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Global analysis of an HIV/AIDS epidemic model*

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Abstract. We present an HIV/AIDS epidemic model that incorporates constant recruitment and sexually active AIDS individuals. The disease-free and endemic equilibria are found and their local as well as global stabilities are investigated. Using a Lyapunov function and LaSalle's invariant set theorem, we proved that the disease-free equilibrium is globally asymptotically stable. Local stability of the endemic equilibrium is determined using the center manifold theory and using the Poincarè-Bendixson property, global asymptotic stability is proved.

Keywords: HIV/AIDS model, reproductive number, equilibria, stability.

1 Introduction

The extensive spread of human immunodeficiency virus (HIV) and the associated acquired immune deficiency syndrome (AIDS) continues around the world since its recognition in the early 1980s. In 2003, almost five million people became newly infected with HIV, the greatest number in any one year since the beginning of the epidemic. At a global level, the number of people living with HIV/AIDS continued to grow from 35 million in 2001 to 38 million in 2003 and in the same year, almost three million were killed by AIDS (UNAIDS [24]). HIV/AIDS had killed more than 25 million people by 2005, making it one of the most destructive epidemics recorded in history. Despite recent, improved access to antiretroviral treatment and care in many regions of the world, the HIV/AIDS epidemic claimed 3.1 million [2.8-3.6 million] lives in 2005, of which more than half a million (570000) were children (UNAIDS/WHO [25]). The HIV/AIDS epidemic has remained one of the leading causes of death in the world and has been destructive in Africa with Sub-Saharan Africa remaining the epidemiological locus of the epidemic. In all affected countries with either high or low HIV/AIDS prevalence, HIV/AIDS hinders development, exacting a devastating toll on individuals and families. In the hardest-hit countries, it is erasing decades of health, economic and social progress, reducing life expectancy by years, deepening poverty, and contributing to and exacerbating food shortages (UNAIDS [24]).

In this study we revisit a simple HIV/AIDS model with constant recruitment and sexually active AIDS individuals partially studied in [18]. Unlike in [18] we present a more systematic and complete qualitative analysis of the HIV/AIDS model. Further, the global stability of endemic equilibrium was investigated for a special case but here we proceed using the higher dimensional Poincarè-Bendixson property[13–16] to prove the global stability of the complete system. Next, in Section 2 we present the model development and analysis and in Section 3 we present numerical simulations with summary and concluding remarks rounding up the paper in Section 4.

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2 Model development and analysis

The HIV/AIDS model divides the sexually-active population into sub-populations of susceptible individuals (S), HIV infected with no clinical symptoms of AIDS (I) and HIV-infected individuals with AIDS symptoms (A). Thus the total sexually interacting adult population is given by $N(t) = S(t) + I(t) + A(t)$. It is assumed that susceptible individuals are recruited into the population at a per capita rate Λ. Susceptible individuals acquire HIV infection following contact with HIV-infected individuals at a rate λ given by

$$\lambda = \frac{pc(I + \eta A)}{N} = \frac{\beta(I + \eta A)}{N}. \tag{1}$$

In Eq. (1), p is the probability of being infected from a sexual partner and c is the rate at which an individual acquires sexual partners per unit time and $\beta = pc$ is the effective contact rate for HIV infection (contact sufficient to result in HIV infection). The parameter $\eta > 1$ captures the fact that individuals in the AIDS stage of HIV infection are more infectious than HIV-infected individuals displaying no clinical symptoms of AIDS. This is due to the fact that individuals in the AIDS stage have higher viral load compared to other HIV-infected individuals with no symptoms. The contribution of AIDS patients in the transmission of HIV is often ignored in most HIV epidemic models by imposing the simplifying assumptions that AIDS mortality is instantaneous or that AIDS patients are not capable of mixing and acquiring new sex partners. But, epidemiological evidence shows that AIDS patients do engage in risky sexual behavior defined in terms of inconsistent condom use or having multiple sex partners[11]. For example, the findings in [21], for a study of HIV-1-infected transfusion male recipients and their female sex partners, shows that advanced AIDS patients are more likely to infect their partners compared to recipient with no advanced immunodeficiency. Similar findings were also noted in [20]. Natural death occurs in all human sub-populations (at a rate μ). Further, there is an AIDS-related death constant δ in the AIDS class. The assumptions result in the following model equations

$$S' = \Lambda - \frac{\beta(I + \eta A)S}{N} - \mu S, \quad I' = \frac{\beta(I + \eta A)S}{N} - (\mu + \kappa)I, \quad A' = \kappa I - (\mu + \delta)A. \tag{2}$$

Consider the region

$$\Phi = \left\{(S, I, A) \in \mathbb{R}^3_+ : N \leq \Lambda/\mu\right\}.$$

It can be shown that all solutions of the system (2) starting in Φ remain in Φ for all $t \geq 0$. Thus, Φ is positively-invariant (hence, it is sufficient to consider the dynamics of (2) in Φ).

2.1 Local stability of the disease-free equilibrium

The HIV/AIDS model (2) has a disease-free equilibrium given by, $\mathcal{E}_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$. It is obvious that $\mathcal{E}_0$ attracts the region (stable manifold of $\mathcal{E}_0$)

$$\Phi_0 = \{(S, I, A) \in \Phi : I = A = 0\}.$$

The linear stability of $\mathcal{E}_0$ is governed by the basic reproductive number $\mathcal{R}_0$[1, 2, 5, 8]. The stability of this equilibrium will be investigated using the *next generation operator*[7, 26]. Using the notation in [26] on model system (2), the matrices F and V, for the new infection terms and the remaining transfer terms are respectively given by

$$\boldsymbol{F} = \begin{pmatrix} \beta & \beta\eta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{V} = \begin{pmatrix} \mu + \kappa & 0 \\ -\kappa & \mu + \delta \end{pmatrix}.$$

It follows that the *basic reproductive number*, denoted by $\mathcal{R}_0$ is given by

$$\mathcal{R}_0 = \rho(\boldsymbol{F}\boldsymbol{V}^{-1}) = \frac{\beta(\delta + \kappa\eta + \mu)}{(\kappa + \mu)(\delta + \mu)}. \tag{3}$$

Using Theorem 2 in [26], the following result is established.

Lemma 1. *The disease-free equilibrium of the HIV/AIDS model system (2) is locally asymptotically stable if* $\mathcal{R}_0 < 1$ *and unstable if* $\mathcal{R}_0 > 1$.

The basic reproductive number measures the average number of new infections generated by a single infected individual in a completely susceptible population. Thus, Lemma 1 implies that HIV/AIDS can be eliminated from the community (when $\mathcal{R}_0 < 1$) if the initial sizes of the sub-populations of the model are in the basin of attraction of the disease-free equilibrium $\mathcal{E}_0$. To ensure that elimination of the disease is independent of the initial sizes of the sub-populations, it is necessary to show that the disease-free equilibrium is globally-asymptotically stable. This is established below.

2.2 Global stability of the disease-free equilibrium

Theorem 1. *If* $\mathcal{R}_0 < 1$, *the disease-free equilibrium of the model (2) given by* $\mathcal{E}_0$ *is globally asymptotically stable in* Φ *and unstable if* $\mathcal{R}_0 > 1$.

Proof. Consider the following Lyapunov function

$$\mathcal{L} = (\mu + \delta + \eta\kappa)I + \eta(\mu + \kappa)A.$$

Its Lyapunov derivative along the solutions to model system (2) is

$$\begin{aligned}
\mathcal{L}' &= (\mu + \delta + \eta\kappa)I' + \eta(\mu + \kappa)A', \\
&= (\mu + \delta + \eta\kappa)\left(\beta(I + \eta A)\frac{S}{N} - (\mu + \kappa)I\right) + \eta(\mu + \kappa)\left(\kappa I - (\mu + \delta)A\right), \\
&= \left(\beta(\mu + \delta + \eta\kappa)\frac{S}{N} - (\mu + \delta)(\mu + \kappa)\right)I + \eta\left(\beta(\mu + \delta + \eta\kappa)\frac{S}{N} - (\mu + \delta)(\mu + \kappa)\right)A, \\
&\leq \left(\beta(\mu + \delta + \eta\kappa) - (\mu + \delta)(\mu + \kappa)\right)I + \eta\left(\beta(\mu + \delta + \eta\kappa) - (\mu + \delta)(\mu + \kappa)\right)A \quad \text{for} \quad S \leq N, \\
&= (\mu + \delta)(\mu + \kappa)(\mathcal{R}_0 - 1)I + \eta(\mu + \delta)(\mu + \kappa)(\mathcal{R}_0 - 1)A < 0 \quad \text{for} \quad \mathcal{R}_0 < 1.
\end{aligned}$$

Since all the model parameters are nonnegative, it follows that $\mathcal{L}' < 0$ for $\mathcal{R}_0 < 1$ with $\mathcal{L}' = 0$ only if $I = A = 0$. Hence, $\mathcal{L}$ is a Lyapunov function on Φ. Since Φ is invariant and attracting, it follows that the largest compact invariant set in $\{(S, I, A) \in \Phi : \mathcal{L}' = 0\}$ is the singleton $\{\mathcal{E}_0\}$. Therefore, by the LaSalle's Invariance Principle[12] every solution to the equations in the HIV-only model (2), with initial conditions in Φ, approaches $\mathcal{E}_0$ as $t \to \infty$. That is, $((I(t), A(t)) \to (0, 0))$ at $t \to \infty$. Substituting $I = A = 0$ in model system (2) gives $S \to \Lambda/\mu$ as $t \to \infty$. Thus, $(S(t), I(t), A(t)) \to (\Lambda/\mu, 0, 0)$ as $t \to \infty$ for $\mathcal{R}_0 < 1$ so that $\mathcal{E}_0$ is globally asymptotically stable in Φ if $\mathcal{R}_0 < 1$.

2.3 Endemic equilibria

The HIV/AIDS model system (2) has a unique endemic equilibrium given by $\mathcal{E}_e = (S^*, I^*, A^*) \in \overset{\circ}{\Phi}$, with

$$S^* = \frac{\Lambda(\kappa + \delta + \mu)}{\beta(\delta + \kappa\eta + \mu) - \kappa\delta}, I^* = \frac{\Lambda(\delta + \mu)(\mathcal{R}_0 - 1)}{\beta(\delta + \kappa\eta + \mu) - \kappa\delta} \text{ and } A^* = \frac{\kappa\Lambda(\mathcal{R}_0 - 1)}{\beta(\delta + \kappa\eta + \mu) - \kappa\delta}.$$

The term in the denominators of S^*, I^* and A^* is positive for $\mathcal{R}_0 > 1$, as shown below. We see that,

$$\begin{aligned}
\beta(\delta + \kappa\eta + \mu) - \kappa\delta &= (\kappa + \mu)(\delta + \mu)\left(\frac{\beta(\delta + \kappa\eta + \mu)}{(\kappa + \mu)(\delta + \mu)} - \frac{\delta}{(\delta + \mu)}\frac{\kappa}{(\kappa + \mu)}\right) \\
&= (\kappa + \mu)(\delta + \mu)\left(\mathcal{R}_0 - \frac{\delta}{(\delta + \mu)}\frac{\kappa}{(\kappa + \mu)}\right) > 0 \quad \text{for} \quad \mathcal{R}_0 > 1. \qquad (4)
\end{aligned}$$

Thus, we have established the following result.

Lemma 2. *The HIV/AIDS model system (2) has a unique endemic equilibrium if and only if* $\mathcal{R}_0 > 1$.

Using the centre manifold theory[4] as described in [6] (Theorem 4), we now investigate the local asymptotic stability of the endemic equilibrium. We make the following change of variables in order to apply the centre manifold theory $S = x_1$, $I = x_2$ and $A = x_3$. We now use the vector notation $X = (x_1, x_2, x_3)^T$. Then model system can be written in the form $\frac{\mathrm{d}x}{\mathrm{d}t} = F = (f_1, f_2, f_3)^T$, such that $\lambda = \frac{\beta(x_2+\eta x_3)}{(x_1+x_2+x_3)}$ giving

$$x_1' = \Lambda - \lambda x_1 - \mu x_1, \quad x_2' = \lambda x_1 - (\mu+\kappa)x_2, \quad x_3' = \kappa x_2 - (\mu+\delta)x_3. \tag{5}$$

The Jacobian matrix of the system (5) is given by,

$$\boldsymbol{J}(\mathcal{E}^0) = \begin{pmatrix} -\mu & -\beta & -\beta\eta \\ 0 & \beta-\kappa-\mu & \beta\eta \\ 0 & \kappa & -\delta-\mu \end{pmatrix}. \tag{6}$$

Taking β as a bifurcation parameter and considering the case $\mathcal{R}_0 = 1$ and solve for β gives

$$\beta = \beta^* = \frac{(\kappa+\mu)(\delta+\mu)}{(\delta+\kappa\eta+\mu)}. \tag{7}$$

The linearised system of the transformed Eq. (5) with $\beta = \beta^*$ has a simple zero eigenvalue. Hence the centre manifold theory[4] can be used to analyze the dynamics of (5) near $\beta = \beta^*$. It can be shown that the Jacobian of (5) at $\beta = \beta^*$ has a right eigenvector associated with the zero eigenvalue given by $u = (u_1, u_2, u_3)^T$, where

$$u_1 = -\frac{\beta^*}{\mu}\left[1 + \frac{\eta\kappa}{\delta+\mu}\right]u_2, \quad u_2 = u_2 > 0, \quad u_3 = \frac{\kappa}{\delta+\mu}u_2. \tag{8}$$

The left eigenvector of $\boldsymbol{J}(\mathcal{E}^0)$ associated with the zero eigenvalue at $\beta = \beta^*$ is given by $\boldsymbol{v} = (v_1, v_2, v_3)^T$, where

$$v_1 = 0, \quad v_2 = v_2 > 0, \quad v_3 = \frac{\beta^*\eta v_2}{\delta+\mu}. \tag{9}$$

In order to investigate the existence of a bifurcation, we use the following theorem.

Theorem 2. *Consider the following general system of ordinary differential equations with a parameter ϕ,*

$$\frac{dx}{dt} = f(x,\phi), f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \text{ and } f \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R}), \tag{10}$$

where 0 is an equilibrium of the system that is $f(0,\phi) = 0$ for all ϕ and assume
A1 $\boldsymbol{A} = D_x f(0,0) = \left(\frac{\partial f_i}{\partial x_j}(0,0)\right)$ is linearisation of system (10) around the equilibrium 0 with ϕ evaluated at 0. Zero is a simple eigenvalue of $\boldsymbol{A}$ and other eigenvalues of $\boldsymbol{A}$ have negative real parts;
A2 Matrix $\boldsymbol{A}$ has a right eigenvector u and a left eigenvector v corresponding to the zero eigenvalue.

Let f_k be the k^{th} f and

$$a = \sum_{k,i,j=1}^{n} v_k u_i u_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0), \quad b = \sum_{k,i=1}^{n} v_k u_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0). \tag{11}$$

The local dynamics of (10) around 0 are totally governed by a and b.

i. $a > 0, b > 0$, When $\phi < 0$ with $|\phi| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \phi \ll 1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;

ii. $a < 0, b < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 unstable; when $0 < \phi \ll 1$, asymptotically stable, and there exists a positive unstable equilibrium;

iii. $a > 0, b < 0$. When $\phi < 0$ with $|\phi| \ll 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi \ll 1$, 0 is stable, and a positive unstable equilibrium appears;

iv. $a < 0, b > 0$. When ϕ changes from negative to positive, 0 changes its stability from stable to unstable. Corresponding a negative equilibrium becomes positive and locally asymptotically stable.

Computations of a and b. From model system (10), the associated non zero partial derivatives of F at the disease free equilibrium are given by,

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \beta^* \text{ and } \frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \beta^* \eta. \tag{12}$$

Using (12) we have

$$\begin{aligned} a &= \sum_{k,i,j=1}^{3} v_k u_i u_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(\mathcal{E}_0) = v_2 u_1 u_2 \frac{\partial^2 f_2}{\partial x_1 \partial x_2}(\mathcal{E}_0) + v_2 u_1 u_3 \frac{\partial^2 f_2}{\partial x_1 \partial x_3}(\mathcal{E}_0) \\ &= -\frac{\beta^{*2}}{\mu}\left(1 + \frac{\eta\kappa}{\delta + \mu}\right)^2 u_2^2 v_2 < 0. \end{aligned} \tag{13}$$

For the sign of b, it is associated with the following non-vanishing partial derivatives of F,

$$\frac{\partial^2 f_2}{\partial x_2 \partial \beta^*} = 1 \text{ and } \frac{\partial^2 f_2}{\partial x_3 \partial \beta^*} = \eta. \tag{14}$$

It follows from expressions in (14) that

$$b = \sum_{k,i=1}^{3} v_k u_i \frac{\partial^2 f_k}{\partial x_i \partial \beta^*}(\mathcal{E}_0) = v_2 u_2 \frac{\partial^2 f_2}{\partial x_2 \partial \beta^*}(\mathcal{E}_0) + v_2 u_3 \frac{\partial^2 f_2}{\partial x_3 \partial \beta^*}(\mathcal{E}_0) = \left(1 + \frac{\eta\kappa}{\delta + \mu}\right) v_2 > 0. \tag{15}$$

Thus, $a < 0$ and $b > 0$. Hence, applying item (iv) of Theorem 2, the unique endemic equilibrium for model system (2) which exists whenever $\mathcal{R}_0 > 1$, is locally asymptotically stable when $\mathcal{R}_0 > 1$ and $\beta^* < \beta$ with β close to β^*. We establish the following theorem.

Theorem 3. *The unique endemic equilibrium ($\mathcal{E}_e$) for model system (2) is locally asymptotically stable whenever $\mathcal{R}_0 > 1$ and is close to 1.*

Numerical simulations of model system (2) with specified parameter values using a programme coded in C^{++} indicate that there is transcritical bifurcation at $\mathcal{R}_0 = 1$, where stability changes from the disease-free equilibrium to endemic-equilibrium (using a set of parameter values in Tab. 3 with variable p). For model system (2), the bifurcation diagram obtained for the number of HIV infected individuals (I) with the basic reproductive number $\mathcal{R}_0$ as the bifurcation parameter is shown in Fig. 1.

2.4 Global stability of the endemic equilibrium

We begin by presenting global stability results for the endemic equilibrium for model system (2) obtained in [18] for the special case with no AIDS induced-death ($\delta = 0$). In the continuing absence of HIV/AIDS cure, this assumption and corresponding analysis has no public health meaningful insights but is presented here as a mathematical exercise in an attempt to illustrate the possible conditions for global stability of the model for such a case.

Theorem 4. *The endemic equilibrium of the HIV/AIDS model with $\delta = 0$ is globally asymptotically stable in $\Phi \setminus \Phi_0$ whenever $\mathcal{R}_0 > 1$.*

Proof. Note, first of all, that the unique endemic equilibrium exists only if $\mathcal{R}_0 > 1$ (Lemma 2). Further, $N = \Lambda/\mu$ as $t \to \infty$. Thus, using $S = \Lambda/\mu - I - A$ and substituting in for the model with $\delta = 0$ gives the following limiting system

$$I' = \lambda(\Lambda/\mu - I - A) - (\mu + \kappa)I, \quad A' = \kappa I - \mu A. \tag{16}$$

Using the Dulac's multiplier $1/IA$, it follows that

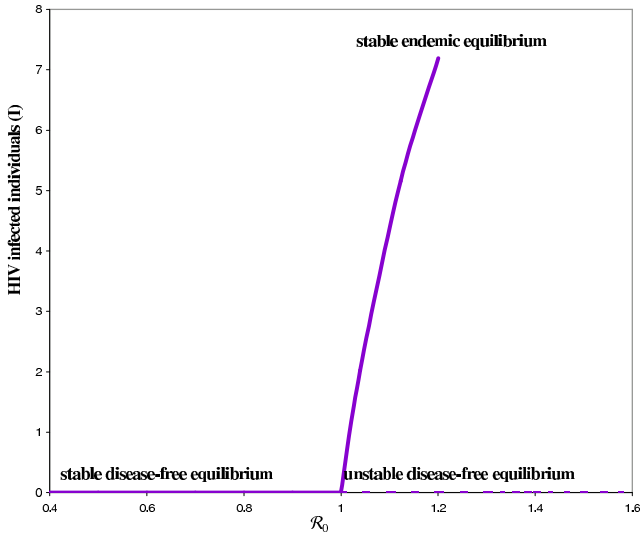


Fig. 1. Bifurcation diagram for the model system (2) obtained from numerical simulations, which show that the disease-free and endemic equilibria exchange stability when $\mathcal{R}_0$ is 1 in Φ using parameter values in Tab. 3 and variable p with $S(0) = 89, I(0) = 10$ and $A(0) = 10$. The bold lines show stability and dashed lines show instability

$$\frac{\partial}{\partial I}\left[\frac{\beta(I+\eta A)}{IA\Lambda/\mu}(\Lambda/\mu - I - A) - \frac{(\kappa+\mu)}{A}\right] + \frac{\partial}{\partial A}\left(\frac{\kappa}{A} - \frac{\mu}{I}\right)$$
$$= -\left[\frac{\beta\mu}{\Lambda} + \frac{\beta\eta\mu}{\Lambda I^2}\left(1 - \frac{A}{\Lambda/\mu}\right) + \frac{\kappa}{A^2}\right] < 0$$

since $A \leq \Lambda/\mu$ in Φ. Thus, by Dulac's criterion, there is no periodic orbit in $\Phi \setminus \Phi_0$. Since Φ is positively invariant, and the endemic equilibrium exists whenever $\mathcal{R}_0 > 1$, then it follows from the Poincaré-Bendixson Theorem[22] that all solutions of the limiting system originating in Φ remain in Φ for all t. Further, the absence of periodic orbits in Φ implies that the unique endemic equilibrium of the special case of the HIV-only model is globally asymptotically stable whenever $\mathcal{R}_0 > 1$.

We proceed to investigate the global stability of the complete model with $\delta > 0$.

Theorem 5. *[16] Let $x \mapsto f(x) \in \mathbb{R}^n$ be a C^1 function for x in an open set $D \subset \mathbb{R}^n$. Consider the differential equation*

$$x' = f(x) \tag{17}$$

Denote by $x(t, x_0)$ the solution to (17) such that $x(0, x_0) = x_0$. A set K is said to be absorbing in D for (17) if $x(t, K_1) \subset K$ for each $K_1 \subset D$ and t is sufficiently large. If the following conditions are satisfied,

(1) there exists a compact absorbing set $K \subset D$ and Eq. (17) has a unique equilibrium $\bar{x}$ in D;
(2) $\bar{x}$ is locally asymptotically stable;
(3) system (17) satisfies a Poincarè-Bendixson Property;
(4) each periodic orbit of the (17) in D is asymptotically orbitally stable.

Then the unique equilibrium $\bar{x}$ is globally asymptotically stable in D.

Theorem 6. *[16, 19] A sufficient condition for a periodic orbit $\Omega = \{p(t) : 0 \leq t \leq \omega\}$ of (17) to be asymptotically orbitally stable with asymptotic phase is that the linear system*

$$z'(t) = \frac{\partial f^{[2]}}{\partial t}(p(t))z(t) \tag{18}$$

be asymptotically stable, where $\frac{\partial f^{[2]}}{\partial t}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial t}$ of f.

Lemma 3. *The trajectory of any nonconstant periodic solution to (2), if it exists, is asymptotically orbitally stable with asymptotic phase.*

Proof. Suppose that the solution $(S(t), I(t), A(t))$ is periodic of least period $\omega > 0$ such that $(S(0), I(0), A(0)) \in \overset{\circ}{\Phi}$. The periodic orbit is $\Omega = \{p(t) : 0 \leq t \leq \omega\}$. We have the second compound system $x' = \boldsymbol{J}^{[2]}(S, I, A)x$ of the differential system $y' = \boldsymbol{J}(S, I, A)y$ in the periodic solution as the following periodic linear system

$$\begin{aligned} X' = & -\left(\frac{\beta(I+\eta A-S)}{N}+(\kappa+2\mu)\right)X+\left(\frac{\beta S(\eta N-(I+\eta A))}{N^2}\right)(Y+Z),\\ Y' = & \kappa X-\left(\frac{\beta(N-S)(I+\eta A)}{N^2}+(\delta+2\mu)\right)Y-\left(\frac{\beta S(N-(I+\eta A))}{N^2}\right)Z,\\ Z' = & \left(\frac{\beta(N-S)(I+\eta A)}{N^2}\right)Y+\left(\frac{\beta S(N-(I+\eta A))}{N^2}-(\kappa+\delta+2\mu)\right)Z \end{aligned} \tag{19}$$

with

$$\boldsymbol{J}^{[2]}(S,I,A)=\begin{pmatrix} -\frac{\beta(I+\eta A-S)}{N}-(\kappa+2\mu) & \frac{\beta S(\eta N-(I+\eta A))}{N^2} & \frac{\beta S(\eta N-(I+\eta A))}{N^2}\\ \kappa & \frac{\beta(N-S)(I+\eta A)}{N^2}-(\delta+2\mu) & -\frac{\beta S(N-(I+\eta A))}{N^2}\\ 0 & \frac{\beta(N-S)(I+\eta A)}{N^2} & \frac{\beta S(N-(I+\eta A))}{N^2}-(\kappa+\delta+2\mu) \end{pmatrix}. \tag{20}$$

If $(X(t), Y(t), Z(t))$ is a solution of system (19) with $(S, I, A) \in \Omega$. Let

$$V(X,Y,Z,S,I,A)=\sup\left\{|X|,\frac{I}{A}(|Y|+|Z|)\right\}. \tag{21}$$

From condition (1) we have that there exist a constant $\epsilon > 0$ such that

$$V(X,Y,Z,S,I,A)\geq\epsilon|(X,Y,Z)|. \tag{22}$$

for all $(X, Y, Z) \in \mathbb{R}^3$ and $(S, I, A) \in \Omega$. This result in the following differential inequality

$$D_+|X(t)|\leq-\left(\frac{\beta(I+\eta A-S)}{N}+(\kappa+2\mu)\right)|X(t)|+\left(\frac{\beta SA(\eta N-(I+\eta A))}{IN^2}\right)\frac{I}{A}(|Y(t)|+|Z(t)|), \tag{23}$$

$$D_+|Y(t)|\leq\kappa|X(t)|-\left(\frac{\beta(N-S)(I+\eta A)}{N^2}+(\delta+2\mu)\right)|Y(t)|-\left(\frac{\beta S(N-(I+\eta A))}{N^2}\right)|Z(t)|, \tag{24}$$

$$D_+|Z(t)|\leq\left(\frac{\beta(N-S)(I+\eta A)}{N^2}\right)|Y(t)|+\left(\frac{\beta S(N-(I+\eta A))}{N^2}-(\kappa+\delta+2\mu)\right)|Z(t)|. \tag{25}$$

From Eqs. (24) and (25) we have

$$D_+\frac{I}{A}(|Y(t)|+|Z(t)|)\leq\frac{\kappa I}{A}|X(t)|+\left(\frac{I'}{I}-\frac{A'}{A}-\delta-2\mu\right)\frac{I}{A}(|Y(t)|+|Z(t)|). \tag{26}$$

Using Eqs. (23) and (26) we obtain

$$D_+V(t)\leq\sup\{g_1,g_2\}\cdot V(t) \tag{27}$$

where,

$$g_1 = -\frac{\beta(I+\eta A-S)}{N}-(\kappa+2\mu)+\frac{\beta SA(\eta N-(I+\eta A))}{IN^2}, \tag{28}$$

$$g_2 = \frac{\kappa I}{A}+\frac{I'}{I}-\frac{A'}{A}-\delta-2\mu. \tag{29}$$

Using model system (2), we obtain

$$\frac{\beta\eta AS}{NI} = \frac{I'}{I} - \frac{\beta S}{N} + (\mu + \kappa), \tag{30}$$

$$\frac{\kappa I}{A} = \frac{A'}{A} + (\mu + \delta). \tag{31}$$

Substituting (30) and (31) into (28) and (29) respectively gives

$$g_1 = \frac{I'}{I} - \mu - \frac{\beta(I + \eta A)}{N} - \frac{\beta AS(I + \eta A)}{N^2 I}, \tag{32}$$

$$g_2 = \frac{I'}{I} - \mu. \tag{33}$$

Therefore, $\sup\{g_1(t), g_2(t)\} \leq \frac{I'}{I} - \mu$ and

$$\int_0^\omega \sup\{g_1(t), g_2(t)\}\mathrm{d}t \leq \log I(t)|_0^\omega - \mu\omega = -\mu\omega < 0.$$

This relationship and (27) imply that $V(t) \to 0$ as $t \to \infty$ and in turn that $(X(t), Y(t), Z(t)) \to 0$ as $t \to \infty$ by (22). As a result, the second compound system (19) is asymptotically stable and the periodic solution $(S(t), I(t), A(t))$ is asymptotically stable by Theorem 6.

Theorem 7. *If $\mathcal{R}_0 > 1$, then the unique endemic equilibrium $\mathcal{E}_e$ is globally asymptotically stable for model system (2) in $\overset{\circ}{\Phi}$.*

Proof. From the condition (1) of Theorem 5, uniformly persistent property of the solution of model system (2) can be concluded[3]. Let $B = \mathcal{E}_0$, Theorem 1 implies that, when $\mathcal{R}_0 > 1$, B^s is just contained in the $S-$axis and thus just in the boundary of Φ. It also implies that B^s is isolated in Φ. When $\mathcal{R}_0 > 1$, model system (2) satisfies the conditions of Theorem 4 of [10], namely, (i) the maximal compact invariant set B in the boundary of Φ is isolated and (ii) the stable set B^s of B is contained in the boundary of Φ. Therefore, model system (2) is uniformly persistent in Φ when $\mathcal{R}_0 > 1$. By looking at the Jacobian matrix of model system (2) and choosing matrix $\boldsymbol{H}$ as $\boldsymbol{H} = \mathrm{diag}(1, -1, 1)$, we can see that $\boldsymbol{HJ}(S, I, A)\boldsymbol{H}$ has non positive of-diagonal elements for all $(S, I, A) \in \Phi$. Thus we can verify that model system (2) is competitive with respect to the partial ordering defined by the orthant $\Phi = \{(S, I, A) \in \mathbb{R}^3 : S \geq 0, I \geq 0, A \geq 0\}$ [23]. From [9, 16, 23], we note that model system (2) satisfies Poincarè-Bendixson property. By Lemma 3 and Theorem 3, we note that model system (2) is satisfied with every condition of Theorem 5, so the unique endemic equilibrium of model system (2) is globally asymptotically stable.

3 Numerical simulations

In order to illustrate some of the analytical results in this study, numerical simulations of model system (2) are carried out using a set of parameter values given in Tab. 3. We use Matlab software for the numerical simulations.

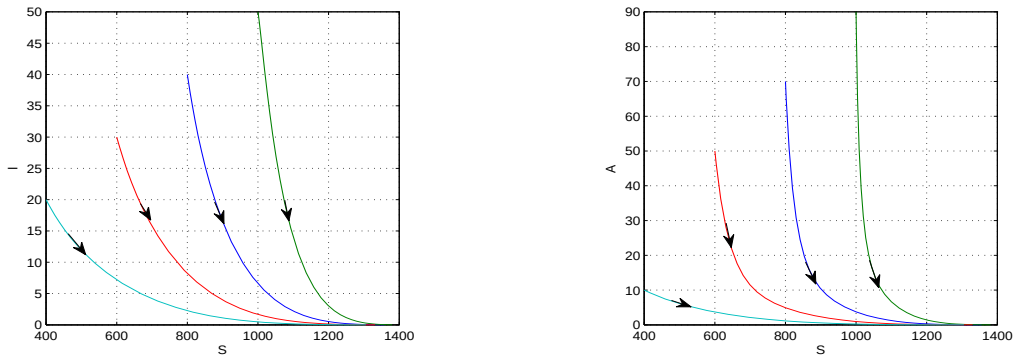
Figs. 3 (a) and 3 (b) illustrate the phase plane portraits of (a) SI-plane, (b) SA-plane for $\mathcal{R}_0 = 0.341 < 1$ $(p = 0.011)$ showing convergence to the disease free equilibrium in line with Lemma 1. Similarly, Figs. 3 (a) and 3 (b) illustrate the phase plane portraits of (a) SI-plane, (b) SA-plane for $\mathcal{R}_0 = 3.41 > 1$ $(p = 0.11)$ showing convergence to the endemic equilibrium in line with Theorem 3.

4 Summary and concluding remarks

In this study an HIV/AIDS epidemic model incorporating constant recruitment and sexually active AIDS individuals is presented and analysed. Using a Lyapunov function and LaSalle's invariant set theorem, we proved the global asymptotic stability of the disease-free equilibrium. Local asymptotic stability of the endemic equilibrium is derived using the center manifold theory and Poincarè-Bendixson property is used to prove its global asymptotic stability. The analytical and numerical findings suggest that the disease dies out (or is controlled) when $\mathcal{R}_0 < 1$ or persists in a population with endemicity when $\mathcal{R}_0 > 1$.

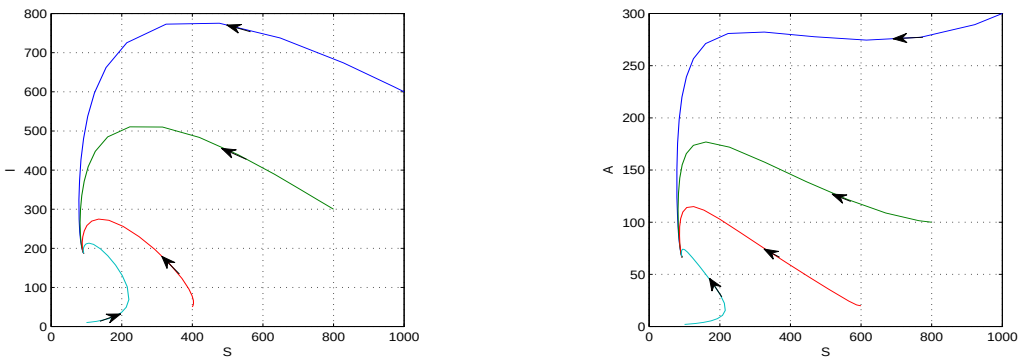
Table 1. Data for the HIV/AIDS model

Parameter	Symbol	Value	Source
Recruitment rate	Λ	29 yr^{-1}	Estimate
Rate of acquiring new sexual partners	c	3 partners/year	[17]
Probability of transmission	p	0.011-0.95	[17]
Rate of progression to AIDS	κ	0.125 yr^{-1}	[17, 18]
Natural death rate	μ	0.02 yr^{-1}	[17, 18]
AIDS-related death rate	δ	0.333 yr^{-1}	[17, 18]
Enhancement factor	η	1.4	[18]



(a) (b)

Fig. 2. Phase plane portraits of (a) SI-plane, (b) SA-plane for $\mathcal{R}_0 = 0.341 < 1$ ($p = 0.011$) showing convergence to the disease free equilibrium using parameter values in Tab. 3 and varying initial conditions



(a) (b)

Fig. 3. Phase plane portraits of (a) SI-plane, (b) SA-plane for $\mathcal{R}_0 = 3.41 > 1$ ($p = 0.11$) showing convergence to the endemic equilibrium using parameter values in Tab. 3 and varying initial conditions

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