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Periodic solution in a delayed predator prey model with Holling type III functional response and harvesting term*

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Abstract. In this paper, a delayed predator-prey system with harvesting and Holling type-III functional response is investigated. The local stability of a positive equilibrium and the existence of Hopf bifurcations are established. By using the normal form theory and center manifold reduction, the explicit algorithm determining the stability, direction of the bifurcating periodic solutions are derived. Finally, numerical simulations are carried out to illustrate the theoretical results.

Keywords: time delay, harvesting, stability, Hopf bifurcation, functional response

1 Introduction

As we known, in the real world all biological resources are renewable. The management and exploitation of these resources, will affect the continuable development of renewable resources. Clark[2] considered the MSY (maximum sustainable yields) and OSY (optimum sustainable yield) for renewable resources. Brauer and Soudack[1], Dai and Tang[4], Xiao and Ruan[10] studied the predator-prey systems with constant-rate harvesting. The stability of the equilibria, existence of Hopf bifurcation, limit cycles was studied. We note that in the biological system the time delay is unavoidable. Time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate. Wangersky and Cunningham[9] studied the following predator-prey systems with time delay

$$\dot{x}(t) = x(t)\left(r_1 - ax(t) - a_1 y(t)\right), \quad \dot{y}(t) = -r_2 y(t) + a_2 x(t-\tau) y(t-\tau), \tag{1}$$

where r_1 is the rate of increase of the prey population, r_2 is the death rate of the predator population, a is the coefficient of competing with inner population of prey, a_1 is the coefficient of effect of predation on prey population, a_2 is the coefficient of effect of predation on predator population. Wangersky and Cunningham studied the effect of time delay on the system. On the basic[9], Annik Martin and Shigui Ruan[8] considered the following predator-prey systems with harvesting

$$\dot{x}(t) = x(t)\left(r_1 - ax(t) - a_1 y(t)\right) - H, \quad \dot{y}(t) = -r_2 y(t) + a_2 x(t-\tau) y(t-\tau), \tag{2}$$

where H is the constant-rate harvesting of the prey species x. Martin and Ruan studied the stability of the equilibria, the effect of the time delay τ and the harvest H on the system.

We note that functional response plays an important role in predator-prey systems. Based on the work developed in [8], in the present paper, we are concerned with the effect of functional response on the dynamics of a predator-prey model. To this end, we consider the following delay differential equations

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$$\dot{x}(t) = x(t)\left(r_1 - ax(t) - \frac{a_1x(t)y(t)}{1+mx^2(t)}\right) - H, \quad \dot{y}(t) = \frac{a_2x^2(t-\tau)y(t-\tau)}{1+mx^2(t-\tau)} - r_2y(t), \tag{3}$$

where the positive r_1, r_2, a, H are meanings to same as system Eq. (2), and $\frac{x^2}{1+mx^2}$ is Holling type-III functional response.

The initial conditions for system Eq. (3) take the form

$$x(\theta) = \phi(\theta), \quad y(\theta) = \psi(\theta), \quad \phi(\theta) \geq 0, \quad \psi(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad \phi(0) > 0, \quad \psi(0) > 0, \tag{4}$$

where $(\phi(\theta), \psi(\theta)) \in C([-\tau, 0], R^2_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^2_{+0}, where $R^2_{+0} = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$.

This paper is organized as follows. In Section 2, we discuss the local stability of a positive equilibrium of system Eq. (3) and the existence of Hopf bifurcations. In Section 3, we study the stability of the bifurcating periodic solutions and the direction of the Hopf bifurcation in system Eq. (3) by using the normal form theory and center manifold reduction. Finally, numerical examples are given.

2 Stability and hopf bifurcations

In this section, we discuss the local stability of equilibria and the existence of Hopf bifurcations of system Eq. (3).

It is easy to show that, if $r_1^2 = 4aH$ then system Eq. (3). has a boundary equilibrium $E_0(\frac{r_1}{2a}, 0)$; if $r_1^2 > 4aH$, then system Eq. (3) has two boundary equilibria

$$E_1\left(\frac{r_1 + \sqrt{r_1^2 - 4aH}}{2a}, 0\right) \text{ and } E_2\left(\frac{r_1 - \sqrt{r_1^2 - 4aH}}{2a}, 0\right).$$

Further, if the following holds

$$(1)\ a_2 - r_2m > 0, \quad 0 < H < \sqrt{r_2/(a_2 - r_2m)}r_1 - ar_2/(a_2 - r_2m).$$

System Eq. (3) has a unique positive equilibrium $E^* = (x^*, y^*)$, where

$$x^* = \sqrt{\frac{r_2}{a_2 - r_2m}}, \quad y^* = \frac{a_2}{a_1r_2}\left(\sqrt{\frac{r_2}{a_2 - r_2m}}r_1 - \frac{ar_2}{a_2 - r_2m} - H\right).$$

We now study the local stability of equilibria of system (3). Linearizing system (3) at E_1, we derive that

$$\dot{x}(t) = k_1x(t) + k_2y(t), \quad \dot{y}(t) = k_3y(t-\tau) - r_2y(t), \tag{5}$$

where

$$k_1 = -\sqrt{r_1^2 - 4aH}, \quad k_2 = -\frac{a_1(2r_1^2 + 2r_1\sqrt{r_1^2 - 4aH} - 4aH)}{4a^2 + m(2r_1^2 + 2r_1\sqrt{r_1^2 - 4aH} - 4aH)},$$
$$k_3 = \frac{a_2(2r_1^2 + 2r_1\sqrt{r_1^2 - 4aH} - 4aH)}{4a^2 + m(2r_1^2 + 2r_1\sqrt{r_1^2 - 4aH} - 4aH)}.$$

The characteristic equation of system (5) takes the form

$$(\lambda - k_1)(\lambda + r_2 - k_3e^{-\lambda\tau}) = 0. \tag{6}$$

Clearly, $\lambda = k_1$ is a negative real root. All other roots are given by the roots of equation

$$\lambda + r_2 - k_3e^{-\lambda\tau} = 0. \tag{7}$$

Let $f(\lambda)=\lambda+r_2-k_3e^{-\lambda\tau}$. If $r_2<k_3$, it is easy to see that for λ real,

$$f(0)=r_2-k_3<0, \qquad \lim_{\lambda+\to\infty} f(\lambda)=+\infty.$$

Hence, $f(\lambda)=0$ has a positive real root. Therefore, the equilibrium E_1 is unstable. If $r_2>k_3$, when $\tau=0$, it is easy to see that the equilibrium E_1 is stable. By Kuang[7], we see that the equilibrium E_1 is locally stable for all $\tau>0$.

Linearizing system (3) at E_2, we derive that

$$\dot{x}(t)=k_4x(t)+k_5y(t), \quad \dot{y}(t)=k_6y(t-\tau)-r_2y(t), \tag{8}$$

where

$$k_4=\sqrt{r_1^2-4aH}, \quad k_5=-\frac{a_1(2r_1^2-2r_1\sqrt{r_1^2-4aH}-4aH)}{4a^2+m(2r_1^2-2r_1\sqrt{r_1^2-4aH}-4aH)},$$
$$k_6=\frac{a_2(2r_1^2-2r_1\sqrt{r_1^2-4aH}-4aH)}{4a^2+m(2r_1^2-2r_1\sqrt{r_1^2-4aH}-4aH)}.$$

The characteristic equation of system (8) takes the form

$$(\lambda-k_4)(\lambda+r_2-k_6e^{-\lambda\tau})=0. \tag{9}$$

Clearly, $\lambda=k_4$ is a positive real root. Hence, E_2 is always unstable.

Linearizing system (3) at E^*, we derive that

$$\dot{x}(t)=p_1x(t)+p_2y(t), \quad \dot{y}(t)=p_3x(t-\tau)+p_4y(t-\tau)+p_5y(t), \tag{10}$$

where

$$p_1=r_1-2ax^*-\frac{2a_1x^*y^*}{(1+m(x^*)^2)^2}, \quad p_2=-\frac{a_1r_2}{a_2}, \quad p_3=\frac{2a_2x^*y^*}{(1+m(x^*)^2)^2}, \quad p_4=r_2, \quad p_5=-r_2.$$

The characteristic equation of system (10) takes the form

$$\lambda^2+A\lambda+B\lambda e^{-\lambda\tau}+C+De^{-\lambda\tau}=0, \tag{11}$$

where $A=-(p_1+p_5), B=-p_4, C=p_1p_5, D=p_1p_4-p_2p_3$.

Flowing Cooke and Grossman[3], we know that if $\mathrm{i}\omega(\omega>0)$ is a root of Eq. (11), then

$$-\omega^2+\mathrm{Ai}\omega+\mathrm{Bi}\omega\mathrm{e}^{-\mathrm{i}\omega\tau}+\mathrm{C}+\mathrm{De}^{-\mathrm{i}\omega\tau}=0. \tag{12}$$

Separating the real and imaginary parts, we obtain

$$C-\omega^2=-B\omega\sin\omega\tau-D\cos\omega\tau, \quad A\omega=-B\omega\cos\omega\tau+D\sin\omega\tau. \tag{13}$$

It follows from Eq. (13) that

$$\omega^4+(A^2-B^2-2C)\omega^2+C^2-D^2=0. \tag{14}$$

Because $A^2-B^2-2C=p_1^2\geq 0$, if $C^2-D^2\geq 0$ (or $p_1\leq -a_1p_3/2a_2$), then Eq. (14) has no positive roots. So if $p_1\geq -a_1p_3/2a_2$, then Eq. (14) has only one positive root ω_0 defined by

$$\omega_0=\left[\frac{1}{2}\left((B^2+2C-A^2)+\sqrt{(B^2+2C-A^2)^2-4(C^2-D^2)}\right)\right]^{\frac{1}{2}}. \tag{15}$$

Define

$$\tau_j = \frac{1}{\omega_0}\left(\arccos\frac{-AB\omega_0^2+(C-\omega_0^2)D}{B^2\omega_0^2+D^2}+2j\pi\right), \quad j=0,1,\cdots. \tag{16}$$

Then (τ_j, ω_0) solves Eq. (12). This means that when $\tau = \tau_j$, Eq. (11) has a pair of purely imaginary roots $\pm \mathrm{i}\omega_0$.

Denote $\lambda = \lambda(\tau)$, from Eq. (11) we have

$$\left(\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\right)^{-1} = \frac{(2\lambda+A)e^{\lambda\tau}+B}{\lambda(B\lambda+D)} - \frac{\tau}{\lambda},$$

which leads to

$$\begin{aligned}
\operatorname{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)_{\tau=\tau_j}\right\} &= \operatorname{sign}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_j}\right\} \\
&= \operatorname{sign}\left\{\mathrm{Re}\left(-\frac{2\lambda+\mathrm{A}}{\lambda(\lambda^2+\mathrm{A}\lambda+\mathrm{C})}\right)_{\tau=\tau_j} + \mathrm{Re}\left(\frac{\mathrm{B}}{\lambda(\mathrm{B}\lambda+\mathrm{D})}\right)_{\tau=\tau_j}\right\} \\
&= \operatorname{sign}\left\{\frac{\mathrm{A}^2-2(\mathrm{C}-\omega_0^2)}{(\mathrm{C}-\omega_0^2)^2+\mathrm{A}^2\omega_0^2} + \frac{-\mathrm{B}^2}{\mathrm{D}^2+\mathrm{B}^2\omega_0^2}\right\} \\
&= \operatorname{sign}\left\{\mathrm{A}^2-2\mathrm{C}-\mathrm{B}^2+2\omega_0^2\right\} \\
&= \operatorname{sign}\left\{\sqrt{(B^2+2C-A^2)^2-4(C^2-D^2)}\right\} > 0.
\end{aligned} \tag{17}$$

Therefore, when the delay τ near τ_j is increased, the root of Eq. (11) crosses the imaginary axis from left to right. In addition, note that when $\tau = 0$, Eq. (11) has roots with negative real parts only if

$$(2)\ p_1 < 0.$$

Applying Theorem 11.1 developed in [5], we obtain the following results.

Theorem 1. *Let ω_0 and $\tau_j (j = 0, 1, \cdots)$ be defined as in Eq. (15) and Eq. (16), respectively.*

(1) If $r_1^2 > 4aH$ and $r_2 < k_3$, the equilibrium E_1 of system (3) is unstable for all τ. If $r_1^2 > 4aH$ and $r_2 > k_3$, the equilibrium E_1 of system (3) is locally stable for all τ.

(2) If $r_1^2 > 4aH$, the equilibrium E_2 of system (3) is unstable for all τ.

(3) Let $(H1)$, $(H2)$ hold, when $p_1 \le -a_1p_3/2a_2$, the positive equilibrium E^* of system (3) is asymptotically stable for all τ .

(4) Let $(H1)$, $(H2)$ hold, when $p_1 > -a_1p_3/2a_2$, the positive equilibrium E^* of system (3) is asymptotically stable for all $\tau \in [0, \tau_0)$, and unstable for $\tau > \tau_0$. System (3) undergoes a Hopf Bifurcation at the positive equilibrium E^* when $\tau = \tau_j (j = 0, 1, \cdots)$.

Remark 1. Theorem 1 is deduction of result in [3].

3 Direction of hopf bifurcation

In this section, we derive explicit formulae to determine the properties of the Hopf bifurcation at critical values τ_j by using the normal form theory and center manifold reduction (see, for example, Hassard et al.[6]).

Without loss of generality, denote the critical values τ_j by $\tilde{\tau}$, and set $\tau = \tilde{\tau} + \mu$. Then $\mu = 0$ is a Hopf bifurcation value of system (3). Thus, we can work in the phase space $C = C([-\tilde{\tau}, 0], R^2)$. Let

$$u_1(t) = x(t) - x^*, \quad u_2(t) = y(t) - y^*.$$

Then system (3) is transformed into

$$\dot{u}_1(t) = p_1u_1(t) + p_2u_2(t) + \sum_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} u_1^i(t)u_2^j(t),$$

$$\dot{u}_2(t) = p_3u_1(t-\tau) + p_4u_2(t-\tau) + p_5u_2(t) + \sum_{i+j+l\geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} u_1^i(t-\tau)u_2^j(t-\tau)u_2^l(t), \tag{18}$$

where

$$f_{ij}^{(1)} = \frac{\partial^{i+j} f^{(1)}}{\partial x^i \partial y^j}\bigg|_{(x^*,y^*)}, \quad f_{ijl}^{(2)} = \frac{\partial^{i+j+l} f^{(2)}}{\partial x^i(t-\tau)\partial y^j(t-\tau)\partial y^l}\bigg|_{(x^*,y^*,y^*)}, \quad i,j,l\geq 0,$$
$$f^{(1)} = x(t)\Big(r_1 - ax(t) - \frac{a_1x(t)y(t)}{1+mx^2(t)}\Big) - H, \quad f^{(2)} = \frac{a_2x^2(t-\tau)y(t-\tau)}{1+mx^2(t-\tau)} - r_2y(t).$$

We rewrite Eq. (18) as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{19}$$

where $u(t) = (u_1(t), u_2(t))^T \in R^2$, $u_t(\theta) \in C$ is defined by $u_t(\theta) = u(t+\theta)$, and $L_\mu : C \to R, f : R\times C \in R$ are given by

$$L_\mu\phi = \begin{pmatrix} p_1 & p_2 \\ 0 & p_5 \end{pmatrix}\phi(0) + \begin{pmatrix} 0 & 0 \\ p_3 & p_4 \end{pmatrix}\phi(-\tilde{\tau}) \tag{20}$$

and

$$f(\mu,\phi) = \begin{pmatrix} \sum\limits_{i+j\geq 2} \frac{1}{i!j!} f_{ij}^{(1)} \phi_1^i(0)\phi_2^j(0) \\ \sum\limits_{i+j+l\geq 2} \frac{1}{i!j!l!} f_{ijl}^{(2)} \phi_1^i(-\tilde{\tau})\phi_2^j(-\tilde{\tau})\phi_2^l(0) \end{pmatrix} \tag{21}$$

respectively. By Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta \in [-\tilde{\tau}, 0]$ such that

$$L_\mu\phi = \int_{-\tilde{\tau}}^0 \mathrm{d}\eta(\theta, 0)\phi(\theta), \quad \text{for} \quad \phi \in C. \tag{22}$$

In fact, we can choose

$$\eta(\theta,\mu) = \begin{pmatrix} p_1 & p_2 \\ 0 & p_5 \end{pmatrix}\delta(\theta) - \begin{pmatrix} 0 & 0 \\ p_3 & p_4 \end{pmatrix}\delta(\theta+\tilde{\tau}), \tag{23}$$

where δ is the Dirac delta function. For $\phi \in C^1([-\tilde{\tau}, 0], R^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta\in[-\tilde{\tau},0) \\ \int_{-\tilde{\tau}}^0 \mathrm{d}\eta(\mu,s)\phi(s), & \theta = 0 \end{cases} \quad \text{and} \quad R(\mu)\phi = \begin{cases} 0, & \theta\in[-\tilde{\tau},0) \\ f(\mu,\phi), & \theta = 0 \end{cases}$$

Then system (19) is equivalent to

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{24}$$

for $\psi \in C^1([0,\tilde{\tau}], (R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in(0,\tilde{\tau}] \\ \int_{-\tilde{\tau}}^0 \mathrm{d}\eta^T(t,0)\psi(-t), & s = 0 \end{cases}$$

and a bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle=\bar{\psi}(0)\phi(0)-\int_{-\tilde{\tau}}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)\,\mathrm{d}\eta(\theta)\phi(\xi)d\xi, \tag{25}$$

where $\eta(\theta)=\eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. By discussions in Section 2 and foregoing assumption, we know that $\pm\mathrm{i}\omega_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $\mathrm{i}\omega_0$ and $-\mathrm{i}\omega_0$, respectively.

Suppose that $q(\theta)=(1,\rho)^T e^{\mathrm{i}\omega_0\theta}$ is the eigenvector of $A(0)$ corresponding to $\mathrm{i}\omega_0$. Then $A(0)q(\theta)=\mathrm{i}\omega_0\mathrm{q}(\theta)$. It follows from the definition of $A(0)$, Eq. (22) and Eq. (23) that

$$\begin{pmatrix} p_1-\mathrm{i}\omega_0 & p_2 \\ p_3e^{-\mathrm{i}\omega_0\tilde{\tau}} & p_4e^{-\mathrm{i}\omega_0\tilde{\tau}}+p_5-\mathrm{i}\omega_0\end{pmatrix}q(0)=\begin{pmatrix}0\\0\end{pmatrix}.$$

We therefore derive that

$$q(0)=(1,\rho)^T=\left(1,\frac{p_3e^{-\mathrm{i}\omega_0\tilde{\tau}}}{-p_4e^{-\mathrm{i}\omega_0\tilde{\tau}}-p_5+\mathrm{i}\omega_0}\right)^T.$$

On the other hand, suppose that $q^*(s)=D(1,\sigma)e^{\mathrm{i}\omega_0\mathrm{s}}$ is the eigenvector of A^* corresponding to $-\mathrm{i}\omega_0$. From the definition of A^*, Eq. (22) and Eq. (23) we have

$$\begin{pmatrix} p_1+\mathrm{i}\omega_0 & p_3e^{\mathrm{i}\omega_0\tilde{\tau}} \\ p_2 & p_4e^{\mathrm{i}\omega_0\tilde{\tau}}+p_5+\mathrm{i}\omega_0\end{pmatrix}(q^*(0))^T=\begin{pmatrix}0\\0\end{pmatrix},$$

then

$$q^*(0)=D(1,\sigma)=D\left(1,-\frac{p_2}{p_4e^{\mathrm{i}\omega_0\tilde{\tau}}+p_5+\mathrm{i}\omega_0}\right).$$

In order to assure $\langle q^*(s),q(\theta)\rangle=1$, we need to determine the value of D. From Eq. (25), we can choose

$$D=\frac{1}{1+\bar{\rho}\sigma+(p_3+p_4\bar{\rho})\sigma\tilde{\tau}e^{\mathrm{i}\omega_0\tilde{\tau}}}.$$

In the remainder of this section, we use the same notations as in Hassard et al.[6]. We first compute the coordinates to describe the center manifold C_0 at $\mu=0$. Let u_t be the solution of Eq. (2) with $\mu=0$. Define

$$z(t)=\langle q^*,u_t\rangle,\quad W(t,\theta)=u_t(\theta)-2\mathrm{Re}\{z(t)q(\theta)\}. \tag{26}$$

On the center manifold C_0 we have

$$W(t,\theta)=W(z(t),\bar{z}(t),\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\bar{z}+W_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots.$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. we consider only real solutions. For the solution $u_t\in C_0$ of Eq. (19), since $\mu=0$, we have

$$\begin{aligned}\dot{z}=&\langle q^*,\dot{u_t}\rangle=\langle q^*,A(\mu)u_t\rangle+\langle q^*,R(\mu)u_t\rangle\\ =&i\omega_0z+\langle\bar{q}^*(\theta),f(0,W(z,\bar{z},\theta)+2\mathrm{Re}\{zq(\theta)\})\rangle\\ =&i\omega_0z+\bar{q}^*(\theta)f(0,W(z,\bar{z},\theta)+2\mathrm{Re}\{zq(\theta)\})\\ =&i\omega_0z+\bar{q}^*(0)f(0,W(z,\bar{z},0)+2\mathrm{Re}\{zq(0)\})\\ \stackrel{\mathrm{def}}{=}&\mathrm{i}\omega_0\mathrm{z}+\bar{\mathrm{q}}^*(0)\mathrm{f}_0(\mathrm{z},\bar{\mathrm{z}}).\end{aligned} \tag{27}$$

We rewrite Eq. (27) as $\dot{z}=\mathrm{i}\omega_0\mathrm{z}+\mathrm{g}(\mathrm{z},\bar{\mathrm{z}})$ with

$$g(z,\bar{z})=\bar{q}^*(0)f_0(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}+\cdots. \tag{28}$$

Noting that $u_t(\theta) = (u_{1t}(\theta), u_{2t}(\theta)) = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta)$ and $q(\theta) = (1,\rho)^T e^{\mathrm{i}\omega_0\theta}$, we have

$$\begin{aligned}
u_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
u_{1t}(-\tilde{\tau}) &= e^{-\mathrm{i}\omega_0\tilde{\tau}}z + e^{\mathrm{i}\omega_0\tilde{\tau}}\bar{z} + W_{20}^{(1)}(-\tilde{\tau})\frac{z^2}{2} + W_{11}^{(1)}(-\tilde{\tau})z\bar{z} + W_{02}^{(1)}(-\tilde{\tau})\frac{\bar{z}^2}{2} + \cdots,\\
u_{2t}(0) &= \rho z + \bar{\rho}\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots,\\
u_{2t}(-\tilde{\tau}) &= \rho e^{-\mathrm{i}\omega_0\tilde{\tau}}z + \bar{\rho}e^{\mathrm{i}\omega_0\tilde{\tau}}\bar{z} + W_{20}^{(2)}(-\tilde{\tau})\frac{z^2}{2} + W_{11}^{(2)}(-\tilde{\tau})z\bar{z} + W_{02}^{(2)}(-\tilde{\tau})\frac{\bar{z}^2}{2} + \cdots.
\end{aligned}$$

Thus, it follows from Eq. (21) and Eq. (28) that

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) = \bar{D}(1,\bar{\sigma})\begin{pmatrix} \sum\limits_{i+j\geq 2}\frac{1}{i!j!}f_{ij}^{(1)}u_{1t}^i(0)u_{2t}^j(0)\\ \sum\limits_{i+j+l\geq 2}\frac{1}{i!j!l!}f_{ijl}^{(2)}u_{1t}^i(-\tilde{\tau})u_{2t}^j(-\tilde{\tau})u_{2t}^l(0)\end{pmatrix}\\
&= \bar{D}\Big\{z^2\Big(\frac{1}{2}f_{20}^{(1)} + f_{11}^{(1)}\rho\Big) + \bar{\sigma}z^2\Big(\frac{1}{2}f_{200}^{(2)}e^{-2\mathrm{i}\omega_0\tilde{\tau}} + f_{110}^{(2)}\rho e^{-2\mathrm{i}\omega_0\tilde{\tau}}\Big)\\
&\quad + z\bar{z}\Big(f_{20}^{(1)} + 2f_{11}^{(1)}Re\{\rho\}\Big) + \bar{\sigma}z\bar{z}\Big(f_{200}^{(2)} + 2f_{110}^{(2)}Re\{\rho\}\Big)\\
&\quad + \bar{z}^2\Big(\frac{1}{2}f_{20}^{(1)} + f_{11}^{(1)}\bar{\rho}\Big) + \bar{\sigma}\bar{z}^2\Big(\frac{1}{2}f_{200}^{(2)}e^{2\mathrm{i}\omega_0\tilde{\tau}} + f_{110}^{(2)}\bar{\rho}e^{2\mathrm{i}\omega_0\tilde{\tau}}\Big)\\
&\quad + z^2\bar{z}\Big(\frac{1}{2}f_{30}^{(1)} + \frac{1}{2}f_{21}^{(1)}\bar{\rho} + f_{21}^{(1)}\rho + \frac{1}{2}f_{11}^{(1)}W_{20}^{(2)}(0) + \frac{1}{2}f_{11}^{(1)}W_{20}^{(1)}(0)\bar{\rho}\Big)\\
&\quad + \bar{\sigma}z^2\bar{z}\Big(\frac{1}{2}f_{300}^{(2)}e^{-\mathrm{i}\omega_0\tilde{\tau}} + \frac{1}{2}f_{210}^{(2)}\Big(2\rho e^{-\mathrm{i}\omega_0\tilde{\tau}} + \bar{\rho}e^{-\mathrm{i}\omega_0\tilde{\tau}}\Big)\\
&\quad + f_{110}^{(2)}\Big(\frac{1}{2}W_{20}^{(1)}(-\tilde{\tau})\bar{\rho}e^{\mathrm{i}\omega_0\tilde{\tau}} + \frac{1}{2}W_{20}^{(2)}(-\tilde{\tau})e^{\mathrm{i}\omega_0\tilde{\tau}} + W_{11}^{(1)}(-\tilde{\tau})\rho e^{-\mathrm{i}\omega_0\tilde{\tau}}\\
&\quad + W_{11}^{(2)}(-\tilde{\tau})e^{-\mathrm{i}\omega_0\tilde{\tau}}\Big)\Big) + \cdots\Big\}.
\end{aligned}$$

Comparing the coefficients in Eq. (28), we get

$$\begin{aligned}
g_{20} &= \bar{D}\Big(f_{20}^{(1)} + 2f_{11}^{(1)}\rho + \bar{\sigma}\Big(f_{200}^{(2)}e^{-2\mathrm{i}\omega_0\tilde{\tau}} + 2f_{110}^{(2)}\rho e^{-2\mathrm{i}\omega_0\tilde{\tau}}\Big)\Big),\\
g_{11} &= \bar{D}\Big(f_{20}^{(1)} + 2f_{11}^{(1)}Re\{\rho\} + \bar{\sigma}\Big(f_{200}^{(2)} + 2f_{110}^{(2)}Re\{\rho\}\Big)\Big),\\
g_{02} &= \bar{D}\Big(f_{20}^{(1)} + 2f_{11}^{(1)}\bar{\rho} + \bar{\sigma}\Big(f_{200}^{(2)}e^{2\mathrm{i}\omega_0\tilde{\tau}} + 2f_{110}^{(2)}\bar{\rho}e^{2\mathrm{i}\omega_0\tilde{\tau}}\Big)\Big),\\
g_{21} &= \bar{D}\Big(f_{30}^{(1)} + f_{21}^{(1)}\bar{\rho} + 2f_{21}^{(1)}\rho + f_{11}^{(1)}W_{20}^{(2)}(0) + f_{11}^{(1)}W_{20}^{(1)}(0)\bar{\rho}\\
&\quad + \bar{\sigma}\Big(f_{300}^{(2)}e^{-\mathrm{i}\omega_0\tilde{\tau}} + f_{210}^{(2)}\big(2\rho e^{-\mathrm{i}\omega_0\tilde{\tau}} + \bar{\rho}e^{-\mathrm{i}\omega_0\tilde{\tau}}\big) + f_{110}^{(2)}\big(W_{20}^{(1)}(-\tilde{\tau})\bar{\rho}e^{\mathrm{i}\omega_0\tilde{\tau}}\\
&\quad + W_{20}^{(2)}(-\tilde{\tau})e^{\mathrm{i}\omega_0\tilde{\tau}} + 2W_{11}^{(1)}(-\tilde{\tau})\rho e^{-\mathrm{i}\omega_0\tilde{\tau}} + 2W_{11}^{(2)}(-\tilde{\tau})e^{-\mathrm{i}\omega_0\tilde{\tau}}\big)\Big)\Big).
\end{aligned} \tag{29}$$

We now compute $W_{20}(\theta)$ and $W_{11}(\theta)$. It follows from Eq. (23) and Eq. (25) that

$$\begin{aligned}
\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} &= \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\}, & \theta\in(0,\tilde{\tau}]\\ AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(0)\} + f_0, & \theta = 0\end{cases}\\
&\overset{\text{def}}{=} AW + H(z,\bar{z},\theta),
\end{aligned} \tag{30}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots . \tag{31}$$

On the other hand, on C_0 near the origin

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}}. \tag{32}$$

We derive from Eq. (30) to Eq. (32) that

$$(A - 2\mathrm{i}\omega_0)\mathrm{W}_{20}(\theta) = -\mathrm{H}_{20}(\theta), \quad \mathrm{AW}_{11}(\theta) = -\mathrm{H}_{11}(\theta), \quad \cdots . \tag{33}$$

It follows from Eq. (28) and Eq. (30) that for $\theta \in [-\tilde{\tau}, 0)$,

$$H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -gq(\theta) - \bar{g}\bar{q}(\theta). \tag{34}$$

Comparing the coefficients in Eq. (31) gives that for $\theta \in [-\tilde{\tau}, 0)$,

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) \tag{35}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) \tag{36}$$

derive from Eq. (33), Eq. (35) and the definition of A that

$$\dot{W}_{20}(\theta) = 2\mathrm{i}\omega_0\mathrm{W}_{20}(\theta) + \mathrm{g}_{20}\mathrm{q}(\theta) + \bar{\mathrm{g}}_{02}\bar{\mathrm{q}}(\theta).$$

Noting that $q(\theta) = q(0)e^{\mathrm{i}\omega_0\theta}$, it follows that

$$W_{20}(\theta) = \frac{\mathrm{i}\mathrm{g}_{20}}{\omega_0}q(0)e^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\bar{\mathrm{g}}_{02}}{3\omega_0}\bar{q}(0)e^{-\mathrm{i}\omega_0\theta} + E_1e^{2\mathrm{i}\omega_0\theta}, \tag{37}$$

where $E_1 = \left(E_1^{(1)}, E_1^{(2)}\right) \in R^2$ is a constant vector. Similarly, from Eq. (33) and Eq. (36), we can obtain

$$W_{20}(\theta) = \frac{\mathrm{i}\mathrm{g}_{20}}{\omega_0}q(0)e^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\bar{\mathrm{g}}_{02}}{3\omega_0}\bar{q}(0)e^{-\mathrm{i}\omega_0\theta} + E_1e^{2\mathrm{i}\omega_0\theta}, \tag{38}$$

where $E_2 = \left(E_2^{(1)}, E_2^{(2)}\right) \in R^2$ is also a constant vector.

In what follows, we seek appropriate E_1 and E_2. From the definition of A and Eq. (33), we obtain

$$\int_{-\tilde{\tau}}^{0} \mathrm{d}\eta(\theta)W_{20}(\theta) = 2\mathrm{i}\omega_0\mathrm{W}_{20}(0) - \mathrm{H}_{20}(0) \tag{39}$$

and

$$\int_{-\tilde{\tau}}^{0} \mathrm{d}\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{40}$$

where $\eta(\theta) = \eta(0,\theta)$. From Eq. (30), it follows that

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \begin{pmatrix} f_{20}^{(1)} + 2f_{11}^{(1)}\rho \\ f_{200}^{(2)}e^{-2\mathrm{i}\omega_0\tilde{\tau}} + 2f_{110}^{(2)}\rho e^{-2\mathrm{i}\omega_0\tilde{\tau}} \end{pmatrix} \tag{41}$$

and

$$H_{11}(0)=-g_{11}q(0)-\bar{g}_{11}\bar{q}(0)+\begin{pmatrix} f_{20}^{(1)}+2f_{11}^{(1)}Re\{\rho\} \\ f_{200}^{(2)}+2f_{110}^{(2)}Re\{\rho\}) \end{pmatrix}. \tag{42}$$

Substituting Eq. (37) and Eq. (41) into Eq. (39) and noticing that

$$\Big(\mathrm{i}\omega_0\mathrm{I}-\int_{-\tilde{\tau}}^{0}\mathrm{e}^{\mathrm{i}\omega_0\theta}\,\mathrm{d}\eta(\theta)\Big)\mathrm{q}(0)=0$$

and

$$\Big(-\mathrm{i}\omega_0\mathrm{I}-\int_{-\tilde{\tau}}^{0}\mathrm{e}^{-\mathrm{i}\omega_0\theta}\,\mathrm{d}\eta(\theta)\Big)\bar{\mathrm{q}}(0)=0,$$

we obtain

$$\Big(2\mathrm{i}\omega_0\mathrm{I}-\int_{-\tilde{\tau}}^{0}\mathrm{e}^{2\mathrm{i}\omega_0\theta}\,\mathrm{d}\eta(\theta)\Big)\mathrm{E}_1=\begin{pmatrix} f_{20}^{(1)}+2f_{11}^{(1)}\rho \\ f_{200}^{(2)}e^{-2\mathrm{i}\omega_0\tilde{\tau}}+2f_{110}^{(2)}\rho e^{-2\mathrm{i}\omega_0\tilde{\tau}} \end{pmatrix},$$

which leads to

$$\begin{pmatrix} 2\mathrm{i}\omega_0-\mathrm{p}_1 & -p_2 \\ -p_3e^{-2\mathrm{i}\omega_0\tilde{\tau}} & 2\mathrm{i}\omega_0-\mathrm{p}_5-\mathrm{p}_4\mathrm{e}^{-2\mathrm{i}\omega_0\tilde{\tau}} \end{pmatrix}E_1=\begin{pmatrix} f_{20}^{(1)}+2f_{11}^{(1)}\rho \\ f_{200}^{(2)}e^{-2\mathrm{i}\omega_0\tilde{\tau}}+2f_{110}^{(2)}\rho e^{-2\mathrm{i}\omega_0\tilde{\tau}} \end{pmatrix}.$$

It follows that

$$E_1^{(1)}=\frac{1}{A}\begin{vmatrix} f_{20}^{(1)}+2f_{11}^{(1)}\rho & -p_2 \\ f_{200}^{(2)}e^{-2\mathrm{i}\omega_0\tilde{\tau}}+2f_{110}^{(2)}\rho e^{-2\mathrm{i}\omega_0\tilde{\tau}} & 2\mathrm{i}\omega_0-\mathrm{p}_5-\mathrm{p}_4\mathrm{e}^{-2\mathrm{i}\omega_0\tilde{\tau}} \end{vmatrix}$$

and

$$E_1^{(2)}=\frac{1}{A}\begin{vmatrix} 2\mathrm{i}\omega_0-\mathrm{p}_1 & f_{20}^{(1)}+2f_{11}^{(1)}\rho \\ -p_3e^{-2\mathrm{i}\omega_0\tilde{\tau}} & f_{200}^{(2)}e^{-2\mathrm{i}\omega_0\tilde{\tau}}+2f_{110}^{(2)}\rho e^{-2\mathrm{i}\omega_0\tilde{\tau}} \end{vmatrix},$$

where

$$A=\begin{vmatrix} 2\mathrm{i}\omega_0-\mathrm{p}_1 & -p_2 \\ -p_3e^{-2\mathrm{i}\omega_0\tilde{\tau}} & 2\mathrm{i}\omega_0-\mathrm{p}_5-\mathrm{p}_4\mathrm{e}^{-2\mathrm{i}\omega_0\tilde{\tau}} \end{vmatrix}.$$

Similarly, we can get

$$\begin{pmatrix} -p_1 & -p_2 \\ -p_3 & 0 \end{pmatrix}E_2=\begin{pmatrix} f_{20}^{(1)}+2f_{11}^{(1)}Re\{\rho\} \\ f_{200}^{(2)}+2f_{110}^{(2)}Re\{\rho\}) \end{pmatrix}$$

and hence,

$$E_2^{(1)}=-\frac{1}{p_2p_3}\begin{vmatrix} f_{20}^{(1)}+2f_{11}^{(1)}Re\{\rho\} & -p_2 \\ f_{200}^{(2)}+2f_{110}^{(2)}Re\{\rho\} & 0 \end{vmatrix},$$

$$E_2^{(2)}=-\frac{1}{p_2p_3}\begin{vmatrix} -p_1 & f_{20}^{(1)}+2f_{11}^{(1)}Re\{\rho\} \\ -p_3 & f_{200}^{(2)}+2f_{110}^{(2)}Re\{\rho\} \end{vmatrix}.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from Eq. (37) and Eq. (38). Furthermore, we can determine g_{21}. Therefore, each g_{ij} in Eq. (29) is determined by the parameters and delay in system Eq. (18). Thus, we can compute the following values:

$$c_1(0)=\frac{\mathrm{i}}{2\omega_0}\Big(g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\Big)+\frac{g_{21}}{2},\quad \mu_2=-\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tilde{\tau})\}},$$
$$\beta_2=2\mathrm{Re}\{\mathrm{c}_1(0)\},\quad \mathrm{T}_2=-\frac{\mathrm{Im}\{c_1(0)\}+\mu_2\mathrm{Im}\{\lambda'(\tilde{\tau})\}}{\omega_0}, \tag{43}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value $\tilde{\tau}$, i.e., μ_2 determines the direction of the Hopf bifurcation: if $\mu_2>0\,(\mu_2<0)$, then the Hopf bifurcation is supercritical (subcritical); β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2<0\,(\beta_2>0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2>0\,(T_2<0)$.

4 Numerical examples

In this section, we give two examples to illustrate the result above.

First, we consider the following system

$$\dot{x}(t) = x(t)\left(1.2 - 0.6x(t) - \frac{0.1x(t)y(t)}{1 + x^2(t)}\right) - 0.4, \quad \dot{y}(t) = \frac{0.4x^2(t-\tau)y(t-\tau)}{1 + x^2(t-\tau)} - 0.2y(t). \tag{44}$$

It is easy to see that system Eq. (44) has a positive equilibrium $E^*(1, 4)$. By computing we know that $p_1 \leq -a_1p_3/2a_2$. By Theorem 1, we see that the positive equilibrium E^* is stable for all $\tau > 0$ (see, Fig. 1).

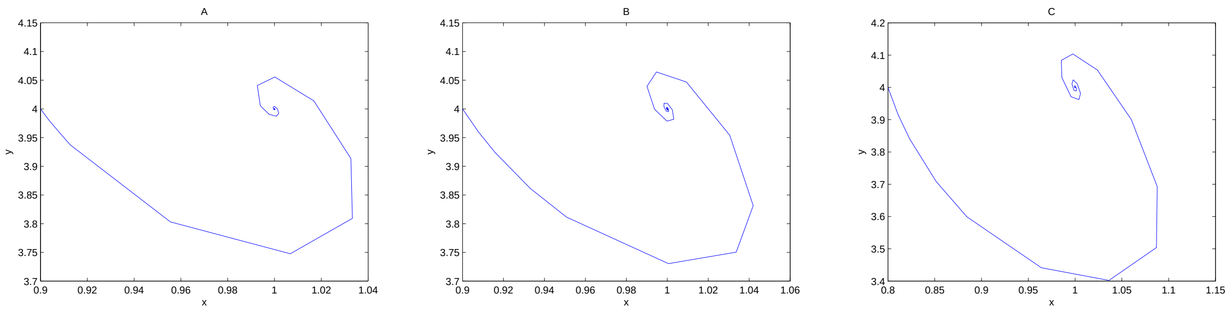

Fig. 1. The positive equilibrium E^* is asymptotically stable for all $\tau > 0$. Here $\tau = 0.1$ in A, $\tau = 1$ in B, $\tau = 10$ in C

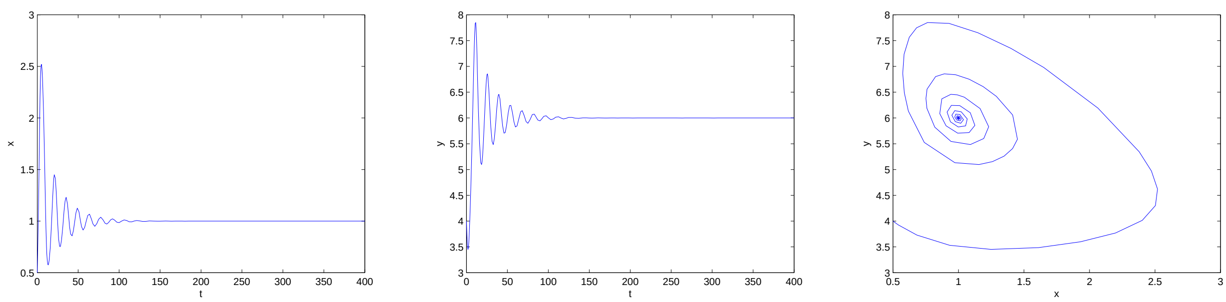

Fig. 2. When $\tau = 0.6 < \tau_0$, the positive equilibrium E^* is asymptotically stable

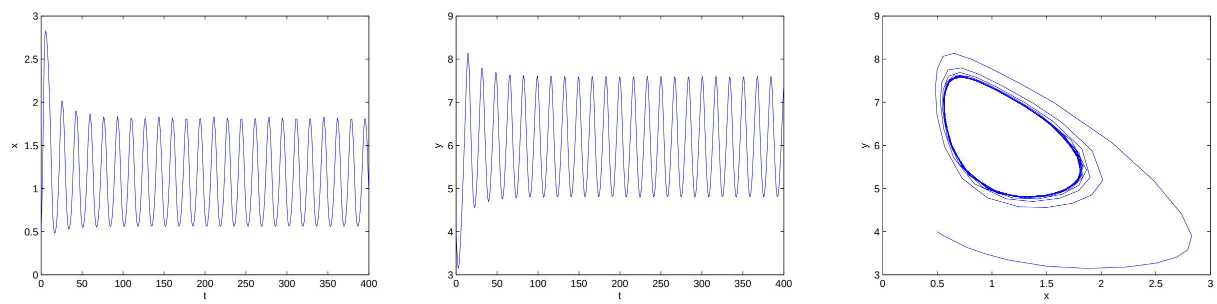

Fig. 3. When $\tau = 1.6 > \tau_0$, bifurcating periodic solutions from E^* occur

Next, we consider the following system:

$$\dot{x}(t) = x(t)\left(2 - 0.5x(t) - \frac{0.4x(t)y(t)}{1 + x^2(t)}\right) - 0.3, \quad \dot{y}(t) = \frac{0.4x^2(t-\tau)y(t-\tau)}{1 + x^2(t-\tau)} - 0.2y(t), \tag{45}$$

then system Eq. (45) has a positive equilibrium $E^*(1, 6)$. It is easy to show that $\tau_0 = 1.1604$. By computing we know that $p_1 > -a_1p_3/2a_2$. By Theorem 2.1, we see that the positive equilibrium E^* is stable when $\tau < \tau_0$ (see, Fig. 3); when $\tau > \tau_0$ E^* is unstable (see, Fig. 3); and system Eq. (45) undergoes a Hopf bifurcation at τ_j. When $\tau = \tau_0$, $c_1(0) = -0.1948 - 0.5621\text{i}$. It follows from Eq. (33) that $\mu_2 > 0$ and $\beta < 0$. Therefore, the Hopf bifurcation of system Eq. (45) is supercritical, and the bifurcating periodic solutions are stable.

In this paper we considered a delayed predator-prey system with harvesting and Holling type-III functional response. The local stability of equilibria of system (3) was studied. The existence of Hopf bifurcations at positive equilibrium was established. From Theorem 2.1 and Fig. 1, we see that if $p_1 \leq -a_1p_3/2a_2$, the time delay does not affect the stability of positive equilibrium of system (3). However, if $p_1 > -a_1p_3/2a_2$, the time delay can induce instability and oscillations in system (3) (see Figs. 2 and 3). On the other hand, the harvesting rate affects the positive equilibrium of system (3), as the harvesting rate increasing, the positive equilibrium may be vanished which means that the population of predator may become extinct.

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