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Three solutions for a quasilinear Neumann boundary value problem

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Abstract. In this note, we deal with the existence of at least three weak solutions for the following Neumann problem

$$\begin{cases} -u'' + u = \lambda f(x,u), \\ u'(0) = u'(1) = 0. \end{cases}$$

The technical approach is mainly based on a three critical points theorem.

Keywords: three solutions, critical point, multiplicity results, Neumann problem

1 Introduction

In this work, we establish some existence theorems for the following Neumann problem

$$\begin{cases} -u'' + u = \lambda f(x,u), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1}$$

where $\lambda > 0$ and $f : [0,1] \times R \to R$ is a continuous function.

Precisely, we deal with the existence of an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q, such that, for each $\lambda \in \Lambda$, the Eq. (1) admits at least three weak solutions whose norms in $W^{1,2}([0,1])$ are less than q.

We say that u is a weak solution to the Eq. (1) if $u \in W^{1,2}([0,1])$ and

$$\int_0^1 u'(x)v'(x)\mathrm{d}x + \int_0^1 u(x)v(x)\mathrm{d}x - \lambda \int_0^1 f(x,u(x))v(x)\mathrm{d}x = 0, \quad \forall v \in W^{1,2}([0,1]).$$

Conditions that guarantee the existence of multiple solutions to differential equations are of interest because physical processes described by differential equations can exhibit more than one solution. For example, certain chemical reactions in tubular reactors can be mathematically described by a nonlinear, two-point boundary value problem with the interest in seeing if multiple steady-states to the problem exist. For a recent treatment of chemical reactor theory and multiple solutions see [6] and references therein.

Problems of the above type with Dirichlet and Neumann boundary conditions were widely studied in these latest years and we refer to [1–5, 7–15, 18, 19] and the reference therein.

Example 1. For example, in [10] using variational methods, the author ensures the existence at least three weak solutions in $W_0^{1,2}([0,1])$ for the problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{2}$$

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where $\lambda > 0$ and $f : R \to R$ is a continuous function, while in their interesting paper [9], R. Avery and J. Henderson studied the Eq. (2) (independent of λ, in that case), where $f : R \to R$ is a continuous function and λ is a real parameter, by using multiple fixed-point theorem to obtain three symmetric positive solutions under growth conditions on f.

Also, M. Ramaswamy and R. Shivaji recently in [19] established the existence of three positive solutions for classes of nondecreasing, p-sublinear functions f belonging to $C^1([0,\infty))$ for a p-Laplacian version of [9], i.e., the problem

$$\begin{cases} -\Delta_p u = \lambda f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $p > 1$, $\lambda > 0$ is a parameter and Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected.

In [14], the authors obtained the existence of an open interval $\Lambda \subseteq [0,\infty[$ such that for each $\lambda \in \Lambda$, problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Delta_p u = \text{div } (|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\lambda \in]0,+\infty[$, $\Omega \subset R^N$ is a non-empty bounded open set with a boundary of class C^1, $a \in L^\infty(\Omega)$, with $\text{essinf}_\Omega a > 0$, $f : \Omega \times R \to R$ a function, $p \geq 2$ and v is the outer unit normal to $\partial\Omega$, and in [4] the authors proved the existence of an open interval $]\lambda', \lambda''[$ for each λ of the problem Eq. (4) depending on λ admit at least three solutions.

In [1], we established an equivalent statement of minimax inequality for a special class of functionals and we obtained existence of at least three weak solutions to the Dirichlet problem

$$\begin{cases} -u''(x) + m(x)u(x) = \lambda f(x,u(x)), & x \in (a,b), \\ u(a) = u(b) = 0, \end{cases} \tag{5}$$

where $\lambda > 0$, $f : [a,b] \times R \to R$ is a continuous function which changes sign on $[a,b] \times R$ and positive function $m(x) \in C([a,b])$.

In [5], we obtained the existence of an interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$ problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

admits at least three weak solutions in whose norms in $W_0^{1,p}(\Omega)$ are less than q, where $\Omega \subset R^N (N \geq 2)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$, $f : \Omega \times R \to R$ is a continuous function and positive weight function $a(x) \in C(\overline{\Omega})$.

For additional approaches to the existence of multiple solutions to boundary value problems, see [6, 16, 17] and references therein.

In the present paper, our approach is based on a three critical points theorem proved in [21], recalled below for the reader's convenience (Theorem 2.1), on a technical lemma (Lemma 2.3) that allow us to apply it.

Theorem 2.4 which is our main result, ensures the existence of an open interval $\Lambda \subseteq [0,\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the Eq. (1) admits at least three weak solutions whose norms in $W^{1,2}([0,1])$ are less than q.

As some consequences of Theorem 2.4, we obtain Corollary 2.5 and Theorem 2.6.

Corollary 2.5 ensures the existence of three weak solutions for the problem

$$\begin{cases} -u'' + u = \lambda h_1(x)h_2(u), \\ u'(0) = u'(1) = 0, \end{cases} \tag{7}$$

where $h_1 : [0,1] \to R$ and $h_2 : R \to R$ are two continuous functions such that $h_1(x) \geq 0$ in $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $h_2(t) \geq 0$ in $[0, \frac{d}{4}]$ for a positive constant $d \in R$.

Finally, Theorem 2.6 ensures the existence of three weak solutions for the problem

$$\begin{cases} -u'' + u = \lambda f(u), \\ u'(0) = u'(1) = 0, \end{cases} \tag{8}$$

where $f : R \to R$ is a continuous function, and we present an example to illustrate the results.

The aim of the present paper is to extend the main result of [10] in the case Neumann boundary condition.

2 Main results

First we here recall for the reader's convenience the three critical points theorem of [21] and Proposition 3.1 of [20]:

Theorem 1. *Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semicontinuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact.*

Assume that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty,$$

for all $\lambda \in [0, +\infty[$, and that there exists a continuous concave function $h : [0, +\infty[\to R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda \Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda \Psi(u) + h(\lambda)).$$

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0,$$

has at least three solutions in X whose norms are less than q.

Proposition 1. *Let X be a non-empty set and $\Phi,\ J$ two real function on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = J(x_0) = 0,\ \Phi(x_1) > r, \sup_{x \in \Phi^{-1}(]-\infty, r])} J(x) < r \frac{J(x_1)}{\Phi(x_1)}.$$

Then, for each ρ satisfying

$$\sup_{x \in \Phi^{-1}(]-\infty, r])} J(x) < \rho < r \frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - J(x))).$$

Here and in the sequel, X will denote the Sobolev space $W^{1,2}([0,1])$ endowed with the norm

$$|| u ||= \left(\int_0^1 |u'(x)|^2 dx + \int_0^1 |u(x)|^2 dx \right)^{1/2}.$$

Put

$$g(x,t) = \int_0^t f(x, \xi) d\xi,$$

for each $(x,t) \in [0,1] \times R$.

Our main results fully depend on the following lemma:

Lemma 1. *Assume that there exist two positive constants* c *and* d *with* $2\sqrt{30}c < 13d$ *such that*
(1) $g(x,t) \geq 0$ *for each* $(x,t) \in ([0,\frac{1}{4}] \cup [\frac{3}{4},1]) \times [0,\frac{d}{4}]$,
(2) $\frac{169}{c^2} \max_{(x,t)\in[0,1]\times[-c,c]} g(x,t) < \frac{120}{d^2} \int_{\frac{1}{4}}^{\frac{3}{4}} g(x,\frac{d}{4})dx$. *Then, there exist* $r > 0$ *and* $w \in X$ *such that* $||w||^2 > 2r$ *and*

$$\max_{(x,t)\in[0,1]\times[-2\sqrt{r},\,2\sqrt{r}]} g(x,t) < 2r\frac{\int_0^1 g(x,w(x))dx}{||w||^2}.$$

Proof. We put

$$w(x) = \begin{cases} 4dx^2, & 0 \leq x < \frac{1}{4} \\ \frac{d}{4}, & \frac{1}{4} \leq x \leq \frac{3}{4} \\ 4d(1-x)^2, & \frac{3}{4} < x \leq 1 \end{cases}$$

and $r = \frac{1}{4}c^2$. It is easy to see that $w \in X$ and, in particular, one has $||w||^2 = \frac{169}{240}d^2$.Hence, taking into account that $2\sqrt{30}c < 13d$, one has$2r < ||w||^2$. Since $0 \leq w(x) \leq \frac{d}{4}$ for each $x \in [0,1]$, condition (i) ensures that

$$\int_0^{\frac{1}{4}} g(x,w(x))\mathrm{d}x + \int_{\frac{3}{4}}^1 g(x,w(x))\mathrm{d}x \geq 0. \tag{9}$$

Moreover, from Eq. (2) and Eq. (9), we have

$$\max_{(x,t)\in[0,1]\times[-2\sqrt{r},2\sqrt{r}]} g(x,t) < \frac{120}{169}(\frac{c}{d})^2 \int_{\frac{1}{4}}^{\frac{3}{4}} g(x,\frac{d}{4})\mathrm{d}x \leq 2r\frac{\int_0^1 g(x,w(x))\mathrm{d}x}{||w||^2}.$$

Namely

$$\max_{(x,t)\in[0,1]\times[-2\sqrt{r},2\sqrt{r}]} g(x,t) < 2r\frac{\int_0^1 g(x,w(x))\mathrm{d}x}{||w||^2}.$$

So, the proof is complete. □

Now, we state our main result:

Theorem 2. *Assume that there exist three positive constants* c, d, s *with* $2\sqrt{30}c < 13d$, $s < 2$ *and a positive function* $a \in L^1$ *such that*
(1) $g(x,t) \geq 0$ *for each* $(x,t) \in ([0,\frac{1}{4}] \cup [\frac{3}{4},1]) \times [0,\frac{d}{4}]$,
(2) (ii) $\frac{169}{c^2} \max_{(x,t)\in[0,1]\times[-c,c]} g(x,t) < \frac{120}{d^2} \int_{\frac{1}{4}}^{\frac{3}{4}} g(x,\frac{d}{4})dx$,
(3) $g(x,t) \leq a(x)(1+|t|^s)$ *almost everywhere in* $[0,1]$ *and for each* $t \in R$. *Then, there exist an open interval* $\Lambda \subseteq [0,+\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *the Eq. (1) admits at least three solutions in* X *whose norms are less than* q.

Proof. For each $u \in X$, we put

$$\Phi(u) = \frac{||u||^2}{2}, \Psi(u) = -\int_0^1 g(x,u(x))\mathrm{d}x.$$

Of course, Φ is a continuously $G\hat{a}teaux$ differentiable and sequentially weakly lower semi continuous functional whose $G\hat{a}teaux$ derivative admits a continuous inverse on X^* and Ψ is a continuously $G\hat{a}teaux$ differentiable functional whose $G\hat{a}teaux$ derivative is compact. In particular, for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_0^1 u'(x)v'(x)\mathrm{d}x + \int_0^1 u(x)v(x)\mathrm{d}x, \Psi'(u)(v) = -\int_0^1 f(x,u(x))v(x)\mathrm{d}x.$$

Hence, the weak solutions of the Eq. (1) are exactly the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

Furthermore, thanks to the assumption (3), for each $\lambda > 0$, one has that $\lim_{||u||\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty$.

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Now, taking into account that for every $u \in X$, one has

$$\max_{x\in[0,1]}|u(x)| \le \sqrt{2}||u||,$$

for each $u \in X$, it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) = \sup_{||u||^2\le 2r}\int_0^1 g(x,u(x))\mathrm{d}x \le \max_{(x,t)\in[0,1]\times[-2\sqrt{r},\,2\sqrt{r}]} g(x,t),$$

and, thanks to Lemma 2.3, there exist $r > 0$ and $w \in X$ such that $||w||^2 > 2r$ and

$$\max_{(x,t)\in[0,1]\times[-2\sqrt{r},\,2\sqrt{r}]} g(x,t) < 2r\frac{\int_0^1 g(x,w(x))\mathrm{d}x}{||w||^2}.$$

So, the claim is true. Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < \rho < r\frac{(-\Psi(w))}{\Phi(w)},$$

and with $J = -\Psi$, $x_0 = 0$ and $x_1 = w$, from Proposition 2.2, we obtain

$$\sup_{\lambda\ge 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\ge 0}(\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 2.1. □

Let $h_1 \in C([0,1])$ and $h_2 \in C(R)$ be two functions. Put

$$H(t) = \int_0^t h_2(\xi)d\xi,$$

for all $t \in R$. We have the following consequence of Theorem 2.4:

Corollary 1. *Assume that there exist three positive constants c, d, s with $2\sqrt{30}c < 13d$, $s < 2$ and a positive function $b \in L^1$ such that*
(1) $h_1(x) \ge 0$ for each $x \in [0,\frac{1}{4}] \cup [\frac{3}{4},1]$ and $h_2(t) \ge 0$ for each $t \in [0,\frac{d}{4}]$,
(2) $\frac{169}{c^2}\max_{(x,t)\in[0,1]\times[-c,c]} h_1(x)H(t) < \frac{120}{d^2}H(\frac{d}{4})\int_{\frac{1}{4}}^{\frac{3}{4}} h_1(x)dx$,
(3) $h_1(x)H(t) \le b(x)(1 + |t|^s)$ almost everywhere in $[0,1]$ and for each $t \in R$.

Then, there exist an open interval $\Lambda \subseteq [0,+\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the Eq. (7) admits at least three solutions in X whose norms are less than q.

Proof. Put $f(x,t) = h_1(x)h_2(t)$ for each $(x,t) \in [0,1] \times R$, and note that

$$\max_{(x,t)\in[0,1]\times[-c,c]} g(x,t) = \max_{(x,t)\in[0,1]\times[-c,c]} h_1(x)H(t),$$

it is easy to verify that all the assumptions of Theorem 2.4 are satisfied. So, by using Theorem 2.4 we have the desired results. □

Here, we point out the following remarkable case of Theorem 2.4.

Theorem 3. *Let $f : R \to R$ be a continuous function. Put $g(t) = \int_0^t f(\xi)d\xi$ for each $t \in R$ and assume that there exist four positive constants c, d, s and η with $2\sqrt{30}c < 13d$, and $s < 2$ such that*
(1) $g(t) \geq 0$ for each $t \in [0, \frac{d}{4}]$,
(2) $169\frac{\max_{t\in[-c,c]} g(t)}{c^2} < 120\frac{g(\frac{d}{4})}{d^2}$,
(3) $g(t) \leq \eta(1 + |t|^s)$ for each $t \in R$.

Then, there exist an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the Eq. (8) admits at least three solutions in X whose norms are less than q.

We conclude this paper by giving the following example to illustrate the results of Theorem 2.6.

Example 2. Consider the problem

$$\begin{cases} -u'' + u = \lambda(e^{-u}u^7(8 - u)), \\ u'(0) = u'(1) = 0. \end{cases} \tag{10}$$

By choosing, for instance $c = 1,\ d = 16,\ s = 1$ and η sufficiently large, all the assumptions of Theorem 2.6 are satisfied. So, Theorem 2.6 is applicable to the Eq. (10).

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