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Adapted variational iteration method and axisymmetric flow over a stretching sheet

H. Mirgolbabaei[1], D. D. Ganji[2*], M. M. Etghani[3], A. Sobati

[1] School of Mechanic, Islamic Azad University, Jouybar Branch, Jouybar 47715-195, Iran
[2] Department of Mechanical Engineering, Mazandaran University, P. O. Box 484, Babol 47148-71167, Iran
[3] Department of Automotive Engineering, Iran University of Technology, Narmak, Tehran 57148–81987, Iran
[4] Department of Civil Engineering, Imam Khomeini International University, Qazvin 34149-16818, Iran

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Abstract. A number of boundary layer flows induced by the axsymmetric stretching of a sheet are studied. The sheet is stretched with a velocity proportional to the distance from the vertical axis. The resulting non-linear ordinary differential equations are solved analytically using the adopted variational iteration method (AVIM). The values for the skin friction obtained numerically are also listed in tables. Comparison of the results with the exact solutions shows that the mentioned method gives excellent results.

Keywords: variational iteration method, axisymmetric flow, stretching sheet

1 Introduction

In fluid dynamics of viscous fluids, the exact analytical solutions of the flow problems are usually catalogued in one chapter of the books on fluid dynamics. This is on account of the fact the governing equations of motion, in general, are highly non-linear partial differential equations in the three components of velocity. In some situations, by means of similarity transformations, the system of partial differential equations are reduced to that of ordinary differential equations, which, on few occasions, for a long time, before the advent computers, the researchers mainly directed their efforts at obtaining some forms of approximate solutions. One of the key issues of approximate solutions has always been the accuracy of the solutions. The accuracy, generally speaking, is measured in terms of the norm of the error in Banach space-the error being the difference of the approximate solution from the exact solution. In the absence of an exact solution, (analytical or numerical) a heuristic approach consisting of the convergence of successive approximate solution.

With the advent of computers, the approximate solutions in fluid dynamics have lost some of their importance as more and better numerical algorithms have been developed to solve the increasingly realistic, but more but more complicated problems numerically. Nevertheless, approximate analytical solutions still have their relevance for the following reasons: Firstly, they they give the solution for each point within the domain of interest, unlike the numerical solutions, which are available, for a particular run, only for a set of discrete points in the domain. Secondly, compared to a numerical solution, a nicely produced approximate solution, requiring a minimal effort and having a reasonable amount of accuracy, is always handy for an engineer, scientist or an applied mathematician, who can obtain a solution quickly, thereby gaining a valuable insight into the essential of the problem. Thirdly, even with most of the scientific packages, some initial guess is required for the solution, as the algorithm, in general, are nor globally convergent. In such situations, approximate solutions can provide an excellent starting guess, that can be readily refined to the exact numerical solution in a few iterations. Finally, aesthetically an approximate solution, if it is analytical, is more pleasing than a

* Corresponding author. *E-mail address*: ddg_davood@yahoo.com.

numerical solution. It is, therefore, not surprising that even after the numerical techniques for obtaining the solutions of the flow problems in fluid dynamics have peaked during the last 3 $\sim$ 4 decades, there still have been attempts to develop new techniques for obtaining analytical solutions which reasonably approximate the exact solutions. Perturbation method is one of the well-known methods to solve nonlinear problems, it is based on the existence of small/large parameters, the so-called perturbation quantity[2, 20]. Many nonlinear problems do not contain such kind of perturbation quantity, and we can use non-perturbation methods, such as the artificial small parameter method[19], the $\delta-$expansion method[18], the Adomian's decomposition method[1], the homotopy perturbation method (HPM)[5, 12–15], and the variational iteration method (VIM)[4, 6, 10, 11].

One of those rare problems in fluid dynamics for which an exact analytical solution has been found in the literature is that of the two-dimensional flow past a stretching sheet. The fluid occupies the space above the sheet and the motion is caused by stretching the sheet in opposite directions with a velocity that is proportional to the distance from a fixed axis. Crane[3] reported an elegant solution for this problem. Even more interesting is the fact that the problem still admits an exact analytical solution when several other effects are taken into account, separately or jointly, such as suction at the sheet (Gupta and Gupta[9]), presence of a transverse magnetic field (Andersson[16]), viscoelasticity of the fluid (Ariel[23, 24]), partial slip at the boundary (Wang[30]). The companion problem of the flow due the radial stretching of the sheet (the velocity of the sheet is proportional to the distance from a vertical rather than a horizontal axis) does not seem to have an exact solution. Apparently for this reason, this problem has received much less attention in the literature. Its numerical solution was first given by Wang[29]. The effects of viscoelasticity were considered by Ariel for an elastico-viscous fluid[22], and a second grade fluid[25]. Recently Ariel[26] has chosen to illustrate the axisymetric flow due to stretching of a sheet in hydromagnetics as the prototype problem for the non-iterative algorithm, he developed for solving the problems of flow induced by the moving boundaries in hydromagnetics.

In the present paper our endeavor is to evolve a adopted variational iteration method for obtaining the solution of the flow problem of the flow motion caused by the axisymmetric stretching of the sheet. Once a variatinal solution is in place, we next wish to examine if it is still suitable for use when the effects of magnetic field and/or suction are included. It is well-known that the boundary lawyers in the form of Hartmann layers due to presence of strong magnetic field, or suction boundary layer due to the large scale mass transfer across the boundary wreak havoc with variational solutions. It would be of interest to find out if the solution obtained by the AVIM can withstand the presence of the large magnetic field or suction.

2 Equations of motion

We consider the motion of an electrically conducting, viscous, incompressible fluid of density ρ and viscosity μ, caused by the radial stretching of a sheet situated at $z = 0$, in the presence of a transverse magnetic field. The stretching results into a velocity of the sheet that is proportional to the distance fro the center of the sheet (see Fig. 1). The flow takes place in the upper half plane $z > 0$.

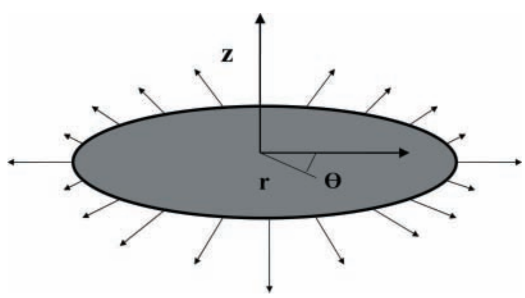

Fig. 1. Schematic diagram of the flow

We shall be working in the cylindrical polar coordinate system (γ, θ, z). In view of the rotational symmetry of the flow all physical quantities are independent of θ i.e., $\partial/\partial\theta \equiv 0$. the equation of the motion for steady, laminar, axisymmetric flow are:

$$\rho\left(\mu\frac{\partial\mu}{\partial\gamma}+\omega\frac{\partial\mu}{\partial z}\right) = -\frac{\partial p}{\partial\gamma}+\mu\left(\frac{\partial^2\mu}{\partial\gamma^2}+\frac{\partial\mu}{\gamma\partial\gamma}+\frac{\partial^2\mu}{\partial z^2}-\frac{\mu}{\gamma^2}\right)-\frac{\sigma B_0}{\rho}\mu, \tag{1}$$

$$\rho\left(\omega\frac{\partial\omega}{\partial\gamma}+\omega\frac{\partial\omega}{\partial z}\right) = -\frac{\partial p}{\partial\gamma}+\mu\left(\frac{\partial^2\omega}{\partial\gamma^2}+\frac{\partial\omega}{\gamma\partial\gamma}+\frac{\partial^2\omega}{\partial z^2}\right), \tag{2}$$

$$\frac{\partial\mu}{\partial\gamma}+\frac{\mu}{\gamma}+\frac{\partial\mu}{\partial z}=0, \tag{3}$$

where P is the pressure, and $(\mu,\ 0,\ \omega)$ are the velocity components along $(\gamma,\ \theta,\ z)$ directions.

The boundary conditions are:

$$\begin{aligned} &u=cr, \quad w=0 \quad \text{at} \quad z=0,\\ &u\to 0 \quad \text{as} \quad z\to\infty, \end{aligned} \tag{4}$$

where $c(c>0)$ is the stretching rate assumed to be constant.

It is shown by Wang[30] that a similarity solution characterizes the flow. Eq. (1) $\sim$ Eq. (4) admit a similarity solution, if we choose:

$$u=cr\varphi^{'}(\xi), \qquad w=-2\sqrt{\tfrac{c\mu}{\rho}}\ \varphi(\xi), \tag{5}$$

where, $\xi=\sqrt{\frac{c\rho}{\mu}}z$. From Eq. (1) we obtain:

$$\frac{\partial p}{\partial r}=\rho c^2 r\left[\varphi^{'''}+2\varphi\varphi^{''}-\varphi^{'2}-M\varphi^{'}\right] \tag{6}$$

where $M=\sigma B_0/\rho c$ is the magnetic parameter. On the other hand, Eq. (2) gives: $\partial p/\partial\xi=-4c\mu\varphi\varphi^{'}-2c\mu\varphi^{''}$, which when integrated with respect to f yields,

$$p=-2c\mu\varphi^2-2c\mu\varphi^{'}+g(r), \tag{7}$$

where $g(r)$ is an arbitrary function of r.

If we substitute for p from Eq. (7) into Eq. (6) we obtain:

$$\frac{g^{'}(r)}{\rho c^2 r}=\varphi^{'''}+2\varphi\varphi^{''}-\varphi^{'2}-M\varphi^{'}. \tag{8}$$

Since in Eq. (8), the left-hand side is a function of r only, and the right-hand side is a function of ξ only, in order for it to be consistent, each side must be a constant, say A. $g^{'}(r)=\rho c^2 rA$. Its integration with respect to r gives:

$$g(r)=p_0+\frac{1}{2}\rho c^2 r^2 A. \tag{9}$$

where p_0 is a constant.

Substitution of $g(r)$ from Eq. (9) into Eq. (7) leads to:

$$p=p_0+\frac{1}{2}\rho c^2 r^2 A-2c\mu\varphi^2-2c\mu\varphi^{'}. \tag{10}$$

Since the entire motion of the fluid is caused due to stretching of the sheet, the pressure far away from the sheet must be given by the Bernoulli's equation, i.e., Matching of the pressure from Eq. (10), therefore, gives $A=0$. Hence from Eq. (8), we obtain the following differential equation for φ:

$$\varphi^{'''}+2\varphi\varphi^{'}-\varphi^{'2}-M\varphi^{'}=0. \tag{11}$$

Also the pressure p at any point in terms of the physical variables is given by:

$$p=p_0-\frac{1}{2}\rho w^2+\mu\frac{\partial w}{\partial z}. \tag{12}$$

The boundary conditions of the problem In terms of u they can be expressed as:

$$\varphi(0)=0,\varphi^{'}(0)=1,\varphi^{'}(\infty)=0. \tag{13}$$

3 The description of adapted variational iteration method

To clarify the basic ideas of He's variaional iteration method, we consider the following differential equation. $L(u) + N(u) = g(\xi)$, where L is a linear operator, N a nonlinear operator, and $g(t)$ a given continuous function. The basic character of the method is to constructed a correction functional for the system, which reads[7, 8, 21].

$$u_{n+1}(\xi) = u_n(\xi) + \int_0^{\xi} \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau, \tag{14}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory[7, 8, 21], μ_n is the n-th approximate solution, and $\widetilde{\mu}_n$ denotes a restricted variation, i.e. $\delta\widetilde{\mu}_n$.

After imposing the variational to Eq. (12) we will have:

$$\delta\left(u_{n+1}(\xi)\right) = \delta\left(u_n(\xi)\right) + \delta\left(\int_0^{\xi} \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau\right). \tag{15}$$

Taking restricted variation into consideration changes Eq. (15) into:

$$\delta\left(u_{n+1}(\xi)\right) = \delta\left(u_n(\xi)\right) + \delta\left(\int_0^{\xi} \lambda(Lu_n(\tau))d\tau\right). \tag{16}$$

More effort makes the previous equation be more tangible, namely:

$$\delta\left(u_{n+1}(\xi)\right) = \delta\left(u_n(\xi)\right) + \lambda(\tau)\delta\left(\int_0^{\tau} (Lu_n(\tau))d\tau\right)\Bigg|_{\tau=\xi} - \delta\left(\int_0^{\xi} \lambda'\delta\left(\int_0^{\tau} Lu_n(\tau)d\tau\right)d\tau\right). \tag{17}$$

In the foregoing, the order of the linear operator is considered as unity; otherwise the manipulation may take more time depending on the order of the operator.

Finally, if the unknown which is the linear operator L for any special problem is correctly defined, the corresponding Lagrangian multiplier will be identified.

It is worth noting that for linear cases ($N \equiv 0$) the first iteration of Eq. (14) leads to the exact solution; because; the exact Lagrangian multiplier is used while in nonlinear cases the most optimum one is considered.

According to Eq. (14), the correction functional of our problem is as follow:

$$\varphi_{n+1}(\xi) = \varphi_n(\xi) + \int_0^{\xi} \lambda(\varphi''' + 2\varphi\varphi' - \varphi'^2 - M\varphi')d\tau, \tag{18}$$

4 Applications of AVIM

4.1 Case1

First we consider the flow of a fluid with no magnetic field. The equation of motion (11) modified to:

$$\varphi''' + 2\varphi\varphi' - \varphi'^2 = 0, \tag{19}$$

and the boundary conditions are as the same of Eq. (13). Making the above correction functional stationary, we have the following stationary conditions: $1 + \lambda'' = 0,\ \lambda'|_{\tau=\xi} = 0,\ \lambda|_{\tau=\xi} = 0$. The Lagrangian multiplier can therefore be identified as:

$$\lambda(\tau) = -\frac{1}{2}(\tau - \xi)^2 \tag{20}$$

.

According to (18), we have the following iteration formulation:

$$\varphi_{n+1}(\xi) = \varphi_n(\xi) + \int_0^{\xi} -\frac{1}{2}(\tau-\xi)^2 \left(\varphi_n^{'''} + 2\varphi_n\varphi_n^{'} - \varphi_n^{'\,2}\right) \mathrm{d}\tau. \tag{21}$$

Now we assume that an initial approximation has the form:

$$\varphi_0(\xi) = \left(a\xi^2 + b\xi + c\right) e^{-\xi}, \tag{22}$$

where a, b, c are unknown constants to be further determined.

By the iteration formula (21), we have the following first-order approximation:

$$\begin{aligned}
\phi_1(\xi) = (a\xi^2+b\xi+c)e^{-\xi} + \int_0^t -\frac{1}{2}(\xi-\tau)^2 \Big\{ 2\left[(a\xi+b)e^{-\xi} - (a\xi^2+b\xi+c)e^{-\xi}\right] \\
(a\xi^2+b\xi+c)e^{-\xi} - \left[(2a\xi+b)e^{-\xi} - (a\xi^2+b\xi+c)e^{-\xi}\right]^2 \Big\} d\tau \\
= a\xi^2 + b\xi + c + \frac{b^2}{8}\xi^2 e^{-2\xi} + \frac{1}{2}ace^{-2\xi} - \frac{3}{8}abe^{-2\xi} + \frac{1}{8}bce^{-2\xi} + \frac{1}{8}\xi^4 a^2 e^{-2\xi} - \frac{3}{8}\xi b^2 + \frac{15}{16}a^2 + \frac{1}{8}b^2 \\
- \frac{1}{8}c^2 + \frac{1}{4}bc\xi e^{-2\xi} + \frac{1}{4}c^2\xi - c\xi + \frac{3}{8}ab\xi^2 e^{-2\xi} + \frac{1}{4}ab\xi^3 e^{-2\xi} - \frac{3}{4}a^2\xi e^{-2\xi} + \frac{1}{4}a^2\xi^3 e^{-2\xi} + \frac{1}{8}\xi b^2 e^{-2\xi} \\
- \frac{1}{8}b^2 e^{-2\xi} - \frac{15}{16}a^2 e^{-2\xi} + \frac{1}{8}c^2 e^{-2\xi} - \xi^2 b - \frac{1}{8}bc - \frac{1}{2}ac + \frac{3}{8}\xi^2 a^2 + \frac{3}{8}ab + \frac{1}{4}bc\xi^2 + \frac{3}{8}ab\xi^2 \\
- \frac{3}{4}ac\xi^2 + \frac{3}{4}ac\xi - \frac{3}{4}ab\xi - \frac{9}{8}a^2\xi + \frac{3}{8}b^2\xi^2 - \frac{1}{4}c^2\xi^2 + \frac{1}{2}c\xi^2 + \frac{1}{4}ac\xi e^{-2\xi} + \frac{1}{4}ac\xi^2 e^{-2\xi}.
\end{aligned} \tag{23}$$

Incorporating the boundary conditions, Eq. (13), into $\varphi_1(\xi)$, and solving the system which includes three equations with three unknowns we obtain: $a = 0.4089373412$, $b = 1$, $c = 0$.

Therefore, we obtain the following first-order approximate solution:

$$\begin{aligned}
\phi_1(\xi) = \; & +0.020930371862\xi^4 e^{-2\xi} + 0.1440417726\xi^3 e^{-2\xi} + 1e - 10\xi^2 \\
& + 0.2783515030\xi^2 e^{-2\xi} + 0.1301635265\xi - 0.0004223118\xi e^{-2\xi} \\
& + 0.4351293927 - 0.4351293927 e^{-2\xi}
\end{aligned} \tag{24}$$

We check the validity of our solution by comparing it with the exact numerical solution. One representative number, which is also measure of the physical interest, namely the stress at the sheet is $-\varphi^{''}(0)$. The exact value of $-\varphi^{''}(0)$ is 1.173721, and the value obtained by AVIM is 1.182125. The error being less than 0.7% which is indeed remarkly small.

4.2 Case2

In this section we consider the flow of an electrically conducting, viscous, incompressible fluid over a radially stretching sheet in the presence of a transverse magnetic field. The differential equation for φ is the same, i.e., Eq. (11).

$\sqrt{M}$ is the Hartmann number. As the value of M is increased, Hartmann layers start setting in at $\xi = 0$ causing great difficulties in obtaining a numerical solution if it is attempted by marching techniques (Kumar et al.[27]). A better approach is to use a variant technique used by Samuel and Hall[28], which also gives the solution non-iteratively (Ariel[26]). However, as we shall see presently, a AVIM solution successfully negotiates arbitrary values of M.

To solve Eq. (11) via VIM, one has to find the Lagrangian multiplier, which can be identified by substituting Eq. (11) into Eq. (14), upon making it stationary leads to the following: $\lambda^{''} - M\lambda + 1 = 0,\ -\lambda^{'}|_{\tau=\xi} = 0,\ \lambda|_{\tau=\xi} = 0.$

The Lagrangian multiplier can therefore be identified as:

$$\lambda(\tau) = -\frac{1}{2M}(-2 + e^{-\sqrt{M}(\xi-\tau)} + e^{\sqrt{M}(\xi-\tau)}). \tag{25}$$

Now we assume that an initial approximation has the same form of Eq. (22). By the iteration formula (18), we have the following first-order approximation:

$$\begin{aligned}\varphi_1(\xi)=&(a\xi^2+b\xi+c)e^{-\xi}+\int_0^t-\frac{1}{2M}(-2+e^{-\sqrt{M}(\xi-\tau)}+e^{\sqrt{M}(\xi-\tau)})\left\{2\left[(a\xi+b)e^{-\xi}-(a\xi^2+b\xi+c)e^{-\xi}\right]\right.\\&(a\xi^2+b\xi+c)e^{-\xi}-\left[(2a\xi+b)e^{-\xi}-(a\xi^2+b\xi+c)e^{-\xi}\right]^2-M\left[(a\xi+b)e^{-\xi}-(a\xi^2+b\xi+c)e^{-\xi}\right]d\tau\\=&a\xi^2+b\xi+c+\frac{b^2}{8}\xi^2e^{-2\xi}+\frac{1}{2}ace^{-2\xi}-\frac{3}{8}abe^{-2\xi}+\frac{1}{8}bce^{-2\xi}+\frac{1}{8}\xi^4a^2e^{-2\xi}-\frac{3}{8}\xi b^2+\frac{15}{16}a^2+\frac{1}{8}b^2\\&-\frac{1}{8}c^2+\frac{1}{4}bc\xi e^{-2\xi}+\frac{1}{4}c^2\xi-c\xi+\frac{3}{8}ab\xi^2e^{-2\xi}+\frac{1}{4}ab\xi^3e^{-2\xi}-\frac{3}{4}a^2\xi e^{-2\xi}+\frac{1}{4}a^2\xi^3e^{-2\xi}+\frac{1}{8}\xi b^2e^{-2\xi}\\&-\frac{1}{8}b^2e^{-2\xi}-\frac{15}{16}a^2e^{-2\xi}+\frac{1}{8}c^2e^{-2\xi}-\xi^2b-\frac{1}{8}bc-\frac{1}{2}ac+\frac{3}{8}\xi^2a^2+\frac{3}{8}ab+\frac{1}{4}bc\xi^2+\frac{3}{8}ab\xi^2-\frac{3}{4}ac\xi^2\\&+\frac{3}{4}ac\xi-\frac{3}{4}ab\xi-\frac{9}{8}a^2\xi+\frac{3}{8}b^2\xi^2-\frac{1}{4}c^2\xi^2+\frac{1}{2}c\xi^2+\frac{1}{4}ac\xi e^{-2\xi}+\frac{1}{4}ac\xi^2e^{-2\xi}-6\sqrt{M}a-\sqrt{M}c\\&-2\sqrt{M}b-\frac{3}{8}abe^{-2\xi}+\xi\sqrt{M}c+\xi\sqrt{M}b+2\xi\sqrt{M}a-\xi c-\frac{1}{2}\xi^2\sqrt{M}c+\sqrt{M}b\xi e^{-\xi}+\sqrt{M}a\xi^2e^{-\xi}\\&+4\xi\sqrt{M}ae^{-\xi}+6\sqrt{M}ae^{-\xi}+2\sqrt{M}ae^{-\xi}+\sqrt{M}ce^{-\xi}.\end{aligned}\tag{26}$$

Incorporating the boundary conditions, Eq. (13), into $\varphi_1(\xi)$, and solving the system which includes three equations with three unknowns, we obtain the values of a, b, c for each value of M.

In Tab. 1, $-\varphi''(0)$ is listed for different values of M, using the exact numerical by Ariel[26] and AVIM described above.

It is satisfying to note that the error in the solution obtained by the AVIM implying that the AVIM is able to successfully negotiate the presence of Hartmann layers near the stretching sheet.

4.3 Case3

When there is a massive transfer of the fluid across the boundary, another type of boundary layer is manifested near the boundary. We next examine if the AVIM is able to handle the suction boundary layer in the same efficient manner as it handles the Hartman layer. When a suction takes place across the sheet, the boundary conditions (13) change to: $\varphi(0) = A$, $\varphi'(0) = 1$, $\varphi'(\infty) = 0$. where A is the suction parameter given by:

$$A=\frac{w_0}{2}\sqrt{\frac{\rho}{c\mu}},\tag{27}$$

w_0 being the suction velocity. The differential equation for φ is the same, i.e., Eq. (19).

The initial approximate has the form of Eq. (22), then our first-order approximate solution is the same as Eq. (23).

Incorporating the boundary conditions, Eq. (13), into $\varphi_1(\xi)$, and solving the system which includes three equations with three unknowns, we obtain the values of a, b, c for each value of A.

In Tab. 2, the values of $-\varphi''(0)$ are presented for various values of A using the exact numerical solution and AVIM described above. The numerical solution was obtained using the Ackroyd method[17].

4.4 Case4

In this section we consider the flow of an electrically conducting, viscous, incompressible fluid due to radial stretching of a sheet in the presence of a transverse magnetic field, when there is also a suction at the sheet. The $\varphi''' + 2\varphi\varphi' - \varphi'^2 - M\varphi' = 0, \varphi(0) = A, \varphi'(0) = 1, \varphi'(\infty) = 0$. We formulate the VIM equation the same as in Eq. (21), and the initial approximate has the form of Eq. (22), then our first-order approximate solution is the same as Eq. (26). Incorporating the boundary conditions, Eq. (20), into $\varphi_1(\xi)$, and solving the

Table 1. Comparison of the first-order approximate solution with exact solution for flow I the presence of magnetic field

M	a	b	c	$-\varphi''_{\text{AVIM}}(0)$	$-\varphi''_{\text{Exact}}(0)$	$\lvert\varphi''_{\text{Exact}}(0)-\varphi''_{\text{AVIM}}(0)\rvert$
0	0.408937341	1	0	1.182125000	1.173721	0.008404
0.01	0.403544842	1	0	1.192910285	1.177834	0.015076
0.04	0.395018813	1	0	1.209962370	1.190100	0.019862
0.25	0.350574316	1	0	1.298851368	1.273033	0.025817
1	0.221582884	1	0	1.556834232	1.535711	0.021123
4	−0.161440022	1	0	2.322880043	2.311719	0.011161
25	−1.575931552	1	0	5.151863104	5.131808	0.020055
100	−4.043836108	1	0	10.08767222	10.066473	0.021199

Table 2. Comparison of the first-order approximate solution with exact solution for flow in the presence of suction

A	a	b	c	$-\varphi''_{\text{AVIM}}(0)$	$-\varphi''_{\text{Exact}}(0)$	$\lvert\varphi''_{\text{Exact}}(0)-\varphi''_{\text{AVIM}}(0)\rvert$
0.0	0.408937341	1.0	0	1.182125000	1.173721	0.008404
0.1	0.385447686	1.1	0.1	1.329104628	1.284242	0.044863
0.2	0.355791949	1.2	0.2	1.488416102	1.402357	0.086059
0.5	0.221367319	1.5	0.5	2.057265362	1.798669	0.258596

Table 3. Comparison of the first-order approximate solution with exact solution for MHD flow with suction

A	a	b	c	$-\varphi''_{\text{AVIM}}(0)$	$-\varphi''_{\text{Exact}}(0)$	$\lvert\varphi''_{\text{Exact}}(0)-\varphi''_{\text{AVIM}}(0)\rvert$	M
0	0.221582884	1.0	0.0	1.556834232	1.535711	0.021123	1
0.1	0.208470237	1.1	0.1	1.683059527	1.643119	0.039940	
0.2	0.189124765	1.2	0.2	1.821750469	1.756636	0.065110	
0.5	0.085783708	1.5	0.5	2.328432584	2.131936	0.196490	
0	−0.161440022	1.0	0.0	2.322880043	2.311719	0.011161	4
0.1	−0.172066549	1.1	0.1	2.444133098	2.415859	0.028274	
0.2	−0.188156645	1.2	0.2	2.576313289	2.524220	0.052093	
0.5	−0.275688473	1.5	0.5	3.051376946	2.874031	0.177346	
0	−1.575931552	1.0	0.0	5.151863104	5.131808	0.020055	25
0.1	−1.594931468	1.1	0.1	5.289862936	5.233199	0.056664	
0.2	−1.617807388	1.2	0.2	5.435614776	5.336531	0.099084	
0.5	−1.712825408	1.5	0.5	5.925650816	5.658119	0.267532	

system which includes three equations with three unknowns, we obtain the values of a, b, c for each values of M and A.

In Tab. 3, the values of $-\varphi''(0)$ are presented for different non-zero values of M and A using the exact numerical solution obtained by the Ackroyd method and the AVIM.

5 Conclusion

In the present paper we have investigated the suitability of the variational iteration method (VIM) for computing the axisymetric flow of a viscous, incompressible fluid due to the stretching of a sheet. It is found that the VIM produces a very accurate solution. The effects of the presence of a transverse magnetic field and suction at the sheet are also taken into account, separately and jointly. The method is able to handle the Hartmann layers due to the magnetic field without degradation of the performance.

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