuals are assumed to have immunity (so called natural immunity) against the disease.

The quantities $\Lambda(t), \beta(t), \beta_1(t), \mu(t), \mu_1(t), \delta(t), \gamma(t), \gamma_1(t)$ are:
$\Lambda(t)$: The recruitment rate function of susceptible population from the larger embedding population.
$\beta(t)$: The transmission rate function of disease when susceptible individuals contact with infected individuals and the rate of transmission is of the form:

$$\beta(t)S(t)\int_0^h I(t-s)\mathrm{d}\eta(s).$$

$\beta_1(t)$: The transmission rate function of disease when vaccinees (before obtaining immunity) contact with infected individuals and the rate of transmission is of the form:

$$\beta_1(t)V(t)\int_0^h I(t-s)\mathrm{d}\eta(s),$$

$\beta_1(t)$ may be assumed to be less than $\beta(t)$ since the vaccinees may have some partial immunity during the process or they may recognize the transmission characters of the disease.
$\mu(t)$: The instantaneous per capita mortality rate function of the susceptible, vaccinating and recovered population.
$\mu_1(t)$: The instantaneous per capita mortality rate function of the infected population. It is natural biologically to assume that $\mu(t) \leq \mu_1(t)$ (that is, epidemics will increase the death rates of the infective).

$\delta(t)$: The instantaneous rate function at which susceptible individuals are moved into the vaccination process.
$\gamma(t)$: The recovery rate function of infected individuals.
$\gamma_1(t)$: Average rate function for vaccinees to gain immunity during or after the vaccination process and move into recovered population.

The nonnegative constant h is the time delay. The function $\eta(s) : [0, h] \to [0, \infty)$ is nondecreasing and has bounded variation such that:

$$\int_0^h \mathrm{d}\eta(s) = \eta(h) - \eta(0) = 1.$$

The time delay is due to intracellular delay between initial infection of a cell and the release of new virions. Those infected at time $t - s$ become infectious at time $s(0 \le s \le h)$ later with different probabilities.

3 Permanence and extinction

In this section, we first introduce the following assumptions for system Eqs. (1) $\sim$ (4): functions $\Lambda(t), \beta(t), \beta_1(t), \mu(t), \mu_1(t), \delta(t), \gamma(t), \gamma_1(t)$ are positive continuous bounded and have positive lower bounds. It is natural biologically to assume that $\mu(t) \le \mu_1(t)$ (that is, epidemics will increase the death rates of the infective).

The initial conditions of Eqs. (1) $\sim$ (4) are given as

$$S(\theta) = \varphi_1(\theta),\ V(\theta) = \varphi_2(\theta),\ I(\theta) = \varphi_3(\theta),\ R(\theta) = \varphi_4(\theta),\ -h \le \theta \le 0, \tag{5}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in C$ such that $\varphi_i(\theta) \ge 0\ (i = 1, 2, 3, 4),\ \forall\ \theta \in [-h, 0]$, and C denotes the Banach space $C([-h, 0], \mathbb{R}^4)$ of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^4$ and the norm of an element φ in C is designated by

$$\|\varphi\| = \sup_{-h \le \theta \le 0} \{|\varphi_1(\theta)|, |\varphi_2(\theta)|, |\varphi_3(\theta)|, |\varphi_4(\theta)|\}$$

. For a biological meaning, we further assume that $\varphi_i(0) > 0,\ i = 1, 2, 3, 4$.

Here we wish to discuss the permanence of the system Eqs. (1) $\sim$ (4), this means that the long-term survival (i.e., will not vanish in time) of all components of the system Eqs. (1) $\sim$ (4), with initial conditions Eq. (5). It demonstrates how the disease will be permanent (i.e., will not vanish in time) under some conditions. Also, we discuss how the disease will be going to extinct under some conditions.

Let

$$f^l = \inf_{t \ge 0} f(t), \quad f^u = \sup_{t \ge 0} f(t),$$

for a continuous and bounded function defined on $[0, +\infty)$.

Definition 1. *The system Eqs. (1) $\sim$ (4) is said to be permanent, i.e., the long-term survival (will not vanish in time) of all components of the system Eqs. (1) $\sim$ (4), if there are positive constants v_i and M_i $(i = 1, 2, 3, 4)$ such that:*

$$v_1 \le \liminf_{t\to\infty} S(t) \le \limsup_{t\to\infty} S(t) \le M_1, v_2 \le \liminf_{t\to\infty} V(t) \le \limsup_{t\to\infty} V(t) \le M_2,$$
$$v_3 \le \liminf_{t\to\infty} I(t) \le \limsup_{t\to\infty} I(t) \le M_3, v_4 \le \liminf_{t\to\infty} R(t) \le \limsup_{t\to\infty} R(t) \le M_4,$$

hold for any solution $(S(t), V(t), I(t), R(t))$ of Eqs. (1) $\sim$ (4) with initial conditions Eq. (5). Here v_i and M_i $(i = 1, 2, 3, 4)$ are independent of Eq. (5).

Theorem 1. *The system Eqs. (1) $\sim$ (4) with initial conditions Eq. (5) is permanent provided*

$$R_0 = \frac{\beta^l}{(\mu_1 + \gamma)^u} \frac{\Lambda^l}{(\mu + \delta)^u} > 1. \tag{6}$$

Proof. We will give the following Propositions 1∼5 to complete the proof of this theorem.

Proposition 1. *The solution* $(S(t), V(t), I(t), R(t))$ *of Eqs. (1) ∼ (4) with initial conditions Eq. (5) is positive for all* $t \geq 0$*, and*

$$\limsup_{t\to\infty} N(t) \leq \left(\frac{\Lambda}{\mu}\right)^u.$$

Proof. Since the right hand side of system Eqs. (1) ∼ (4) is completely continuous and locally Lipschitzian on C, the solution $(S(t), V(t), I(t), R(t))$ of Eqs. (1) ∼ (4) with initial conditions Eq. (5) exists and is unique on $[0, \alpha)$, where $0 < \alpha \leq +\infty$[12]. Now,

$$\begin{aligned}
S(t) &= S(0)\exp\left[-\int_0^t \{\beta(\theta)\int_0^h I(\theta-s)\mathrm{d}\eta(s) + \mu(\theta) + \delta(\theta)\}\mathrm{d}\theta\right] \\
&\quad + \int_0^t \Lambda(u)\exp\left[\int_t^u \{\beta(\theta)\int_0^h I(\theta-s)\mathrm{d}\eta(s) + \mu(\theta) + \delta(\theta)\}\mathrm{d}\theta\right]\mathrm{d}u > 0,\ \forall\, t \geq 0. \\
V(t) &= V(0)\exp\left[-\int_0^t \{\beta_1(\theta)\int_0^h I(\theta-s)\mathrm{d}\eta(s) + \mu(\theta) + \gamma_1(\theta)\}\mathrm{d}\theta\right] \\
&\quad + \int_0^t \delta(u)S(u)\exp\left[\int_t^u \{\beta_1(\theta)\int_0^h I(\theta-s)\mathrm{d}\eta(s) + \mu(\theta) + \gamma_1(\theta)\}\mathrm{d}\theta\right]\mathrm{d}u > 0,\ \forall\, t \geq 0.
\end{aligned}$$

We claim that $I(t) > 0,$ for all $t \in [0, \alpha)$, where $0 < \alpha \leq +\infty$. If this is not true, then there exists a $t_1 \in (0, \alpha)$ such that $I(t_1) = 0,\ \dot{I}(t_1) \leq 0$ and $I(t) > 0$ for all $t \in [0, t_1)$. Integrating the Eq. (3) from 0 to t_1, we have:

$$\begin{aligned}
I(t_1) &= I(0)\exp\left\{-\int_0^{t_1}(\mu_1(s) + \gamma(s))\mathrm{d}s\right\} + \int_0^{t_1}\int_0^h \{(\beta(u)S(u) + \beta_1(u)V(u))I(u-s)\} \\
&\quad \exp\left\{\int_{t_1}^u (\mu_1(s) + \gamma(s))\mathrm{d}s\right\}\mathrm{d}\eta(s)\mathrm{d}u > 0,
\end{aligned}$$

which is a contradiction with $I(t_1) = 0$. So $I(t) > 0$ for all $t \geq 0$.

From the Eq. (4), we also have:

$$R(t) = R(0)\exp\left\{-\int_0^t \mu(\theta)\mathrm{d}\theta\right\} + \int_0^t \{\gamma_1(u)V(u) + \gamma(u)I(u)\}\exp\left\{\int_t^u \mu(\theta)\mathrm{d}\theta\right\}\mathrm{d}u > 0,\ \forall\, t \geq 0.$$

Therefore, $S(t) > 0, V(t) > 0, I(t) > 0, R(t) > 0,\ \forall\, t \geq 0$. Thus $\forall\, t \in [0, +\infty)$,

$$\dot{N}(t) \leq \Lambda(t) - \mu(t)N(t),\ (\because \mu(t) \leq \mu_1(t)) \Rightarrow \limsup_{t\to\infty} N(t) \leq \left(\frac{\Lambda}{\mu}\right)^u. \tag{7}$$

That is, $(S(t), V(t), I(t), R(t))$ is uniformly bounded on $[0, +\infty)$.

This completes the proof. □

Proposition 2. *The solution* $(S(t), V(t), I(t), R(t))$ *of Eqs. (1) ∼ (4) with initial conditions Eq. (5) satisfies*

$$\liminf_{t\to\infty} S(t) \geq \left\{\frac{\Lambda}{\beta\left(\frac{\Lambda}{\mu}\right)^u + \mu + \delta}\right\}^l \equiv v_1 > 0. \tag{8}$$

Proof. By Proposition 1, for any $\epsilon > 0$ (no matter however small), there exists a $t_1 > 0$ such that:

$$I(t) \leq \left(\frac{\Lambda}{\mu}\right)^u + \epsilon,\ \text{as } t \geq t_1.$$

Thus, from Eq. (1), when $t \geq t_1 + h$,

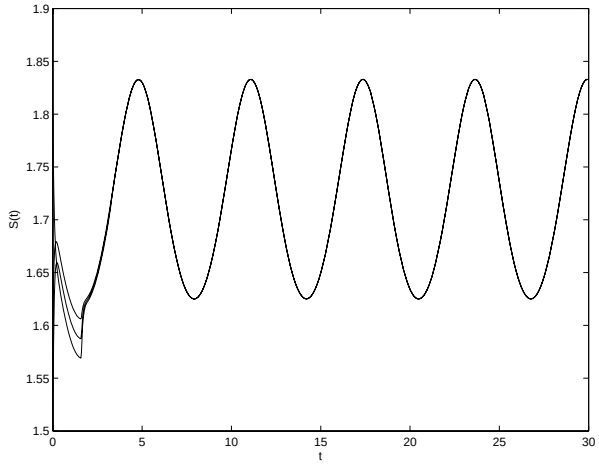

Fig. 1. Trajectories of S(t) for different initial conditions

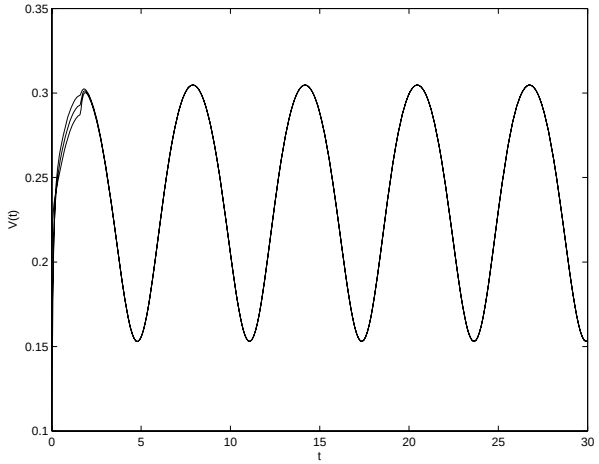

Fig. 2. Trajectories of V(t) for different initial conditions

$$\dot{S}(t) \geq \Lambda(t) - \left\{\beta(t)\left(\left(\frac{\Lambda}{\mu}\right)^u + \epsilon\right) + \mu(t) + \delta(t)\right\} S(t) \Rightarrow \liminf_{t\to\infty} S(t) \geq \left\{\frac{\Lambda}{\beta\left(\left(\frac{\Lambda}{\mu}\right)^u + \epsilon\right) + \mu + \delta}\right\}^l.$$

Since $\epsilon > 0$ can be made arbitrarily small, the result of this proposition is valid. This completes the proof. □

Proposition 3. *it The solution* $(S(t), V(t), I(t), R(t))$ *of Eqs. (1)* $\sim$ *(4) with initial conditions Eq. (5) satisfies*

$$\liminf_{t\to\infty} V(t) \geq \left\{\frac{\delta}{\beta_1\left(\frac{\Lambda}{\mu}\right)^u + \mu + \gamma_1}\right\}^l v_1 \equiv v_2 > 0,$$

where $v_1 > 0$ *is given in the Proposition 2.*

Proof. By Proposition 1, for any $\epsilon > 0$ (no matter however small), there exists a $t_1 > 0$ such that:

$$I(t) \leq \left(\frac{\Lambda}{\mu}\right)^u + \epsilon, \text{ as } t \geq t_1.$$

Thus, from Eq. (2), when $t \geq t_1 + h$,

$$\dot{V}(t) \geq \delta(t)S(t) - \left\{\beta_1(t)\left(\left(\frac{\Lambda}{\mu}\right)^u + \epsilon\right) + \mu(t) + \gamma_1(t)\right\} V(t).$$

Since $\epsilon > 0$ can be made arbitrarily small and by Proposition 2, we easily have:

$$\liminf_{t\to\infty} V(t) \geq \left\{\frac{\delta}{\beta_1\left(\frac{\Lambda}{\mu}\right)^u + \mu + \gamma_1}\right\}^l v_1 \equiv v_2 > 0,$$

where $v_1 > 0$ is given in the Proposition 2. This completes the proof.

Proposition 4. *Assume that* $R_0 > 1$, *then for any solution* $(S(t), V(t), I(t), R(t))$ *of Eqs. (1)* $\sim$ *(4) with initial conditions Eq. (5) we have*

$$\liminf_{t\to\infty} I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)} \equiv v_3 > 0, \tag{9}$$

where $\alpha > 0$ *and* $\rho > 0$ *will be given in the proof.*

Proof. Since $R_0 > 1$, and it is obvious that

$$\frac{\Lambda^l}{G} \to \frac{\Lambda^l}{(\mu+\delta)^u} \text{ as } \alpha \to 0, \text{ where } G = (\mu+\delta)^u + \alpha\beta^u, \tag{10}$$

then there exists two positive constants α and ρ such that

$$\frac{\Lambda^l}{G}\{1-\exp(-G\rho)\}\frac{\beta^l}{(\mu_1+\gamma)^u} > 1.$$

Let us consider the following differential function $L(t)$,

$$L(t) = I(t) + \int_0^h \int_{t-s}^t \beta(u+s)S(u+s)I(u)\mathrm{d}u\mathrm{d}\eta(s) + \int_0^h \int_{t-s}^t \beta_1(u+s)V(u+s)I(u)\mathrm{d}u\mathrm{d}\eta(s). \tag{11}$$

The derivative of $L(t)$ along solution of Eqs. (1) $\sim$ (4) is

$$\begin{aligned}\dot{L}(t) &= \left[\int_0^h \beta(t+s)S(t+s)\mathrm{d}\eta(s) - (\mu_1(t)+\gamma(t))\right] I(t) + I(t)\int_0^h \beta_1(t+s)V(t+s)\mathrm{d}\eta(s) \\ &\geq [\beta^l \int_0^h S(t+s)\mathrm{d}\eta(s) - (\mu_1+\gamma)^u]I(t).\end{aligned} \tag{12}$$

We claim that it is impossible that $I(t) \leq \alpha$, $\forall\ t \geq t_1$ (t_1 is any nonnegative constant). Suppose the contrary, then as $t \geq t_1 + h$,

$$\dot{S}(t) = \Lambda(t) - \beta(t)S(t)\int_0^h I(t-s)\mathrm{d}\eta(s) - \{\mu(t)+\delta(t)\}S(t) \geq \Lambda^l - GS(t), \tag{13}$$

where G is given in Eq. (9). For $t > t_1 + h$, integrating the above inequality from $t_1 + h$ to t, we obtain

$$S(t) \geq S(t_1+h)\exp\left(\int_t^{t_1+h} G\mathrm{d}s\right) + \int_{t_1+h}^t \Lambda^l \exp\left(\int_t^s G\mathrm{d}\theta\right)\mathrm{d}s \geq \left(\frac{\Lambda^l}{G}\right)\frac{\int_{t_1+h}^t G\exp\left(\int_0^s G\mathrm{d}\theta\right)\mathrm{d}s}{\exp\left(\int_0^t G\mathrm{d}\theta\right)}. \tag{14}$$

Hence,

$$S(t) \geq \left(\frac{\Lambda^l}{G}\right)[1-\exp\{-G(t-t_1-h)\}].$$

Therefore,

$$S(t) \geq \left(\frac{\Lambda^l}{G}\right)[1-\exp\{-G\rho\}] \equiv S^\Delta, \forall\ t \geq t_1+h+\rho \equiv t_2. \tag{15}$$

From Eq. (12) and (15), we have

$$\dot{L}(t) \geq (\mu_1+\gamma)^u\left[\frac{\beta^l S^\Delta}{(\mu_1+\gamma)^u} - 1\right] I(t),\ \forall t \geq t_2. \tag{16}$$

Let us take $\underline{i} = \min_{t_2 \leq t \leq t_2+h} I(t)$. Next we shall prove that $I(t) \geq \underline{i}$, $\forall t \geq t_2$. Suppose that it is not true, then $\exists T \geq 0$, such that $I(t) \geq \underline{i}$, for all $t_2 \leq t \leq t_2 + h + T$, $I(t_2+h+T) = \underline{i}$ and $\dot{I}(t_2+h+T) \leq 0$. On the other hand, by Eq. (3), as $t = t_2 + h + T$,

$$\begin{aligned}\dot{I}(t) &\geq \beta(t)S(t)\int_0^h I(t-s)\mathrm{d}\eta(s) - \{\mu_1(t)+\gamma(t)\}I(t) \\ &\geq \{\beta^l S^\Delta - (\mu_1+\gamma)^u\}\underline{i} = (\mu_1+\gamma)^u\left[\frac{\beta^l S^\Delta}{(\mu_1+\gamma)^u} - 1\right]\underline{i} > 0,\end{aligned} \tag{17}$$

since from Eq. (10), we have

$$\frac{\beta^l S^{\Delta}}{(\mu_1+\gamma)^u} > 1.$$

This is a contradiction. Hence, $I(t) \geq \underline{i}\,,\ \forall\, t \geq t_2$. Consequently, from Eq. (16), we have

$$\dot{L}(t) \geq (\mu_1+\gamma)^u \left[\frac{\beta^l S^{\Delta}}{(\mu_1+\gamma)^u} - 1\right] \underline{i} > 0,\ \forall t \geq t_2, \tag{18}$$

which implies $L(t) \to +\infty$ as $t \to +\infty$. From Proposition 1, $L(t)$ is bounded. This is a contradiction. Hence, the claim is proved. From this claim, we will discuss the following two possibilities:
(1) $I(t) \geq \alpha$ for all large t.
(2) $I(t)$ oscillates about α for all large t. Finally, we will show that $I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)}$ for sufficiently

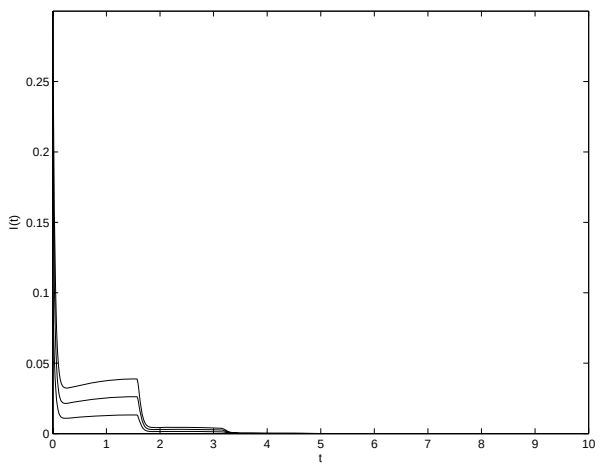

Fig. 3. Trajectories of I(t) for different initial conditions

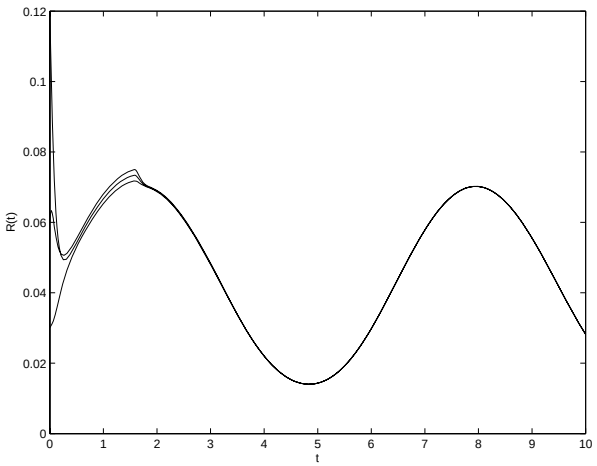

Fig. 4. Trajectories of R(t) for different initial conditions

large t. Evidently, we only need to consider the case (2). Let t_1 and t_2 be sufficiently large times satisfying:

$$I(t_1) = I(t_2) = \alpha, I(t) < \alpha \text{ as } t \in (t_1, t_2).$$

If $t_2 - t_1 \leq h + \rho$, since $\dot{I}(t) \geq -(\mu_1+\gamma)^u I(t)$ and $I(t_1) = \alpha$ which implies

$$I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)},\ \forall\, t \in [t_1, t_2].$$

If $t_2 - t_1 > h + \rho$, then it is obvious that

$$I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)}, \forall\, t \in [t_1, t_1 + h + \rho].$$

By Eq. (15), we have $S(t) \geq S^{\Delta},\ \forall\, t \in [t_1 + h + \rho, t_2]$. Thus, proceeding exactly as the proof of the above claim, we see that

$$I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)},\ \forall\, t \in [t_1 + h + \rho, t_2].$$

If it is not true, then there exists a $T^* \geq 0$ such that

$$I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)},\ \forall\, t \in [t_1, t_1 + h + \rho + T^*],\ I(t_1 + h + \rho + T^*) = \alpha e^{-(\mu_1+\gamma)^u(h+\rho)}$$

and $\dot{I}(t_1 + h + \rho + T^*) \leq 0$. Using Eq. (3), as $t = t_1 + h + \rho + T^*$, we have

$$\begin{aligned}\dot{I}(t) &\geq \beta(t)S(t)\int_0^h I(t-s)\mathrm{d}\eta(s) - (\mu_1(t)+\gamma(t))I(t) \geq \{\beta^l S^{\Delta} - (\mu_1+\gamma)^u\}\alpha e^{-(\mu_1+\gamma)^u(h+\rho)} \\ &= (\mu_1+\gamma)^u\left[\frac{\beta^l S^{\Delta}}{(\mu_1+\gamma)^u} - 1\right]\alpha e^{-(\mu_1+\gamma)^u(h+\rho)} > 0,\end{aligned} \tag{19}$$

since from Eq. (10), we have

$$\frac{\beta^l S^{\Delta}}{(\mu_1+\gamma)^u} > 1.$$

This is a contradiction. Therefore, $I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)},\ \forall\, t \in [t_1, t_2]$. Hence,

$$\liminf_{t\to\infty} I(t) \geq \alpha e^{-(\mu_1+\gamma)^u(h+\rho)} \equiv v_3 > 0.$$

This completes the proof of Proposition 4. □

Proposition 5. *Assume that $R_0 > 1$, then for any solution $(S(t), V(t), I(t), R(t))$ of Eqs. (1) ∼ (4) with initial conditions Eq. (5) we have*

$$\liminf_{t\to\infty} R(t) \geq \left\{ \frac{\gamma_1^l v_2 + \gamma^l v_3}{\mu^u} \right\} \equiv v_4 > 0, \tag{20}$$

where $v_2 > 0$ and $v_3 > 0$ are given in the Proposition 3 and Proposition 4 respectively.

Proof. From Eq. (4) and by Propositions 3 and 4, the result follows.

Thus, the system Eqs. (1) ∼ (4) with initial conditions Eq. (5) is permanent provided

$$R_0 = \frac{\beta^l}{(\mu_1+\gamma)^u} \frac{\Lambda^l}{(\mu+\delta)^u} > 1. \square$$

Next, we shall use the following lemma to discuss the extinction of the disease.

Lemma 1. *Consider an autonomous delay differential equation*

$$\dot{x}(t) = a_1 \int_0^h x(t-s) d\eta(s) - a_2 x(t), \tag{21}$$

where a_1, a_2 are two constants. If $0 \leq a_1 < a_2$, then for any solution $x(t)$ with initial condition $\varphi(\theta) \geq 0,\ \theta \in [-h, 0]$, we have

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Let us define the following Lyapunov functional:

$$V(t) = \frac{x^2(t)}{2} + \frac{a_1}{2} \int_0^h \int_{t-s}^t x^2(u) \mathrm{d}u \mathrm{d}\eta(s).$$

Then the time derivative along system Eq. (21) is given by

$$\begin{aligned}
\dot{V}(t) &= a_1 \int_0^h x(t)x(t-s) \mathrm{d}\eta(s) + \frac{a_1}{2} \int_0^h \{x^2(t) - x^2(t-s)\} d\eta(s) - a_2 x^2(t) \\
&= -\frac{a_1}{2} \int_0^h \{x(t) - x(t-s)\}^2 \mathrm{d}\eta(s) + a_1 x^2(t) - a_2 x^2(t) \leq -(a_2 - a_1) x^2(t) \\
&\Rightarrow \lim_{t\to\infty} x(t) = 0.
\end{aligned}$$

□

Theorem 2. *If*

$$R^* = \frac{(\beta+\beta_1)^u}{(\mu_1+\gamma)^l} \left(\frac{\Lambda}{\mu}\right)^u < 1, \tag{22}$$

then

$$\lim_{t\to\infty} I(t) = 0,$$

i.e. the disease in system Eqs. (1) ∼ (4) will be going to extinction.

Proof. By Eq. (22), there exists a sufficiently small $\epsilon > 0$, such that:

$$\frac{(\beta+\beta_1)^u}{(\mu_1+\gamma)^l}\left\{\left(\frac{\Lambda}{\mu}\right)^u+\epsilon\right\}<1.$$

From Proposition 1, given $\epsilon > 0$ (no matter however small), there exists a $t_1 > 0$ such that:

$$S(t), V(t) \leq \left(\frac{\Lambda}{\mu}\right)^u + \epsilon, \text{ as } t \geq t_1.$$

Thus, from Eq. (3), when $t \geq t_1$,

$$\begin{aligned}\dot{I}(t) &= \beta(t)S(t)\int_0^h I(t-s)\mathrm{d}\eta(s) + \beta_1(t)V(t)\int_0^h I(t-s)\mathrm{d}\eta(s) - \{\mu_1(t)+\gamma(t)\}I(t)\\ &\leq (\mu_1+\gamma)^l\{\frac{(\beta+\beta_1)^u}{(\mu_1+\gamma)^l}\left(\left(\frac{\Lambda}{\mu}\right)^u+\epsilon\right)\int_0^h I(t-s)\mathrm{d}\eta(s) - I(t)\}.\end{aligned}$$

Using the comparison theorem of functional differential equations and Lemma 1, we have $\lim_{t\to\infty} I(t) = 0$, i.e. the disease in system Eqs. (1) $\sim$ (4) will be going to extinction. □

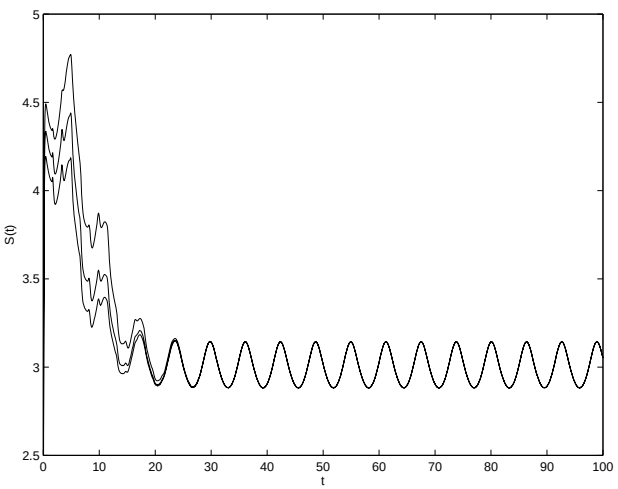

Fig. 5. Trajectories of S(t) for different initial conditions

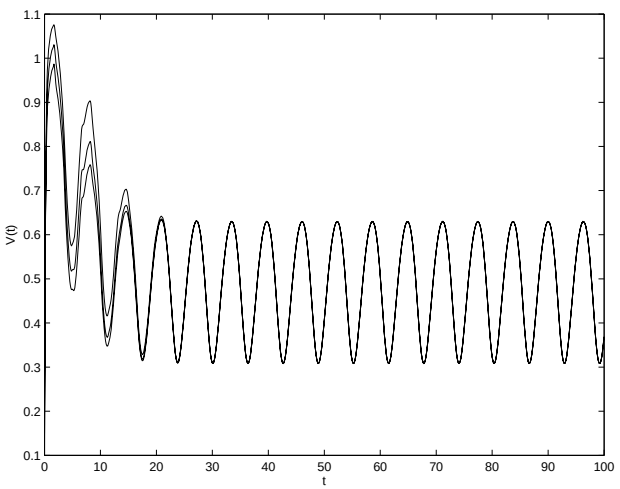

Fig. 6. Trajectories of V(t) for different initial conditions

From Eq. (22) we conclude that the spread of the disease should be controlled by way of suitable protective measures of the society to reduce the values of $\beta(t)$ (transmission rate function of disease when susceptible individuals contact with infected individuals), $\beta_1(t)$ (transmission rate function of disease when vaccinees, before obtaining immunity, contact with infected individuals) and thereby to decrease R^*. If the rate of migration or recruitment is restricted into susceptible community, the spread of the disease can also be kept under control by reducing $\Lambda(t)$ and thereby decreasing R^*. The spread of the disease can also be kept under control by increasing $\gamma(t)$ (recovery rate function of the infected population) and thereby decreasing R^*. Proper treatment should be taken for infected individuals.

4 Global asymptotic stability

In this section, we derive sufficient conditions for global asymptotic stability of system Eqs. (1) $\sim$ (4) with initial conditions Eq. (5). We now state a definition of global asymptotic stability of solutions of system Eqs. (1) $\sim$ (4).

Definition 2. *System Eqs. (1) ∼ (4) with initial conditions Eq. (5) is said to be globally asymptotically stable if*

$$\lim_{t\to\infty} \mid S_1(t) - S_2(t) \mid = 0,\ \lim_{t\to\infty} \mid V_1(t) - V_2(t) \mid = 0,\ \lim_{t\to\infty} \mid I_1(t) - I_2(t) \mid = 0,\ \lim_{t\to\infty} \mid R_1(t) - R_2(t) \mid = 0,$$

hold for any two solutions $(S_1(t), V_1(t), I_1(t), R_1(t))$ *and* $(S_2(t), V_2(t), I_2(t), R_2(t))$ *of Eqs. (1) ∼ (4) with initial conditions of type Eq. (5).*

Assume that $(S(t), V(t), I(t), R(t))$ *is a solution of Eqs. (1) ∼ (4). By the uniform boundedness of solutions of Eqs. (1) ∼ (4), there is an* $A > 0$ *(in fact,* $A = \left(\frac{\Lambda}{\mu}\right)^u + \epsilon$ *where* $\epsilon > 0$ *can be made arbitrarily small) independent of initial conditions Eq. (5) such that*

$$0 \le S(t) \le A,\ 0 \le V(t) \le A,\ 0 \le I(t) \le A,\ 0 \le R(t) \le A,\ \textit{for large enough } t.$$

Without loss of generality, we may assume that

$$0 \le S(t) \le A,\ 0 \le V(t) \le A,\ 0 \le I(t) \le A,\ 0 \le R(t) \le A,\ \forall\, t \ge 0.$$

Theorem 3. *If there exist* $c_1 > 0, c_2 > 0$ *and* $c_3 > 0$ *such that the functions* $B_i(t)$ $(i = 1, 2, 3, 4)$ *are nonnegative on* $[0, \infty)$ *and for any interval sequence* $\{[a_i, b_i]\}_1^\infty, [a_i, b_i] \cap [a_j, b_j] = \phi$ *and* $b_i - a_i = b_j - a_j > 0$, *for all* $i, j = 1, 2, \ldots$ *and* $i \ne j$, *one has* $\sum_{k=1}^{\infty} \int_{a_k}^{b_k} B_i(t)dt = \infty$, *then system Eqs. (1) ∼ (4) with initial conditions Eq. (5) is globally asymptotically stable. Here,*

$$B_1(t) = c_1\mu(t) - c_2 A\beta(t), \tag{23}$$

$$B_2(t) = c_1\mu(t) - c_2 A\beta_1(t) + (c_1 - c_3)\gamma_1(t), \tag{24}$$

$$B_3(t) = c_2\mu_1(t) + (c_2 - c_3)\gamma(t) - (c_1 + c_2)A\int_0^h \{\beta(t+s) + \beta_1(t+s)\}d\eta(s), \tag{25}$$

$$B_4(t) = c_3\mu(t). \tag{26}$$

Proof. Assume that $(S_1(t), V_1(t), I_1(t), R_1(t))$ and $(S_2(t), V_2(t), I_2(t), R_2(t))$ are any two solutions of system Eqs. (1) ∼ (4) with initial conditions of type Eq. (5).

Define $L_1(t) = \mid S_1(t) - S_2(t) \mid + \mid V_1(t) - V_2(t) \mid$. Then the right-upper derivative of $L_1(t)$ along the solution of system Eqs. (1) ∼ (4) and Eq. (5) is given by

$$\begin{aligned}
D^+L_1(t) = {} & sgn(S_1(t) - S_2(t))\{-\beta(t)(S_1(t) - S_2(t))\int_0^h I_1(t-s)\mathrm{d}\eta(s) \\
& + \beta(t)S_2(t)\int_0^h (I_2(t-s) - I_1(t-s))\mathrm{d}\eta(s) - \{\mu(t) + \delta(t)\}(S_1(t) - S_2(t)) \\
& + sgn(V_1(t) - V_2(t))\{\delta(t)(S_1(t) - S_2(t)) - \beta_1(t)(V_1(t) - V_2(t))\int_0^h I_1(t-s)\mathrm{d}\eta(s) \\
& + \beta_1(t)V_2(t)\int_0^h (I_2(t-s) - I_1(t-s))\mathrm{d}\eta(s) - \{\mu(t) + \gamma_1(t)\}(V_1(t) - V_2(t)) \\
D^+L_1(t) \le {} & -\{\mu(t) + \delta(t)\} \mid S_1(t) - S_2(t) \mid + \beta(t)A\int_0^h \mid I_1(t-s) - I_2(t-s) \mid \mathrm{d}\eta(s) \\
& - \{\mu(t) + \gamma_1(t)\} \mid V_1(t) - V_2(t) \mid + \delta(t) \mid S_1(t) - S_2(t) \mid \\
& + \beta_1(t)A\int_0^h \mid I_1(t-s) - I_2(t-s) \mid \mathrm{d}\eta(s).
\end{aligned} \tag{27}$$

Define $L_2(t) = \mid I_1(t) - I_2(t) \mid$. Calculating the right-upper derivative of $L_2(t)$ along the solution of system Eqs. (1) ∼ (4) and Eq. (5), we have

$$
\begin{aligned}
D^+L_2(t) = sgn(I_1(t)-I_2(t))\{&\beta(t)(S_1(t)-S_2(t))\int_0^h I_1(t-s)\mathrm{d}\eta(s) \\
&+\beta(t)S_2(t)\int_0^h (I_1(t-s)-I_2(t-s))\mathrm{d}\eta(s)+\beta_1(t)(V_1(t)-V_2(t))\int_0^h I_1(t-s)\mathrm{d}\eta(s) \\
&+\beta_1(t)V_2(t)\int_0^h (I_1(t-s)-I_2(t-s))\mathrm{d}\eta(s)-(\mu_1(t)+\gamma(t))(I_1(t)-I_2(t))\} \\
\Rightarrow D^+L_2(t) \leq &-(\mu_1(t)+\gamma(t))\mid I_1(t)-I_2(t)\mid +\beta(t)A\mid S_1(t)-S_2(t)\mid +\beta_1(t)A\mid V_1(t)-V_2(t)\mid \\
&+\beta(t)A\int_0^h \mid I_1(t-s)-I_2(t-s)\mid \mathrm{d}\eta(s)+\beta_1(t)A\int_0^h \mid I_1(t-s)-I_2(t-s)\mid \mathrm{d}\eta(s).
\end{aligned} \tag{28}
$$

Define $L_3(t) = \mid R_1(t)-R_2(t)\mid$. Calculating the right-upper derivative of $L_3(t)$ along the solution of system Eqs. (1) $\sim$ (4) and Eq. (5), we have

$$
D^+L_3(t) \leq \gamma_1(t)\mid V_1(t)-V_2(t)\mid +\gamma(t)\mid I_1(t)-I_2(t)\mid -\mu(t)\mid R_1(t)-R_2(t)\mid . \tag{29}
$$

Define $L_4(t)$ as

$$
L_4(t) = \int_0^h\int_{t-s}^t \beta(u+s)A\mid I_1(u)-I_2(u)\mid \mathrm{d}u\mathrm{d}\eta(s)+\int_0^h\int_{t-s}^t \beta_1(u+s)A\mid I_1(u)-I_2(u)\mid \mathrm{d}u\mathrm{d}\eta(s).
$$

The right-upper derivative of $L_4(t)$ along the solution of system Eqs. (1) $\sim$ (4) and Eq. (5) is given below:

$$
\begin{aligned}
D^+L_4(t) = &\mid I_1(t)-I_2(t)\mid A\int_0^h \beta(t+s)\mathrm{d}\eta(s)-\beta(t)A\int_0^h \mid I_1(t-s)-I_2(t-s)\mid \mathrm{d}\eta(s) \\
&+\mid I_1(t)-I_2(t)\mid A\int_0^h \beta_1(t+s)\mathrm{d}\eta(s)-\beta_1(t)A\int_0^h \mid I_1(t-s)-I_2(t-s)\mid \mathrm{d}\eta(s).
\end{aligned} \tag{30}
$$

Let $L(t) = c_1L_1(t)+c_2L_2(t)+c_3L_3(t)+(c_1+c_2)L_4(t)$, then by using Eqs. (27) $\sim$ (30), we have

$$
\begin{aligned}
D^+L(t) \leq &-B_1(t)\mid S_1(t)-S_2(t)\mid -B_2(t)\mid V_1(t)-V_2(t)\mid \\
&-B_3(t)\mid I_1(t)-I_2(t)\mid -B_4(t)\mid R_1(t)-R_2(t)\mid ,\ \forall\, t\geq h,
\end{aligned} \tag{31}
$$

where $B_i(t),\ (i=1,2,3,4)$ are defined Eqs. (23) $\sim$ (26).

Integrating Eq. (31) from h to t, we have

$$
\begin{aligned}
&\int_h^t \{B_1(t)\mid S_1(t)-S_2(t)\mid +B_2(t)\mid V_1(t)-V_2(t)\mid \\
&+B_3(t)\mid I_1(t)-I_2(t)\mid +B_4(t)\mid R_1(t)-R_2(t)\mid\}dt \leq V(h)-V(t) \\
\Rightarrow &\int_h^t \{B_1(t)\mid S_1(t)-S_2(t)\mid +B_2(t)\mid V_1(t)-V_2(t)\mid \\
&+B_3(t)\mid I_1(t)-I_2(t)\mid +B_4(t)\mid R_1(t)-R_2(t)\mid\}dt < \infty.
\end{aligned} \tag{32}
$$

By assumptions about $B_i(t),\ (i=1,2,3,4)$ and the boundedness of $(S_1(t),V_1(t),I_1(t),R_1(t))$ and $(S_2(t),V_2(t),I_2(t),R_2(t))$ on $[0,\infty)$, we obtain from system (2.1) that $\mid S_1(t)-S_2(t)\mid$, $\mid V_1(t)-V_2(t)\mid$, $\mid I_1(t)-I_2(t)\mid$ and $\mid R_1(t)-R_2(t)\mid$ are bounded and uniformly continuous on $[0,\infty)$. It follows from Eq. (32) that,

$$
\lim_{t\to\infty}\mid S_1(t)-S_2(t)\mid =0,\ \lim_{t\to\infty}\mid V_1(t)-V_2(t)\mid =0,\ \lim_{t\to\infty}\mid I_1(t)-I_2(t)\mid =0,\ \lim_{t\to\infty}\mid R_1(t)-R_2(t)\mid =0.
$$

This shows that system Eqs. (1) $\sim$ (4) with initial conditions Eq. (5) is globally asymptotically stable. This completes the proof.□

Corollary 1. *If there exist $c_1 > 0, c_2 > 0$ and $c_3 > 0$ such that*

$$\liminf_{t\to\infty}\{c_1\mu(t) - c_2A\beta(t)\} > 0,$$
$$\liminf_{t\to\infty}\{c_1\mu(t) - c_2A\beta_1(t) + (c_1 - c_3)\gamma_1(t)\} > 0,$$
$$\liminf_{t\to\infty}\{c_2\mu_1(t) + (c_2 - c_3)\gamma(t) - (c_1 + c_2)A\int_0^h \{\beta(t+s) + \beta_1(t+s)\}d\eta(s)\} > 0,$$

then system Eqs. (1) $\sim$ (4) with initial conditions Eq. (5) is globally asymptotically stable.

We observe that the lower values of $\beta(t)$ (transmission rate function of disease when susceptible individuals contact with infected individuals), $\beta_1(t)$ (transmission rate function of disease when vaccinees, before obtaining immunity, contact with infected individuals), $\Lambda(t)$ (recruitment rate function of susceptible population from the larger embedding population) and higher value of $\gamma(t)$ (recovery rate function of the infected population) are leading to make $B_i(t) > 0$ $(i = 1, 2, 3)$ which also keep the spread of the epidemic under control. The results of the Theorem 3 and Corollary 1 indicate that these parametric functions and also time delay have an effect on the global asymptotic stability, which may rule out any complicated behavior (eg. limit cycles, chaos) of the proposed model.

From our everyday experience we know that the biological and environmental parameters are subject to fluctuation in time, the effects of a periodically varying environment have an important selective forces on systems in a fluctuating environment. To investigate this kind of phenomenon, in the model, the coefficients should be periodic functions of time. Let us state a theorem related to this.

Theorem 4. *If system Eqs. (1) $\sim$ (4) is $\psi-$periodic and there are positive constants v_i and M_i $(i = 1, 2, 3, 4)$ such that:*

$$v_1 \le \liminf_{t\to\infty} S(t) \le \limsup_{t\to\infty} S(t) \le M_1, v_2 \le \liminf_{t\to\infty} V(t) \le \limsup_{t\to\infty} V(t) \le M_2,$$
$$v_3 \le \liminf_{t\to\infty} I(t) \le \limsup_{t\to\infty} I(t) \le M_3, v_4 \le \liminf_{t\to\infty} R(t) \le \limsup_{t\to\infty} R(t) \le M_4,$$

hold for any solution $(S(t), V(t), I(t), R(t))$ of Eqs. (1) $\sim$ (4) with initial conditions Eq. (5), then system Eqs. (1) $\sim$ (4) has positive periodic solution with period ψ.

Corollary 2. *If system Eqs. (1) $\sim$ (4) is $\psi-$periodic and conditions in Theorems 3.1 and 4.1 are valid, then there exists a unique positive $\psi-$periodic solution which is globally asymptotically stable.*

5 Numerical simulation

In this section we present computer simulation of some solution of the system Eqs. (1) $\sim$ (4) using MATLAB.

Example 1. Let $\Lambda(t) = 24 + 2\sin t,\ \mu(t) = 12 + \sin t,\ \beta(t) = 2 + \sin t,\ \beta_1(t) = 2 + \cos t,\ \gamma(t) = \gamma_1(t) = \delta(t) = 2 + \sin t,\ \mu_1(t) = 23 + \cos t,\ \eta(s) = \frac{s}{h}, h = \frac{\pi}{2}$.

In this case $R_0 < 1$ and $R^* < 1$. Therefore, system Eqs. (1) $\sim$ (4) is not permanent and the disease in system Eqs. (1) $\sim$ (4) will be going to extinction. If you choose $c_1 = c_2 = c_3 = 1$ then system Eqs. (1) $\sim$ (4) satisfies all the assumptions in Theorem 4.1 and Corollary 4.1 and hence system Eqs. (1) $\sim$ (4) with initial conditions of type Eq. (5) is globally asymptotically stable. Fig. 1 $\sim$ 4 show trajectories of $S(t), V(t), I(t)$ and $R(t)$ respectively for different initial conditions. See Fig. 1 $\sim$ 4.

Example 2. Let $\Lambda(t) = 48 + 6\sin t,\ \mu(t) = 8 + \sin t,\ \beta(t) = 6 + \sin t,\ \beta_1(t) = 2 + \cos t,\ \gamma(t) = \gamma_1(t) = \delta(t) = 2 + \sin t,\ \mu_1(t) = 10 + \cos t,\ \eta(s) = \frac{s}{h}, h = \frac{\pi}{2}$.

In this case $R_0 > 1$ and $R^* > 1$. Therefore, system Eqs. (1) $\sim$ (4) is permanent. Fig. 5 $\sim$ 8 show trajectories of $S(t), V(t), I(t)$ and $R(t)$ respectively for different initial conditions. See Fig. 5 $\sim$ 8.

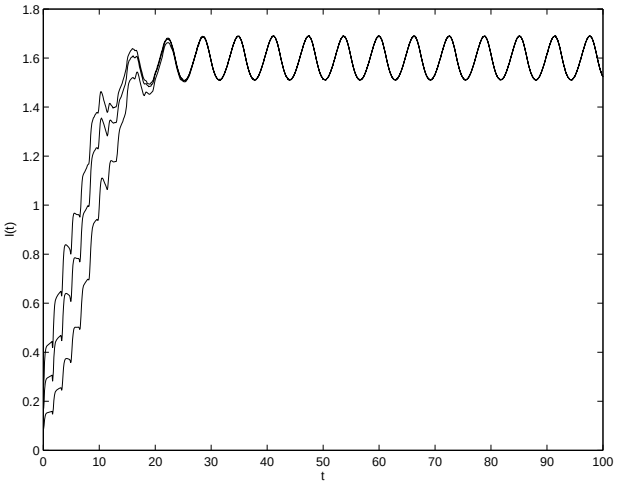

Fig. 7. Trajectories of I(t) for different initial conditions

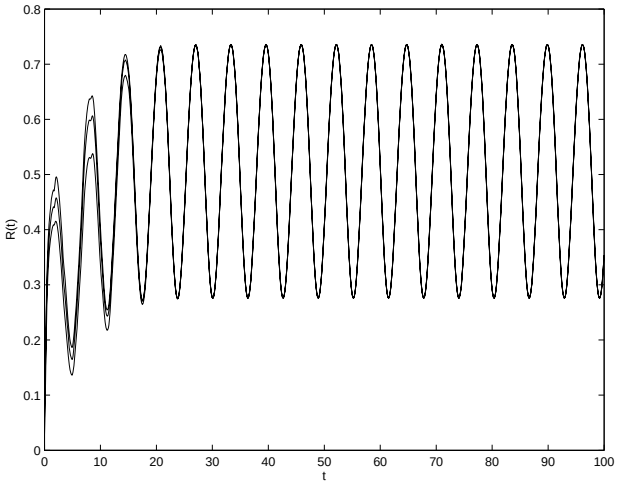

Fig. 8. Trajectories of R(t) for different initial conditions

6 Conclusions

Research on epidemic models that incorporates time dependent biological and environmental parameters, disease related death, varying total population, and time delay is becoming one of the important areas in the mathematical theory of epidemiology. To the best of our knowledge, the research works on the nonautonomous epidemic dynamical models are very few[14, 23, 27, 28, 36, 37]. In this paper we have considered a nonautonomous $SVIR$ epidemic model with varying total population size and distributed time delay to become infectious. Instead of assuming that vaccinees gain immunity immediately, we have assumed that they are different from susceptible and recovered persons and it takes some time for them to gain immunity and then enter into the recovered class. The most basic and important questions to ask for the systems in the theory of mathematical epidemiology are the persistence, extinctions, the existence of periodic solutions, global stability, etc.[2, 6, 8, 15, 21, 22]. Here, we have established some sufficient conditions on the permanence and extinction of the disease by using inequality analytical technique. We have obtained the explicit formula of the eventual lower bound of infected persons. We have introduced some new threshold values

$$R_0 = \frac{\beta^l}{(\mu_1+\gamma)^u}\frac{\Lambda^l}{(\mu+\delta)^u} \text{ and } R^* = \frac{(\beta+\beta_1)^u}{(\mu_1+\gamma)^l}\left(\frac{\Lambda}{\mu}\right)^u,$$

and further obtained that the disease will be permanent when $R_0 > 1$ and the disease will be going to extinct when $R^* < 1$. By Lyapunov functional method, we have also obtained some sufficient conditions for global asymptotic stability of this model. Our mathematical analysis suggests that vaccination process is helpful for disease control by decreasing the threshold value R^*. This process accelerates to decrease $\beta_1(t)$ (disease transmission rate function among vaccinees while contacting with infected individuals), since the vaccinees may have some partial immunity during the process or they may realize the transmission characters of the disease, and thereby helps to decrease the threshold value R^*. Our analysis also suggests that isolation of infected humans and there by reducing infected contacts called 'quarantine policy' decreases the transmission parameters $\beta(t)$ and $\beta_1(t)$ is helpful for disease control by decreasing R^*. If the rate of migration or recruitment is restricted into susceptible community, the spread of the disease can also be kept under control by reducing $\Lambda(t)$ and thereby decreasing R^*. The spread of the disease can also be kept under control by increasing $\gamma(t)$ (recovery rate function of the infected population) and thereby decreasing R^*. Proper treatment should be taken for infected individuals. We have also observed that the lower values of $\beta(t)$, $\beta_1(t)$, $\Lambda(t)$ and higher value of $\gamma(t)$ are leading to make $B_i(t) > 0$ $(i = 1, 2, 3)$ which also keep the spread of the epidemic under control. The results of the Theorem 4.1 and Corollary 4.1 indicate that these parametric functions and also time delay have an effect on the global asymptotic stability, which may rule out any complicated behavior (eg. limit cycles, chaos) of the proposed model. We have observed that the time delay decreases the lower

bound of the infective. When $\delta(t) = 0$ ($\delta(t)$ be the rate function at which susceptible persons are moved into the vaccination process), $\forall\, t$, which means that there are no vaccinations, then $\lim_{t\to\infty} V(t) = 0$. In the case, if all the coefficients are independent of time; $\delta = 0$, the vaccine is totally useless, hence $\beta_1 = 0$, and $\Lambda = \mu = \mu_1$, i.e., the recruitment rate and natural death rate of the population are equal, then our model reduces to the standard $SIRS$ model without vaccination with $R_0 = R^* = \frac{\beta}{\mu+\gamma}$. This is the classical basic reproduction number in the $SIRS$ model, namely, the average number of new infected persons caused by one infective (in a completely susceptible population) during the infective period. We have observed that the time delay has no effect on the permanence of the system but it has an effect on the global asymptotic stability of this model. Our analytical results are illustrated through computer simulations. The aim of the analysis of this model is to identify the parameters of interest for further study, with a view to informing and assisting policy-makers in targeting prevention and treatment resources for maximum effectiveness.

Here, all the coefficients in system Eqs. (1) $\sim$ (4) are time-dependent, i.e., system Eqs. (1) $\sim$ (4) is nonautonomous. Usually, such systems have not any disease-free equilibrium and endemic equilibrium. There are many methods to deal with autonomous systems, but they may not be suitable to nonautonomous systems. Therefore, it is more difficult to study the dynamical behaviours in nonautonomous case. By improving the Lyapunov functionals, we have studied global stability behaviour of system Eqs. (1) $\sim$ (4). Furthermore, by using the inequality analytical technique, we have obtained the ultimate lower bounds of the infected individuals.

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