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Equilibrium structure of differentially rotating and tidally distorted polytropic gaseous spheres

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Abstract. In this paper we propose a method for computing the equilibrium structure and various physical parameters of differentially rotating and tidally distorted polytropic stars. The law of differential rotation has been assumed to be in the form of $\omega(s) = (\sum_{i=1}^{3} a_i e^{-b_i s^2})^{\frac{1}{2}}$. The proposed method utilizes the averaging approach of Kippenhahn and Thomas and concepts of Roche- Equipotential in a manner earlier used by Mohan et al. to incorporate the effects of differential rotation and tidal distortion on the equilibrium structure of polytropic stellar models. The inner structure and various physical parameters of differentially rotating and tidally distorted polytropic models with various polytropic indices have been computed with suitable combination of the parameters. The local stability criterion has also been used for different stellar models with suitable combination of parameters.

Keywords: Roche-Equipotential, differential rotation, tidal distortion, polytropes, equilibrium structure

1 Introduction

Observation of Tassoul [23] shows that many of the stars are rotating stars and rotating about their axis. Some of the stars are observed to be binary stars in which primary component is rotating about their axis. Such stars are rotating differentially instead of solid body rotation (see Welty et al. [24]). In the case of a rotating star it is but natural to expect that rotation will distort its spherical-symmetric configuration, influence the inner structure and dynamical stability. To determine the equilibrium structure of rotationally distorted star is quite a complex problem. The complexity of the problem increases if the star is being influenced by differential rotation and tidal distortions. Therefore, approximate methods have been often used in literature to study such problem.

The inner structures of the realistic stars are generally described by the polytropes with different polytropic indices. Chandrasekhar [1] developed a theory of distorted polytropes. Since then several investigators have addressed themselves to this problem. Whereas Monaghan and Roxburgh [18], Roberts [21], Kopal [12], Geroyannis and Valvi [6] etc., considered the effects of solid body rotation on the equilibrium structures of the polytropic models of the stars, Mohan et al. [17] assumed these to be members of binary system and incorporated the effects of tidal forces as well. Some of the investigators such as Pariah [19], Linnel [13], Geroyannis et.al. [5] and Galli [4] have also considered the problems of differentially rotating stars. Stoeckly [22] obtained the numerical solution of the hydrostatic equilibrium equations for the stellar models with pressure density relation of the type $p \propto \rho^{\frac{3}{2}}$ for the case of non-uniform rotation having no meridianal currents and axial rotation. Ireland [9] presented results for gravity-darkening and limb-darkening in a rapidly rotating Roche model of a star subject to non-uniform rotation and demonstrated that the effect of small non-uniform rotation

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is likely to be of greater significance than the actual value of the rotational velocities themselves. Authors such as Haris and Clement [7] have also tried to analyze the problems of differentially rotating stars. Horedt [8] also analyzed polytropic models and their applications in astrophysics and related field.

For obtaining the equilibrium structure of polytropic stars, we used the averaging technique of Kippenhahn and Thomas [10], Mohan and Saxena [16] and Mohan, Lal and Singh [14] to account for the distortional effects caused by rotation and tidal forces. For computing the distortional effects, the actual equipotential surfaces of star are approximated by Roche-equipotential and Kopal's [11] results on the Roche-equipotential are then used to express the problem in a form convenient for numerical work. However, the problem is still far from having been satisfactorily answered.

Table 1. Differential rotation parameters for the various Polytropes (see Clement [3])

	$N = 2.00$	$N = 2.50$	$N = 3.00$	$N = 3.25$
a_1	0.546668	0.263144	0.095155	0.048836
a_2	0.544726	0.720053	0.555735	0.400167
a_3	−0.091395	0.016858	0.350959	0.550992
b_1	0.117936	0.097485	0.051248	0.037318
b_2	0.387444	0.290017	0.203307	0.15363
b_3	0.714485	0.021676	0.594146	0.490194

In this paper we have tried to investigate the general problem of determining the equilibrium structure of a class of differentially rotating and tidally distorted polytropic models of stars following the approach earlier used by Mohan et. al [15]. The primary component of the binary system is supposed to be rotating differentially according to the law of the type $\omega(s) = \sqrt{\sum_{i=1}^{3} a_i e^{-b_i s^2}}$, where $\omega(s)$ is the angular velocity of rotation, s is a non-dimensional cylindrical coordinate while a_i and b_i are numerical constants. This law was proposed by Clement [3] for studying the oscillations of differentially rotating stars. A differentially rotating model of star is to be stable against local perturbation, if the assumed law of differential rotation satisfy the stability criteria as one obtained by Stoeckly [22]. According to this criterion, a model rotating differentially according to the law $\omega = \omega(s)$ is stable if $\frac{\mathrm{d}}{\mathrm{d}s}[s^2\omega(s)] > 0$, for all s varying from the center to surface.

The paper is organized as follows: In Section 1, we present the modified Roche-equipotential surface of differentially rotating and tidally distorted stars. The methodology developed in Section 1 is further used in Section 2 to determine equilibrium structures of various differentially rotating and tidally distorted polytropes. Volume, surface area and other physical parameters for differentially rotating and tidally distorted polytropic models are also presented in this section. Numerical results and their analysis for different polytropic indices are reported in Section 3. The conclusions based on the present study are finally analyzed in Section 4.

2 Roche equipotential of differentially rotating gaseous spheres

In this section we investigate the problems of equilibrium structure of a polytropic model rotating differentially according to the law.

$$\omega(s) = \sqrt{\sum_{i=1}^{3} a_i e^{-b_i s^2}}, \tag{1}$$

which is earlier proposed by Clement [3] and the values of the parameters and are presented in Tab. 1.

Let M_0 and M_1 be the total masses of the primary and secondary components of a binary system, which are assumed to be gaseous spheres. Let D be the mutual separation between centers of these two masses.

Table 2. Combinations of rotational parameters for different polytropic stellar models and value of r_{os} for different polytropic indices of these models

Model No.	Parameters						q	Nature of stellar model	Polytropic index, values of r_{os}			
									$N = 2.0$	$N = 2.5$	$N = 3.0$	$N = 3.25$
1	0	0	0	0	0	0	0.1	Stable	0.499954	0.499966	0.499976	0.49998
2	$a1$	0	a_3	b_1	0	b_3	0.1	Stable	0.496064	0.498016	0.49757	0.497176
2-A	$a1$	0	a_3	b_1	0	b_3	0.01	Stable	0.496115	0.498056	0.497618	0.49733
2-B	$a1$	0	a_3	b_1	0	b_3	0.05	Stable	0.496096	0.498042	0.4976	0.49721
2-C	$a1$	0	a_3	b_1	0	b_3	0.15	Stable	0.496022	0.497983	0.497533	0.497139
3	0	a_2	a_3	b_1	b_2	0	0.1	Stable	0.496151	0.494712	0.494817	0.495302
3-A	0	a_2	a_3	b_1	b_2	0	0.01	Stable	0.496216	0.494772	0.494868	0.495348
3-B	0	a_2	a_3	b_1	b_2	0	0.05	Stable	0.495512	0.494748	0.494848	0.49533
3-C	0	a_2	a_3	b_1	b_2	0	0.15	Stable	0.496102	0.494667	0.494779	0.495267
4	$a1$	a_2	0	0	0	b_3	0.1	Stable	0.490244	0.492743	0.496369	0.497903
4-A	$a1$	a_2	0	0	0	b_3	0.05	Stable	0.490281	0.492775	0.496393	0.497923
4-B	$a1$	a_2	0	0	0	b_3	0.15	Stable	0.490302	0.492704	0.496337	0.497877
5	$a1$	a_2	a_3	0	0	0	0.1	Stable	0.491117	0.492608	0.494194	0.495021
5-A	$a1$	a_2	a_3	0	0	0	0.01	Stable	0.491176	0.492659	0.494239	0.495064
5-B	$a1$	a_2	a_3	0	0	0	0.05	Stable	0.491154	0.492639	0.494222	0.495048
5-C	$a1$	a_2	a_3	0	0	0	0.15	Stable	0.491073	0.492569	0.494159	0.494988
6	$a1$	a_2	a_3	b_1	b_2	b_3	0.1	Stable	0.49119	0.492658	0.49423	0.495044
6-A	$a1$	a_2	a_3	b_1	b_2	b_3	0.01	Stable	0.491261	0.492724	0.494297	0.495109
6-B	$a1$	a_2	a_3	b_1	b_2	b_3	0.05	Stable	0.491233	0.492698	0.49427	0.495082
6-C	$a1$	a_2	a_3	b_1	b_2	b_3	0.15	Stable	0.491139	0.492611	0.494184	0.494998
7	0	0	0	b_1	b_2	b_3	0.1	Stable	0.499954	0.499966	0.499976	0.499978
7-A	0	0	0	b_1	b_2	b_3	0.01	Stable	0.499998	0.499998	0.499999	0.499999
7-B	0	0	0	b_1	b_2	b_3	0.05	Stable	0.499982	0.499987	0.499992	0.499994
7-C	0	0	0	b_1	b_2	b_3	0.15	Stable	0.499915	0.499936	0.499532	0.499959

A stellar model rotating differentially according to the law $\omega = \omega(s)$ is stable if $\mathrm{d}[s^2.\omega(s)]/\mathrm{d}s > 0$, for all s from the center of surface[22].

Further suppose that the position of two components of this binary system is referred to a rectangular system of Cartesian coordinates having the origin at the centre of gravity of mass M_0.

In order to introduce the concept of Roche equipotential, the total potential Ω of a fluid element is given by

$$\mathrm{d}\Omega = \mathrm{d}V_0 + \mathrm{d}V_1 + \frac{1}{2}\omega^2 \mathrm{d}(s^2). \tag{2}$$

Following the Eq. (1), let us continue to expand the Eq. (2), and may be written as

$$\Omega = \frac{GM_0}{r} + \frac{GM_1}{r_1} + \frac{1}{2}\sum_{i=1}^{3} \frac{a_i}{b_i}\left[1 - e^{-b_i^{s^2}}\right], \tag{3}$$

where G is universal gravitational constant, again r and r_1 represent the distance of point P from the center of gravity, the terms GM_0/r and GM_1/r are the disturbing potentials of its companion to represent the potential arising from the mass of the components of masses M_0 and M_1 respectively. In Roche-equipotential approximation, let (r, θ, ϕ) be the spherical polar co-ordinates of the point with center of star as the origin, θ being measured from the axis of rotation then, we get $s^2 = r^2(1-v^2)$, where r is the distance of fluid element from the centre.

On multiplying both sides of Eq. (3) by D/GM and substituting the value of s, we get

$$\psi = \frac{1}{r} + q + q\sum_{j=2}^{\infty} r^j p_j(\lambda) + \frac{1}{2}r^2(1-v^2)\left[\sum_{i=1}^{3} a_i - \frac{1}{2}\sum_{i=1}^{3} a_i b_i^2 (1-v^2)^2 r^4 + \cdots\right], \tag{4}$$

where $\psi = D\Omega/GM$ is non-radial dimensional parameter, D the mutual separation between centers of these two mass components M_0 and M_1, and $\omega^2(s)$ is non-dimensional parameters of angular velocity of GM/D^3. In our present study, we shall assume that $\omega^2(s)$, a_i and b_i are the unit of GM/D^3 and s is a non-dimensional measure of the distance from the axis of rotation in unit of R_e, where R_e is the equatorial radius under investigation.

Table 3. Differentially rotating and/or Tidally distorted Polytropic models of star with index 2.0

	Model-1			Model-2			Model-3		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	3.45474	3.45559	3.45559	4.41184	3.5828	3.58513	3.95045	3.57373	3.57579
$S_\psi \times 10^{-2}$	2.38101	2.3813	2.38185	2.79953	2.43905	2.43992	2.7652	2.43543	2.4362
σ	0	0.027	0.042	0.288	0.0607	0.0774	0.2281	0.0585	0.0751
ε	0	0.0263	0.0403	0.2236	0.0572	0.0718	0.1857	0.0553	0.0699
ω_p	0	0	0	0.6747	0.6747	0.6747	0.6733	0.6733	0.6733
ω_e	0	0	0	0.1537	0.2146	0.2063	–	–	–
T_e/T_p	1	0.9805	0.97	0.9201	0.9766	0.9655	0.9791	0.9793	0.9683
L_e/L_p	1	0.8999	0.8495	0.5565	0.8575	0.8064	0.7484	0.8688	0.8175
	Model-4			Model-5			Model-6		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	4.12249	3.77893	3.78515	4.12249	3.7508	3.7541	4.01007	3.74118	3.74421
$S_\psi \times 10^{-2}$	2.63348	2.5264	2.52889	2.70287	2.51387	2.51508	2.67506	2.51046	2.51156
σ	0.3555	0.1119	0.1305	0.3582	0.1044	0.1227	1	0.1021	0.1203
ε	1	0.1007	0.1155	1.0447	0.0945	0.1093	0.6747	0.0926	0.1074
ω_p	1.0447	1.0474	1.0474	1	1	1	1	1	1
ω_e	1.0447	1.0447	1.0474	1	1	1	0.1438	0.1999	0.1913
T_e/T_p	0.8189	0.9652	0.9536	0.8257	0.9671	0.9556	0.8706	0.97	0.9586
L_e/L_p	0.2827	0.7804	0.7313	0.3021	0.7921	0.7427	0.3852	0.8033	0.7536

Kopal [11] developed the Roche-equipotential, assuming ψ = constant. On assuming this approach of analysis, we have developed the relation for co-ordinates (r, θ, ϕ) of an element of Roche-equipotential as

$$r = r_0 D\left[1 + \left(qp_2 + \frac{1}{2}\sum_{i=1}^{3} a_i x\right) r_0^3 + qp_3 r_0^4 + \left(qp_4 + \frac{1}{4}\sum_{i=1}^{3} a_i b_i x^2\right) r_0^5 + \left(qp_5 + 3q^2p_2^2 + 3qp_2\sum_{i=1}^{3} a_i x_i \right.\right.$$
$$\left. + \frac{3}{4}\sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j x^2\right) r_0^6 + \left(qp_6 + 7q^2p_2p_3 + \frac{7}{2}qp_3\sum_{i=1}^{3} a_i x_i + \frac{1}{12}\sum_{i=1}^{3} a_i b_i^2 x^3\right) r_0^7 + \left(qp_7 + 4q^2p_2^3\right.$$

$$+8q^2p_2p_4+4qp_4\sum_{i=1}^{3}a_ix_i-2qp_2\sum_{i=1}^{3}a_ib_ix^2-\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_ia_jx^3\Bigg)r_0^8+\big(qp_8+9q^2p_2p_5+9q^2p_3p_4$$
$$+\frac{9}{2}qp_2^2\sum_{i=1}^{3}a_ix+\frac{9}{2}qp_5\sum_{i=1}^{3}a_ix+\frac{9}{4}qp_3\sum_{i=1}^{3}\sum_{j=1}^{3}a_ia_jx^2-\frac{9}{4}qp_3\sum_{i=1}^{3}a_ib_ix^2+\frac{3}{8}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}a_ia_ja_kx^3$$
$$-\frac{1}{48}\sum_{i=1}^{3}a_ib_i^3x^4\Bigg)r_0^9+\Bigg(qp_9+10q^2p_2p_6+10q^2p_3p_5+5q^2p_4^2+6q^2p_2p_3\sum_{i=1}^{3}a_ix-5qp_6\sum_{i=1}^{3}a_ix$$
$$+3q^2p_2p_4\sum_{i=1}^{3}a_ix+\frac{3}{2}qp_3\sum_{i=1}^{3}\sum_{j=1}^{3}a_ia_jx^2-\frac{5}{2}qp_4\sum_{i=1}^{3}a_ib_ix^2+\frac{5}{6}qp_2\sum_{i=1}^{3}a_ib_i^2x^3+\frac{5}{12}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_i^2a_jx^4$$
$$-\frac{5}{16}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_ia_jb_jx^4\Bigg)r_0^{10}+\cdots\Bigg], \tag{5}$$

where $r_0=\frac{1}{\psi}, x=(1-v^2), p_j=p_j(\lambda)$ and D is the distance of mutual separation of stars. On setting $r=$ constant, we get shapes of various equipotential surface of the star distorted by differential rotation.

3 Equilibrium structures of differentially rotating polytropic gaseous spheres

We shall approximate the equipotential surfaces of distorted model by Roche-equipotential. Let P_ψ denote the pressure and p_ψ the density on the equipotential surface $\psi=$ constant of the distorted model. Then the value of density and pressure on the equivalent equipotential surface of the corresponding spherical model will also be p_ψ and P_ψ respectively. We shall assume that the distorted model also behaves like a polytropic model so that P_ψ and p_ψ are also connected through the polytropic type of relation:

$$P_\psi=P_{c\psi}\theta_\psi^{N+1}\quad\text{and}\quad\rho_\psi=\rho_{c\psi}\theta_\psi^N, \tag{6}$$

where P_c and ρ_c are respectively, the value of pressure and density at the center and θ is the parameter depending upon the distance of the chosen point from the center. N used in relation Eq. (6) is called polytropic index of model. It measures the central condensation of the model. Generally, in case of practical use, the value of N lies between zero and five. Polytropic model of index zero has a homogeneous structure, in which density is uniform throughout the model, while a polytropic model of index five is highly centrally condensed model for which radius extends to infinity. Following Chandrasekhar [2] and Mohan et.al. [15], the differential equation governing the equilibrium structure of differentially rotating and tidally distorted polytropic model of star can be written explicitly in the non-dimensional form as

$$\frac{1}{r_0^2}\frac{\mathrm{d}}{\mathrm{d}r_0}\left(\frac{r_0^2}{\rho_\psi f_2}\frac{\mathrm{d}P_\psi}{\mathrm{d}r_0}\right)=-4\pi GD^2\rho_\psi f_1. \tag{7}$$

On substituting the value of P_ψ and ρ_ψ from Eq. (6), the equation can be expressed as in the non-dimensional form as

$$\frac{1}{r_0^2}\frac{\mathrm{d}}{\mathrm{d}r_0}\left(\frac{r_0^2}{f_2}\frac{\mathrm{d}\theta_\psi}{\mathrm{d}r_0}\right)=-\frac{D^2}{\alpha^2}f_1\theta_\psi^N, \tag{8}$$

where

$$\alpha^2=\frac{(N+1)P_{P_{c\psi}}}{4\pi G\rho_{c\psi}^2}.$$

α is in the dimension of length. The boundary conditions for Eq. (8) are as follows:

$$r_0=0,\quad\theta_\psi=1,\quad\frac{\mathrm{d}\theta_\psi}{\mathrm{d}r_0}=0\quad\text{at the center and}\quad r_0=r_{os},\quad\theta_\psi=0\quad\text{at the surface.}$$

Table 4. Differentially rotating and/or Tidally distorted Polytropic models of star with index 2.5

	Model-1			Model-2			Model-3		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	6.43332	6.43537	6.43841	7.78412	6.59282	6.59689	7.78412	6.86203	6.86722
$S_\psi \times 10^{-2}$	3.60391	3.60453	3.60548	4.09883	3.66293	3.66419	4.32116	3.76227	3.76383
σ	0	0.027	0.042	0.167	0.0473	0.0634	0.2838	0.0813	0.099
ε	0	0.0963	0.043	0.1431	0.0451	0.0596	0.8584	0.0752	0.0901
ω_p	0	0	0	0.5292	0.5292	0.5292	0.8584	0.8584	0.8584
ω_e	0	0	0	0.1241	0.1454	0.1408	0.1299	0.13	0.13
T_e/T_p	1	0.9805	0.97	0.9672	0.9788	0.9678	0.9116	0.9748	0.9634
L_e/L_p	1	0.8999	0.8495	0.75	0.8764	0.8251	0.4947	0.835	0.7838
	Model-4			Model-5			Model-6		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	7.9862	7.04545	7.05522	7.92127	7.05661	7.06323	7.65094	7.0339	7.03993
$S_\psi \times 10^{-2}$	4.20812	3.82726	3.82923	4.18725	3.83129	3.83328	4.13112	3.82465	3.82646
σ	1	0.1041	0.1227	0.9916	0.1056	0.1241	1	0.1029	0.1214
ε	0.9947	0.0943	0.1093	0.8223	0.0955	0.1104	0.5292	0.0933	0.1082
ω_p	0.9916	0.9916	0.9916	1	1	1	1	1	1
ω_e	0.9916	0.9916	0.9916	1	1	1	0.1033	0.1335	0.1288
T_e/T_p	0.8223	0.9669	0.9552	0.821	0.9666	0.9549	0.872	0.97	0.9583
L_e/L_p	0.2954	0.7917	0.7416	0.2927	0.7895	0.7395	0.3866	0.8025	0.7522

r_{os} being the value of r_0 at the free surface.

If we set $r_\psi = \alpha\xi$, then ξ will be a non-dimensional variable defined for equivalent spherical model. It corresponds to the usual Emden variable ξ of Lane-Emden equation for an undistorted spherical polytropic model. If we take $D = \alpha\xi_u$ (where ξ_u is the value of ξ at the outermost surface of the undistorted polytropic model) in Eq. (8), the differential equation governing the equilibrium structure of a differentially rotating polytropic model may be written in non-dimensional form as

$$\frac{\mathrm{d}}{\mathrm{d}r_0}\left(A(a_1,a_2,a_3,b_1,b_2,b_3,r_0)\frac{\mathrm{d}\theta_\psi}{\mathrm{d}r}\right) = -\frac{\xi_u^2}{k^2}\theta_\psi^N r_0^2 B(a_1,a_2,a_3,b_1,b_2,b_3,r_0), \tag{9}$$

where

$$\begin{aligned} A =& r_0^2\left[1-\frac{1}{15}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ia_jr_0^6+\frac{8}{105}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_ia_jr_0^8+\left(\frac{64}{189}\sum_{i=1}^{3}a_ib_i^3-\frac{8986}{2835}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}a_ia_ja_k\right)r_0^9\right.\\ &\left.-\left(\frac{8}{315}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_i^2a_j+\frac{8}{315}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_ia_jb_j\right)r_0^{10}+\cdots\right], \end{aligned}$$

$$\begin{aligned} B =& 1+2\sum_{i=1}^{3}a_ir_0^3-\frac{16}{15}\sum_{i=1}^{3}a_ib_ir_0^5+\frac{24}{5}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ia_jr_0^6+\frac{8}{21}\sum_{i=1}^{3}a_ib_i^2r_0^7-\frac{44}{7}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_ia_jr_0^8-\left(\frac{32}{315}\right.\\ &\left.\sum_{i=1}^{3}a_ib_i^3-\frac{32}{5}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}a_ia_ja_k\right)r_0^9+\left(\frac{832}{315}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_i^2a_j+\frac{208}{105}\sum_{i=1}^{3}\sum_{j=1}^{3}a_ib_ia_jb_j\right)r_0^{10}+\cdots, \end{aligned}$$

Following the approach adopted by Mohan and Saxena [16], the volume, surface area and equatorial radius are given by

$$V_\psi = \frac{4}{3}\pi r_{os}^3 D^3 \Bigg[1+\sum_{i=1}^{3} a_i r_{os}^3 - \frac{2}{5}\sum_{i=1}^{3} a_i b_i r_{os}^5 + \left(\frac{12}{5}q^2 + \frac{4}{5}q\sum_i^3 a_i + \frac{8}{5}\sum_i^3\sum_j^3 a_i a_j\right) r_{os}^6 + \frac{4}{35}\sum_i^3 a_i b_i^2 r_{os}^7$$
$$+\left(\frac{15}{7}q^2 - \frac{4}{7}q\sum_i^3 a_i b_i - \frac{12}{7}\sum_i^3\sum_j^3 a_i b_i a_j\right) r_{os}^8 + \left(\frac{297}{198}q^2 + \frac{224}{35}q^2\sum_i^3 a_i + \frac{38}{35}q\sum_i^3\sum_j^3 a_i a_j\right.$$
$$\left. -\frac{8}{315}\sum_i^3 a_i b_i^3 + \frac{8}{5}q\sum_i^3\sum_j^3\sum_k^3 a_i a_j a_k\right) r_{os}^9 + \left(2q^2 - \frac{41}{30}q\sum_i^3 a_i b_i + \frac{3}{35}q\sum_i^3 b_i^2\right.$$
$$\left. +\frac{64}{105}\sum_i^3\sum_j^3 a_i b_i^2 a_j + \frac{16}{35}q\sum_i^3\sum_j^3 a_i b_i a_j b_j\right) r_{os}^{10} + \cdots \Bigg], \tag{10}$$

$$S_\psi = 4\pi r_{os}^2 D^2 \Bigg[1+\frac{2}{3}\sum_{i=1}^{3} a_i r_{os}^3 - \frac{4}{15}\sum_{i=1}^{3} a_i b_i r_{os}^5 + \left(\frac{7}{5}q^2 + \frac{4}{15}q\sum_i^3 a_i + \frac{14}{15}\sum_i^3\sum_j^3 a_i a_j\right) r_{os}^6 + \frac{8}{105}\sum_i^3 a_i$$
$$b_i^2 r_{os}^7 + \left(\frac{9}{7}q^2 - \frac{12}{35}q\sum_i^3 a_i b_i - \frac{36}{35}\sum_i^3\sum_j^3 a_i b_i a_j\right) r_{os}^8 + \left(\frac{45}{32}q^2 + \frac{96}{35}q^2\sum_i^3 a_i + \frac{12}{35}q\sum_i^3\sum_j^3 a_i a_j\right.$$
$$\left. -\frac{32}{1890}\sum_i^3 a_i b_i^3 + \frac{24}{35}q\sum_i^3\sum_j^3\sum_k^3 a_i a_j a_k\right) r_{os}^9 + \left(\frac{11}{9}q^2 - \frac{41}{210}q\sum_i^3 a_i b_i + \frac{44}{315}q\sum_i^3 b_i^2\right.$$
$$\left. +\frac{352}{945}\sum_i^3\sum_j^3 a_i b_i^2 a_j + \frac{88}{315}q\sum_i^3\sum_j^3 a_i b_i a_j b_j\right) r_{os}^{10} + \cdots \Bigg], \tag{11}$$

$$R_e = r_{os} D \Bigg[1+\left(q+\frac{1}{2}\sum_{i=1}^{3} a_i\right) r_{os}^3 + q r_{os}^4 + \left(q - \frac{1}{4}\sum_{i=1}^{3} a_i b_i\right) r_{os}^5 + \left(q + 3q^2 + 3q\sum_i^3 a_i + \frac{3}{4}\sum_i^3\sum_j^3 a_i\right.$$
$$\left. a_j\right) r_{os}^6 + \left(q + 7q^2 + \frac{7}{2}q\sum_i^3 a_i + \frac{1}{12}\sum_i^3 a_i b_i^2\right) r_{os}^7 + \left(q + 12q^2 + 4q\sum_i^3 a_i - 2q\sum_i^3 a_i b_i\right.$$
$$\left. -\frac{1}{12}\sum_i^3\sum_j^3 a_i b_i^2 - \sum_i^3\sum_j^3 a_i b_i a_j\right) r_{os}^8 + \left(q + 18q^2 + \frac{9}{2}q^2\sum_i^3 a_i - \frac{9}{4}q\sum_i^3\sum_j^3 a_i b_i - \frac{1}{48}\sum_i^3 a_i b_i^3\right.$$
$$\left. +\frac{3}{8}q\sum_i^3\sum_j^3\sum_k^3 a_i a_j a_k\right) r_{os}^9 + \left(q + 25q^2 - 5q\sum_i^3 a_i + \frac{5}{6}q\sum_i^3 b_i^2 - \frac{5}{2}\sum_i^3\sum_j^3 a_i a_j\right.$$
$$\left. +\frac{5}{12}\sum_i^3\sum_j^3 a_i b_i^2 a_j \frac{5}{16}q\sum_i^3\sum_j^3 a_i b_i a_j b_j\right) r_{os}^{10} + \cdots \Bigg], \tag{12}$$

Following Geroyannis and Valvi [6], obletness σ and ellipticity ε which measure the departure of the shape of star from spherical symmetry may be computed using

Table 5. Differentially rotating and/or Tidally distorted Polytropic models of star with index 3.0

	Model-1			Model-2			Model-3		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	1.37417	1.37469	1.37192	1.71827	1.43257	1.43353	1.7287	1.50541	1.50679
$S_\psi \times 10^{-2}$	5.97738	5.97865	5.97031	7.14171	6.14616	6.14846	7.11767	6.35171	6.35495
σ	0	0.027	0.0419	0.2239	0.0585	0.0753	0.492	0.0973	0.1159
ε	0	0.0263	0.0402	0.1829	0.0553	0.07	0.3298	0.0887	0.1038
ω_p	0	0	0	0.6679	0.6679	0.6679	0.9522	0.9522	0.9522
ω_e	0	0	0	0.0608	0.0798	0.0764	0.5924	0.5924	0.5924
T_e/T_p	1	0.9805	0.9701	0.979	0.9791	0.9679	0.8596	0.9698	0.958
L_e/L_p	1	0.8999	0.8501	0.7505	0.8684	0.8164	0.3659	0.8061	0.7549
	Model-4			Model-5			Model-6		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	1.65666	1.46811	1.46935	1.96033	1.52337	1.52488	1.67474	1.50566	1.51751
$S_\psi \times 10^{-2}$	6.95695	6.24513	6.24806	7.61622	6.40046	6.40421	6.97218	6.35471	6.38742
σ	0.5063	0.0772	0.0949	0.437	0.1069	0.1258	0.5239	0.1023	0.1219
ε	0.3361	0.0717	0.0867	0.3041	0.0966	0.1117	0.3438	0.0928	0.1086
ω_p	1.0009	0.8068	0.8068	0.8068	1.0009	1.0009	1.0009	1.0009	1.0009
ω_e	0.8068	0.8068	0.8068	0.8068	1.0009	1.0009	0.6679	0.073	0.069
T_e/T_p	0.861	0.9731	0.9615	0.8611	0.966	0.9541	0.8467	0.971	0.9589
L_e/L_p	0.3648	0.8323	0.7806	0.3825	0.7869	0.7361	0.3373	0.8065	0.7537

$$\sigma = \frac{R_e - R_p}{R_p} \quad \text{and} \quad \varepsilon = \frac{R_e - R_p}{R_e}. \tag{13}$$

The value of gravitational force g_p at the pole and g_e at the equator are given by

$$g_p = \frac{GM_0}{R_p^2}, \tag{14}$$

and

$$\begin{aligned} g_e =& \frac{GM_0}{R_p^2}\Bigg[1 + \left(2q + \sum_{i=1}^{3} a_i\right) r_{os}^3 - \left(4q - \sum_{i=1}^{3} a_i b_i\right) r_{os}^5 + \left(5q + 6q^2 + 6q\sum_{i}^{3} a_i + \frac{3}{2}\sum_{i}^{3}\sum_{j}^{3} a_i a_j\right) r_{os}^6 \\ & - \left(6q + \frac{1}{2}\sum_{i}^{3} a_i b_i^2\right) r_{os}^7 - \left(7q + 2q^2 - \frac{13}{2}q\sum_{i}^{3} a_i b_i - \sum_{i}^{3}\sum_{j}^{3} a_i b_i a_j -\right) r_{os}^8 - \big(9q + 18q^2 \\ & + \frac{1}{24}q\sum_{i}^{3}\sum_{j}^{3}\sum_{k}^{3} a_i a_j a_k\Bigg) r_{os}^9 - \Bigg(9q + 20q^2 - 10q\sum_{i}^{3} a_i b_i + 4q\sum_{i}^{3} a_i b_i^2 + 2\sum_{i}^{3}\sum_{j}^{3} a_i b_i^2 a_j \\ & + \frac{3}{2}q\sum_{i}^{3}\sum_{j}^{3} a_i b_i a_j b_j\Bigg) r_{os}^{10} + \cdots\Bigg], \end{aligned} \tag{15}$$

Following Ireland [9], the effective temperature at any point on the surface of the star can be obtained as

$$\left(\frac{T}{T_p}\right) = \left(\frac{g}{g_p}\right)^{\frac{1}{4}}, \tag{16}$$

Table 6. Differentially rotating and/or Tidally distorted Polytropic models of star with index 3.25

	Model-1			Model-2			Model-3		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	2.159929	2.16077	2.16194	2.71955	2.29253	2.29425	2.70275	2.391	2.39334
$S_\psi \times 10^{-2}$	8.080603	8.08238	8.08488	9.8163	8.40995	8.41351	9.60619	8.64649	8.6512
σ	0	0.027	0.042	0.2957	0.0703	0.0877	0.5191	0.1023	0.1211
ε	0	0.0263	0.0403	0.2282	0.0657	0.0807	0.3417	0.0928	0.108
ω_p	0	0	0	0.7745	0.7745	0.7745	0.9753	0.9753	0.9753
ω_e	0	0	0	0.0409	0.0568	0.0543	0.7423	0.7423	0.7423
T_e/T_p	1	0.9805	0.97	0.9608	0.9776	0.9662	0.8409	0.9679	0.9559
L_e/L_p	1	0.8999	0.8495	0.6577	0.8533	0.8011	0.3292	0.7961	0.7448
	Model-4			Model-5			Model-6		
	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$	$q=0.0$	$q=0.1$	$q=0.15$
$V_\psi \times 10^{-2}$	2.56532	2.26459	2.26634	3.06418	2.40647	2.40893	2.65637	2.39449	2.39663
$S_\psi \times 10^{-2}$	9.38343	8.33818	8.34182	10.2154	8.6827	8.68766	9.49841	8.65921	8.6635
σ	0.485	0.061	0.078	0.3151	0.1073	0.1263	0.5344	0.1034	0.1223
ε	0.3266	0.0575	0.0724	0.2396	0.0969	0.1122	0.3483	0.0937	0.109
ω_p	1	0.6701	0.6701	0.6701	1	1	1	1	1
ω_e	0.6701	0.6701	0.6701	0.6701	1	1	0.7745	0.0528	0.0503
T_e/T_p	0.8772	0.9761	0.9648	0.9053	0.9658	0.9538	0.8351	0.9707	0.9588
L_e/L_p	0.3987	0.8557	0.8038	0.5107	0.7859	0.7748	0.317	0.8045	0.753

where T_p is the polar temperature. Once temperature is known as, the radiative flux L at any point on the surface may be estimated using

$$L = -\frac{4ac}{3\rho\chi} T^3 \, grad \, T, \tag{17}$$

where χ is the opacity, T is the gas temperature, a is the radiative constant, c is the velocity of light.

4 Analysis of results

The numerical solution of nonlinear differential Eq. (9) has been obtained in this section. The values of rotational parameters have been taken from Tab. 1. The value of r_{os} thus obtained may be used in the above formulae to determine the volume, the surface area and the shape of outermost equipotential surface of differentially rotating polytropes. Various models obtained by the suitable combination of the parameters a_i, b_i and values of r_{os} for different polytropic indices of these models are given in Tab. 2. The Eq. (9) has been integrated subject to the boundary conditions Eq. (8) for the specified values of the parameters N and ξ_u which denote respectively the polytropic index and the radius of undistorted polytropic model. We have integrated the Eq. (9) for obtaining numerical solution by fourth order Runge-Kutta method for the specified values of the input parameters. On taking starting values from developed series solution at $r_0 = 0.005$, numerical integration has been carried forward using step length 0.005 and continued till θ_ψ first becomes zero. By this approach, we have found the values of r_{os} for different differentially rotating polytropic model for different polytropic indices. Relations Eqs. (10) and (11) were then used to determine the volume and shape of the distorted polytropic model (due to the differential rotation and tidal distortion), like volume and surface area. In this computation we use the value of α equal to one.

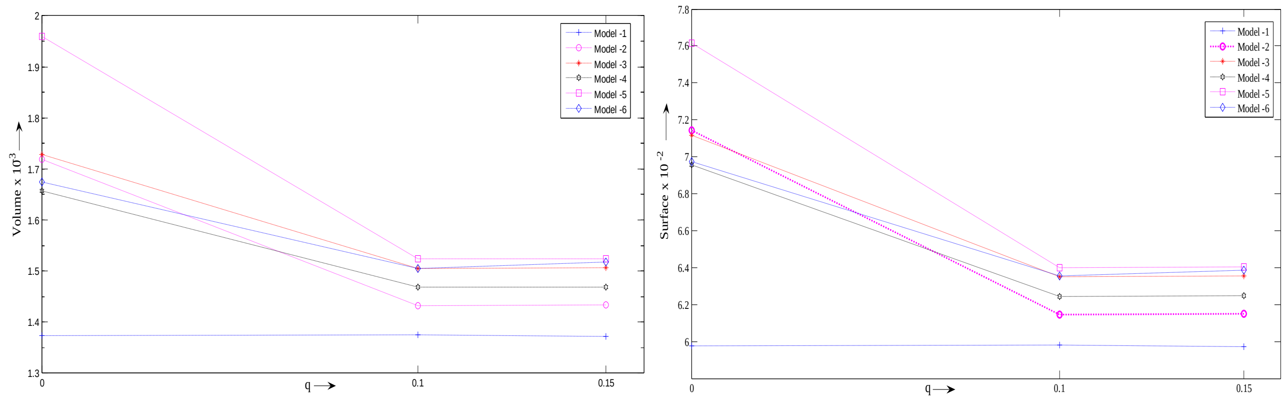

Fig. 1. Graphs of variations for different quantities versus tidal distortion for polytropic index 3.0

The values of V_ψ, S_ψ, σ, ε, ω_p, ω_e, T_e/T_p and L_e/L_p which represent volume, surface area, obletness , ellipticity, angular velocity at the pole and equator, ratio of temperature at an equator and pole and ratio of luminosity at the equator and pole respectively, for various differentially rotating and/or tidally distorted polytropic models with polytropic indices 2.0, 2.5, 3.0 and 3.25 are shown in Figs. 1 $\sim$ 3.

5 Concluding observations

Model 1 is a non-rotating model which becomes non-rotating but tidally distorted if we include the effect of parameter q. In comparison of undistorted model, due to the effect of tidal distortion, volume, surface area, obletness and ellipticity increase and the values of T_e/T_p and L_e/L_p decrease. The stellar models which are only rotating under differential rotation give maximum volume, surface area, obletness and ellipticity, while these quantities decrease due to the effect of tidal distortion. But the stellar models of such types give least T_e/T_p and L_e/L_p, which increase with tidal effect.

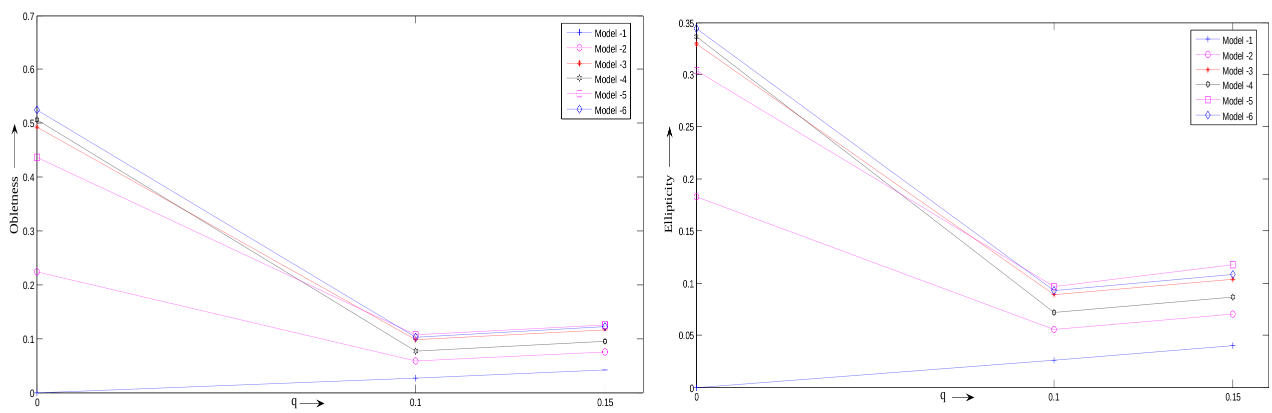

Fig. 2. Graphs of variations for different quantities versus tidal distortion for polytropic index 3.0

In our observations we have noticed that the polytropic stellar models of each polytropic indices have more volume, surface area, obletness and ellipticity for $q = 0.1$, but, the values of these quantities are less with tidal effect of $q = 0.15$. In this study we have found that the influence of tidal distortion does not affect the polar angular velocity while the angular velocity at the equator increases. It has also been noticed that the volume and surface area increase on increasing the polytropic indices.

We can drawn more conclusions for practical significance with the help of Tab. 1 $\sim$ 6 and Figs. 1 $\sim$ 3 which show the variations in the values of different quantities and parameters due to the effects of differential rotations and tidal distortions.

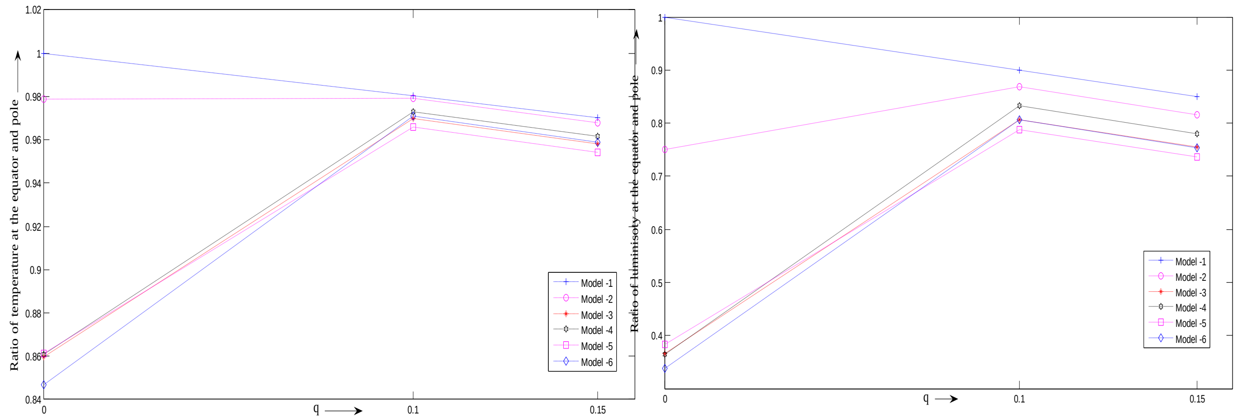

Fig. 3. Graphs of variations for different quantities versus tidal distortion for polytropic index 3.0

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