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The Particle Swarm Optimization approach applied to the Von-Kàrmàn equation*

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Abstract. The B-spline method together with the particle swarm optimization (PSO for short) strategy is presented. It is applied to circular thin flexible plates within the context of Von-Kàrmàn equations. Our main result shows that the low orders of approximation have been sufficient to build the solutions from the basis functions in a high quality. Comparison of solutions obtained to variational approach with known ones is performed.

Keywords: B-spline, variational, Galerkin, Kàrmàn equation, Particle Swarm Optimization

1 Introduction

Many efforts are focused on developing either numerical or analytical methods for solving a variety of mechanical structures. Among these, are the applications of practical and general interest involve equation of thin plate[11, 13–15]. In general, the variational method requires the construction of an appropriate functional of the problem. We introduce therefore the non variational Galerkin-B-spline method. B-Spline functions have been evolved successfully in the major areas solid mechanics and theoretical physics during the past several years[5, 10]. The aim of this paper is to explore the utility and efficiency of the non variational Galerkin-B-spline method together with the help of the algorithm PSO for the numerical solutions of the von-Karman equation.

2 The B-spline functions

The B-Spline functions or basis spline functions are defined in every local segment and constitute a basis for piecewise polynomial functions on the domain $[a, b]$. This domain is broken into subdomains in which a set of number $\{x_i\}, i = 1, 2, \cdots, N$, are defined. The i-th B-spline basis function of order $k + 1$ (degree k) satisfies a recursive relationship between $B_i^k(x)$ and $B_{i+1}^{k-1}(x)$ given by:

$$B_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i} B_i^{k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1}^{k-1}(x), \quad \text{and} \quad B_i^0(x) = \begin{cases} 1, & x_i \le x < x_{i+1} \\ 0, & \text{otherwise.} \end{cases}$$

3 Particles Swarm Optimization strategy

A new stochastic algorithm has recently appeared, called 'particles swarm optimization' PSO. The PSO model is a particle simulation concept, and was first proposed by Eberhart and Kennedy [2, 7]. Since then,

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several variants of the PSO have been developed [3, 6, 8]. It has been shown that, the question of convergence of the PSO algorithm is implicitly guaranteed if the parameters are adequately selected .

The strategy of the PSO algorithm is summarized as follows: We assume that each agent (particle) i can be represented in a N-dimension space by its current position $X_i = (x_{i1}, x_{i2}, \cdots , x_{iN})$ and its corresponding velocity $V_i = (v_{i1}, v_{i2}, \cdots , v_{iN})$. Also a memory of its personal (previous) best position is represented by $P_i = (p_{i1}, p_{i2}, \cdots , p_{iN})$, called (pbest), the subscript i range from 1 to s, where s indicates the size of the swarm. Commonly, each particle localizes its best value so far (pbest) and its position, and consequently identifies its best value in the group (swarm), called also (sbest) among the set of values (pbest).

The velocity and position are updated as:

$$v_{ij}^{k+1} = w_j v_{ij}^{k} + c_1 r_1^k [(pbest)_{ij}^k - x_{ij}^k] + c_2 r_2^k [(sbest)_j^k - x_{ij}^k] \tag{1}$$

$$x_{ij}^{k+1} = v_{ij}^{k+1} + x_{ij}^{k}, \tag{2}$$

where x_i^{k+1} , v_i^{k+1} are the position and the velocity vector of particle i respectively at iteration $k+1$, c_1 and c_2 are acceleration coefficients for each term exclusively situated in the range of 2 to 4, w_j is the inertia weight with its value that ranges from 0.9 to 1.2, whereas r_1^k , r_2^k are uniform random numbers between zero and one. For more detail, the double subscript in the relations (1) and (2) means that, the first subscript for the particle i and the second one for the dimension j. The role of a suitable choice of the inertia weight w_j is important in the PSO success. The following algorithm should give us the general idea how to generate the particles in the swarm:

Step 1. Set the values of the dimension space N, and the size s of the swarm (s can be taken randomly).
Step 2. Initialize the iteration number k (in the general case is set equal to zero).
Step 3. Evaluate for each agent, the velocity vector using its memory and Eq. (1), where pbests and sbest can be modified.
Step 4. Each agent must be updated by applying its velocity vector and its previous position using Eq. (2).
Step 5. Repeat the above steps (3, 4 and 5) until a convergence criterion is reached.

4 Solution for von karman plate: a B-spline basis

The nonlinear equations of the von Karman type describing a thin circular elastic plate are formulated in dimensionless form as [1, 9, 16].

$$3\left(1-\nu^2\right) Q_{ra}(x)\frac{\mathrm{d}W(x)}{dx} - \left(2\frac{\mathrm{d}^2W(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}^3W(x)}{\mathrm{d}x^3}\right) = -\frac{3P}{4}, \tag{3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(xQ_{ra}(x)\right) + \frac{1}{2}\left(\frac{\mathrm{d}W(x)}{\mathrm{d}x}\right)^2 = 0, \tag{4}$$

$$Q_{ta}(x) = Q_{ra}(x) + 2x\frac{\mathrm{d}Q_{ra}(x)}{\mathrm{d}x}. \tag{5}$$

Here the quantities W, Q_s; $(s = ra, ta)$, P and x are the dimensionless forms represented by:

$$W(x) = \frac{w}{H},\ Q_s = \frac{R^2}{EH^3}M_s,\ P = \frac{\mathrm{d}R^4}{EH^4}\left(1-\nu^2\right) \quad \text{and} \quad x = \left(\frac{r}{R}\right)^2 (0 \le r \le R),$$

in which w is the normal displacement from the center of the plate at radial position r, M_{ra} (M_{ta}) is the radial (tangential) force, R is the radius of the plate , H is the thickness of the plate , E is elasticity module, d is the specific uniform load and ν is Poisson number.

The Eqs. (3) $\sim$ (5) subject to the following boundary conditions:

$$\begin{cases} \frac{\mathrm{d}W}{\mathrm{d}x} \text{ and } Q_{ra} \text{ take finite values at } x = a, \\ W(b) = \frac{\mathrm{d}W(b)}{\mathrm{d}x} = 0 \\ 2\frac{\mathrm{d}Q_{ra}(b)}{\mathrm{d}x} + (1-\nu)\, Q_{ra}(b) = 0 \end{cases} \tag{6}$$

The systematic development of the non variational Galerkin-B-splines method can be found in [12]. A suitable knot sequence is chosen like $a = 0 = x_0 < x_1 < \cdots < x_{N+k} = b = 1$, together with the boundary conditions for the displacement. The solution $W(x)$ is then expanded as a linear combination of B-splines functions: $W(x) = \sum_{i=0}^{N-1} q_i B_i(x)$.

We require the residual of the Eq. (3) to be orthogonal to the function space of test functions $B_i(x)$, to yield

$$\left\langle \begin{array}{c} \left[3\left(1-\nu^2\right) Q_{ra}(x)\frac{\mathrm{d}}{\mathrm{d}x} - \left(2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}^3}{dx^3}\right)\right] W \\ +\frac{3P}{4}, B_j \end{array} \right\rangle = 0, \qquad \text{for } j = 0, \cdots, N-1. \tag{7}$$

With the number of coefficients embedded in W and after evaluating all integrals in (7) we obtain a nonlinear algebraic system in term q_i.

Before attempting a solution via non- variational Galerkin B-spline method, we first transform the relation (4) in a more practical form, because the function $Q_{ra}(x)$ contains implicitly the desired displacement $W(x)$. We will obtain an integration of the (4) in a more compact formula as

$$Q_{ra}(x) = \frac{1}{2x}\int_0^1 K(x,y)\left(\frac{\mathrm{d}W}{\mathrm{d}y}\right)^2 \mathrm{d}y, \tag{8}$$

in which the kernel $K(x, y)$ is given by

$$K(x,y) = \begin{cases} (1+\alpha x)\, y & y < x \\ (1+\alpha y)\, x & y > x \end{cases} \tag{9}$$

where $\alpha = (1 + \nu)/(1 - \nu)$.

The elements $(B)_{ij}$ are constructed from (7) and have the following expression

$$(B)_{ij} = \int_a^b B_i(r) B_j(r)\mathrm{d}r. \tag{10}$$

A number of representative numerical tests for several uniform loading are presented and the convergence is met with $N = 35$ for $k = 4$ almost for all the cases and the best knot sequences of interpolation using particles swarm optimization approach are displayed in Tab. 1 and Tab. 2. For the sake of comparison, we use the variational approach for a simple trial form.

5 The variational approach

Using the semi-inverse method[4], J. H. He has shown that the correspondent Lagrangian for the Eqs. (3) and (4) has the form:

$$L(Q_{ra}, W) = \frac{1}{2}xQ_{ra}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 - \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}x}(xQ_{ra})\right]^2 + \frac{1}{6(1-\nu^2)}\left[\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}W\right]^2 - \frac{1}{4(1-\nu^2)}PW \tag{11}$$

Now it is quite common to suppose a simple variational trial function for the displacement of the form:

$$W(x) = A(x-1)^2\left(c_0 + c_1 x + c_2 x^2\right) \tag{12}$$

The c_i's are the variational parameters of the problem, and A is a free quantity. The Lagrangian mechanics starts from the variational problem as:

$$\int_a^b L(c_0, c_1, c_2, \cdots; x)\mathrm{d}x = F(c_0, c_1, c_2, \cdots) = \text{ stationary!}, \tag{13}$$

The desired optimal values of the parameters $c's$ must be estimated by solving the nonlinear algebraic system. We plot out some curves for $W(x)$ across a range of values of P shown in Fig. 1 and Fig. 2 for the present method and the variational treatment respectively. We see from Fig. 2, the shape of the variational solution $W(x)$ is respected.

Table 1. The best knot sequences of interpolation using the particles swarm optimization for $P = 20$ and $P = 30$

i	x_i , $P = 20$	x_i , $P = 30$	i	x_i , $P = 20$	x_i , $P = 30$
1	0.0000	0.0000	13	0.4450	0.4611
2	0.0340	0.0200	14	0.4834	0.4809
3	0.0411	0.0354	15	0.5521	0.5323
4	0.0554	0.0421	16	0.5902	0.5956
5	0.0737	0.0665	17	0.6250	0.6031
6	0.0798	0.0744	18	0.6473	0.6233
7	0.0923	0.0931	19	0.6858	0.7108
8	0.1373	0.1388	20	0.7351	0.7865
9	0.1686	0.1785	21	0.7601	0.7931
10	0.2697	0.2514	22	0.7968	0.8001
11	0.3964	0.3996	23	0.7991	0.8290
12	0.4158	0.4067	24	0.8052	0.8555

Table 2. Continuous Tab. 1: The best knot sequences of interpolation using the particles swarm optimization for $P = 20$ and $P = 30$

i	x_i , $P = 20$	x_i , $P = 30$
25	0.8311	0.8643
26	0.8500	0.8723
27	0.8722	0.8901
28	0.9058	0.9310
29	0.9590	0.9778
30	0.9712	0.9800
31	0.9732	0.9845
32	0.9811	0.9901
33	0.9922	0.9991
34	0.9996	0.9996
35	1.0000	1.0000

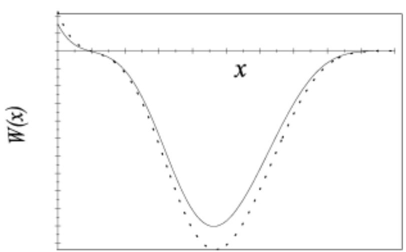

Fig. 1. The dimensionless displacement $W(x)$: The solid and dots curves for $P = 20$ and 30 respectively. Present work with PSO strategy

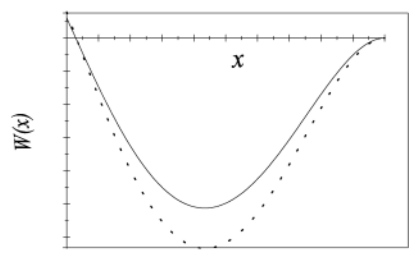

Fig. 2. The dimensionless displacement $W(x)$: The solid and dots curves for $P = 20$ and 30 respectively. Variational case

6 Comment and conclusion

We have presented the B-spline method together with PSO strategy, and its application to the von – Karman equations describing a one- dimensional elastic circular thin plate. It is shown that the solutions for the displacement obtained through the B-spline scheme and suitable knot sequences selected by the PSO algorithm converge quickly and give an excellent result for a set of uniform loads. It is important to emphasize that the B-spline method and the PSO strategy provide us with a way of finding not only the desired good numerical solutions of the problem but also the question of convergence is reached with a surprising manner. It is noted also that, with a restricted basis set, was sufficient to describe the best behavior of displacements for different values of loads. In some situations, the PSO strategy considerably avoids

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