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Application of homotopy perturbation method to non-homogeneous parabolic partial and non linear differential equations

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Abstract. In this paper homotopy perturbation method (HPM) is employed to solve two kinds of differential equations: one dimensional non homogeneous parabolic partial differential equation and non linear differential equation. Using the HPM, an exact analytical solution to non homogeneous parabolic partial differential equation and an approximate explicit solution for a non linear differential equation were obtained. The results obtained by HPM for the non linear differential equation were compared with those results obtained by the exact analytical solution. The comparison shows a complete agreement between results and also shows that this new method may be applicable for solving engineering problem because it needs less computations efforts and is easier than others.

Keywords: homotopy peturbation method, non linear differential equation, non homogeneous partial differential equation

1 Introduction

Most physical phenomena that occurred in nature such as heat transfer are governed by non-linear partial differential equations (NLPDE). To understand such phenomena, one must solve the corresponding NLPDE. Howeverv, most of them do not have exact analytical solutions. Therefore, these NLPDE should be solved using other methods such as numerical methods or semi-analytical method. Some investigators also proposed the combination of these two methods for obtaining the approximate solution to NLPDE.

Another known method to solve the NLPDE is a perturbation method which is studied by several investigators for solving some physical problems[9, 10]. Nevertheless, a perturbation method necessitates the existence of a small parameter, which limits its use for different applications. This limitation is overcome using the homotopy perturbation method (HPM) which was first proposed by He[5–8]. Comparatively to classical methods, the HPM method, presents some advantages: obtaining explicit solution with high accuracy, minimal calculations without loss of physical verification. This method has found application in different fields of non linear equations such as fluid mechanics and heat transfer[1–4].

The objective of the present study is to implement the HPM for finding: (1) the exact analytical solution to one dimensional non-homogeneous parabolic partial differential equation with a variable coefficient and (2) the approximate solution of a non linear differential equation that governs the cooling process of a body, with a variable specific heat with temperature, immerged in a fluid with a given temperature.

The first equation is as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = (\cos t - \sin t)\, e^{-2x}, \tag{1}$$

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with the initial and boundary conditions:

$$u(x,0) = x^2, \tag{2}$$

$$u(0,t) = 2t + \frac{3e^{4t} + 5\sin t - 3\cos t}{17}, \tag{3}$$

$$u(1,t) = 1 + 2t + \frac{3e^{4t} + 5\sin t - 3\cos t}{17e^2}. \tag{4}$$

The second equation is a non-linear differential equation that governs the cooling of a body, with a variable specific heat with temperature, which is exposed to an environment of a given temperature T_a. Suppose that the temperature is uniform throughout the body (the Biot number, $Bi \leq 0.1$) (lumped system) and its initial temperature is T_i. The differential equation that governs the cooling process of a body is:

$$\rho V C \frac{dT}{dt} + hA(T - T_a) = 0, \tag{5}$$

with an initial condition: $T = 0,\ T(0) = T_i$. If the specific heat varies linearly with temperature, its expression is given as follows: $C = C_a(1 + \beta(T - T_a))$, where C_a is a specific heat at the environment temperature and β is a constant. Using the following dimensionless temperature, θ, time, τ, and parameter, ξ:

$$\theta = \frac{T - T_a}{T_i - T_a};\ \tau = \frac{hA\,t}{\rho V C_a};\ \varepsilon = \beta(T_i - T_a) \tag{6}$$

The corresponding dimensionless non linear differential equation is:

$$(1 + \varepsilon\,\theta)\frac{d\theta}{d\tau} + \theta = 0, \tag{7}$$

with

$$\tau = 0,\ \theta = 1. \tag{8}$$

2 Fundamentals of the homotopy-perturbation method

The basic idea of the HPM method is illustrated as follows using the NLPDE:

$$O(u) - f(r) = 0, r \in \mathcal{D}. \tag{9}$$

With its corresponding boundary conditions:

$$\mathcal{B}(u, \frac{\partial u}{\partial n}) = 0, r \in \mathcal{T}, \tag{10}$$

where O, $\mathcal{B}, f(r), r$ and $\mathcal{T}$ are a differential operator, a boundary operator, a known analytical function, a coordinate and the boundary of the domain $\mathcal{D}$, respectively.

In general, the operator O can be divided into two parts $\mathcal{L}$ (linear part) and $\mathcal{N}$ (nonlinear part). Therefore, Eq. (9) can be written as $\mathcal{L}(u) + \mathcal{N}(u) - f(r) = 0$.

Considering the homotopy technique, a homotopy $v(r,p)$ can be constructed:

$$H\,(\nu,\,p) = (1-p)\,[L(\nu) - L(\nu_o)] + p\ [O(\nu) - f(r)] = 0, \tag{11}$$

where $P \in [0,1]$ is an embedding parameter and ν_0 is an initial approximation of Eq. (9), which satisfies the boundary conditions, Eq. (10).

According to HPM, the embedding parameter can first used as an expanding parameter, and the solution of Eq. (11) can be written as a power series in p:

$$\nu(r,p) = \nu_0 + p\nu_1 + p^2\nu_2 + p^3\nu_3 + \cdots \tag{12}$$

The approximate solution of Eq. (9) can be obtained by setting $p = 1$ in Eq. (12):

$$u(x,t) = \lim_{p\to 1}\ \nu(x,t). \tag{13}$$

Generally, the series (13) is convergent and its convergent rate depends on the nonlinear operator $O(v)$.

2.1 Application of the HPM method for Eq. (1)

To solve Eq. (1), a homotopy should be constructed by separating the linear and nonlinear parts of the Eq. (1); hence by applying HPM to Eq. (1) using Eq.(11), one obtain:

$$H(\nu,\, p) = (1-p)\left[\frac{\partial \nu}{\partial t} - \frac{\partial \nu_o}{\partial t}\right] + p\left[\frac{\partial \nu}{\partial t} - \frac{\partial^2 \nu}{\partial x^2} - (\cos t - \sin t)e^{-2x}\right] = 0. \tag{14}$$

By choosing an initial approximation solution $v_o = u(x,0) = x^2$ and substituting Eq. (12) in to Eq. (14) and rearranging the resultant equation based on powers of p-terms, one has:

$$\begin{aligned}
&p^0 : \frac{\partial \nu_o}{\partial t} = 0\\
&p^1 : \frac{\partial \nu_1}{\partial t} + \frac{\partial \nu_o}{\partial t} - \frac{\partial^2 \nu_o}{\partial x^2} - (\cos t - \sin t)e^{-2x} = 0\\
&p^2 : \frac{\partial \nu_2}{\partial t} - \frac{\partial^2 \nu_1}{\partial x^2} = 0\\
&p^3 : \frac{\partial \nu_3}{\partial t} - \frac{\partial^2 \nu_2}{\partial x^2} = 0\\
&\cdots\\
&p^n : \frac{\partial \nu_n}{\partial t} - \frac{\partial^2 \nu_{n-1}}{\partial x^2} = 0.
\end{aligned} \tag{15}$$

Solving the previous equations and considering the initial and boundary conditions (2)-(4), results of the following solutions:

$$\begin{aligned}
&\nu_o = x^2\\
&\nu_1 = 2t + (\cos t + \sin t - 1\,)e^{-2x}\\
&\nu_2 = 4(\sin t - \cos t - t + 1\,)e^{-2x}\\
&\nu_3 = 4^2(-\cos t - \sin t + t - \frac{t^2}{2} + 1\,)e^{-2x}\\
&\nu_4 = 4^3(\cos t - \sin t + t + \frac{t^2}{2} - \frac{t^3}{3!} - 1\,)e^{-2x}\\
&\cdots
\end{aligned} \tag{16}$$

In the same manner, the rest of components can be expressed using the following expressions:

$$\begin{aligned}
\nu_{2n} &= (-1)^n 4^{2n-1}\left[\cos t - \sin t - 1 + \delta_n \sum_{k=1}^{n-1}(-1)^{k+1}\frac{t^{2k}}{(2k)!} + \sum_{k=0}^{n-1}(-1)^k \frac{t^{2k+1}}{(2k+1)!}\right] e^{-2x},\\
\delta_n &= \begin{cases} 1 & n \geq 2 \\ 0 & n = 1 \end{cases}\\
\nu_{2n+1} &= (-1)^n 4^{2n}\left[\cos t + \sin t - 1 + \sum_{k=1}^{n}(-1)^{k+1}\frac{t^{2k}}{(2k)!} + \sum_{k=0}^{n-1}(-1)^{k+1}\frac{t^{2k+1}}{(2k+1)!}\right] e^{-2x}\\
&\text{with} \quad n \geq\ 1.
\end{aligned} \tag{17}$$

According to HPM, one can conclude that:

$$u(x,t) = \lim_{p\to 1}\ \nu\ (x,t) = \nu(x,t) = \nu_0 + \nu_1 + \nu_2 + \nu_3 + \cdots \tag{18}$$

By substituting the expressions of $v_0(x,t), v_1(x,t), v_2(x,t), \cdots$ from Eq. (17) in to Eq. (18) yields an exact analytical solution:

$$u\,(x,t) = x^2 + 2t + \frac{e^{-2x}}{17}\left(3e^{4t} + 5\sin t - 3\cos t\right) \tag{19}$$

2.2 Application of the HPM method for Eq. (2)

To solve Eq. (7), a homotopy should be constructed by separating the linear and nonlinear parts of the Eq. (7); hence by applying HPM to Eq. (7) using Eq. (11), one obtains:

$$H(\nu,\, p) = (1-p)\left[\frac{\partial \nu}{\partial \tau} - \frac{\partial \nu_o}{\partial \tau}\right] + p\left[\frac{\partial \nu}{\partial \tau} + \upsilon + \varepsilon\, \nu\, \frac{\partial \nu}{\partial \tau}\right] = 0. \tag{20}$$

By choosing an initial approximation solution, $V_0 = \theta(0) = 1$, we have:

$$\frac{\partial \nu}{\partial \tau} + p\left[\upsilon + \varepsilon\, \nu\, \frac{\partial \nu}{\partial \tau}\right] = 0, \tag{21}$$

where $\nu = \nu(0)$.

By considering ν as following:

$$\nu(\tau) = \nu_0 + p\nu_1 + p^2\nu_2 + p^3\nu_3 + \cdots \tag{22}$$

substituting Eq. (22) in to Eq. (21) and rearranging the resultant equation based on powers of p-terms, we have:

$$\begin{aligned}
&P^0 : \frac{\partial \nu_o}{\partial \tau} = 0\\
&p^1 : \frac{\partial \nu_1}{\partial \tau} + \nu_o + \varepsilon\, \nu_o\, \frac{\partial \nu_o}{\partial \tau} = 0\\
&p^2 : \frac{\partial \nu_2}{\partial \tau} + \nu_1 + \varepsilon(\, \nu_o\, \frac{\partial \nu_1}{\partial \tau} + \nu_1\, \frac{\partial \nu_o}{\partial \tau}) = 0\\
&p^3 : \frac{\partial \nu_3}{\partial \tau} + \nu_2 + \varepsilon(\, \nu_o\, \frac{\partial \nu_2}{\partial \tau} + \nu_1\, \frac{\partial \nu_1}{\partial \tau} + \nu_2\, \frac{\partial \nu_o}{\partial \tau}) = 0\\
&p^4 : \frac{\partial \nu_4}{\partial \tau} + \nu_3 + \varepsilon(\, \nu_o\, \frac{\partial \nu_3}{\partial \tau} + \nu_1\, \frac{\partial \nu_2}{\partial \tau} + \nu_2\, \frac{\partial \nu_1}{\partial \tau} + \nu_3\, \frac{\partial\, \nu_o}{\partial \tau}) = 0\\
&\cdots\\
&p^n : \frac{\partial \nu_n}{\partial \tau} + \nu_{n-1} + \varepsilon \sum_{i=o}^{n-1} \nu_i\, \frac{\partial\, \nu_{n-1-i}}{\partial \tau} = 0.
\end{aligned} \tag{23}$$

Solving the previous equations and considering the initial condition (8), results of the following solutions:

$$\begin{aligned}
&\nu_o = 1\\
&\nu_1 = -\tau\\
&\nu_2 = \varepsilon\, \tau + \frac{\tau^2}{2!}\\
&\nu_3 = -\varepsilon^2\, \tau - \frac{3}{2!}\varepsilon\tau^2 - \frac{\tau^3}{3!}\\
&\nu_4 = \varepsilon^3\, \tau + \frac{6}{2!}\varepsilon^2\tau^2 + \frac{7}{3!}\varepsilon\tau^3 + \frac{\tau^4}{4!}\\
&\nu_5 = -\varepsilon^4\, \tau - \frac{10}{2!}\,\varepsilon^3\tau^2 - \frac{25}{3!}\varepsilon^2\tau^3 - \frac{15}{4!}\varepsilon\tau^4 - \frac{\tau^5}{5!}\\
&\nu_6 = \varepsilon^5\, \tau + \frac{15}{2}\,\varepsilon^4\tau^2 + \frac{65}{3!}\varepsilon^3\tau^3 + \frac{90}{4!}\varepsilon^2\tau^4 + \frac{31}{5!}\varepsilon\tau^5 + \frac{\tau^6}{6!}\\
&\nu_7 = -\varepsilon^6\, \tau - \frac{21}{2!}\,\varepsilon^5\tau^2 - \frac{70}{3!}\varepsilon^4\tau^3 - \frac{361}{4!}\varepsilon^3\tau^4 - \frac{301}{5!}\varepsilon^2\tau^5 - \frac{63}{6!}\varepsilon\tau^6 - \frac{\tau^7}{7!}.
\end{aligned} \tag{24}$$

Collecting the results, the solution $\theta(\tau)$ obtained by the homotopy perturbation method is: $\theta(\tau) = \lim_{p\to 1} \nu(\tau)$, which is equivalent to: $\upsilon(\tau) = \upsilon_0 + \upsilon_1 + \upsilon_2 + \cdots$

Finally, incorporating the expressions for $v_0, v_1, v_2, v_3 \cdots$ from Eq. (24), the solution, $\theta(\tau)$ is given by:

$$\theta(\tau) = \left[1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \cdots\right] + \varepsilon\left[\tau - \frac{3}{2!}\tau^2 + \frac{7}{3!}\tau^3 - \frac{15}{4\,!}\tau^4 + \cdots\right]$$
$$+\, \varepsilon^2\left[-\tau + \frac{6}{2!}\tau^2 - \frac{25}{3!}\tau^3 + \frac{90}{4!}\tau^4 + \cdots\right] + \cdots \tag{25}$$

which can be written as:

$$\theta(\tau) = e^{-\tau} + \varepsilon\ \varphi_1(\tau) + \varepsilon^2\ \varphi_2(\tau) \cdots + \varepsilon^n\ \varphi_n(\tau), \tag{26}$$

with $\varphi_1(\tau) = e^{-\tau}\ C_1(\tau), \quad \varphi_2(\tau) = e^{-\tau}\ C_2(\tau), \cdots \varphi_n(\tau) = e^{-\tau}\ C_n(\tau)$.

The functions $C_1(\tau), C_2(\tau), \cdots C_i(\tau)$ can be obtained by substituting Eq. (26) into Eq. (7) and rearanging the resulting equation, once obtain the following equations:

$$\begin{aligned}
&C_1 = a_{10} + a_{11}e^{-\tau}, \text{ with } \quad a_{10} = 1 \quad \text{and} \quad a_{11} = -1\\
&C_2' = (2C_1 - C_1')e^{-\tau}\\
&C_3' = (2C_2 - C_2')e^{-\tau} - \left[\frac{\varphi_1^2}{2}\right]' e^{-\tau}\\
&C_4' = (2C_3 - C_3')e^{-\tau} - [\varphi_1\varphi_2]' e^{-\tau}\\
&\ \cdots\\
&C_n' = (2C_{n-1} - C_{n-1}')e^{-\tau} - \left[\varphi_1\varphi_{n-2} + \varphi_2\varphi_{n-3} + \cdots + \varphi_{\frac{n}{2}-1}\varphi_{\frac{n}{2}}\right]' e^{-\tau}, \quad \text{if } n \text{ is pair}\\
&C_n' = (2C_{n-1} - C_{n-1}')e^{-\tau} - \left[\varphi_1\varphi_{n-2} + \varphi_2\varphi_{n-3} + \cdots + \frac{\varphi^2_{\frac{n-1}{2}}}{2}\right]' e^{-\tau}, \quad \text{if } n \text{ is impair}
\end{aligned} \tag{27}$$

The solution of Eq. (27) yields:

$$C_n = a_{n0} + a_{n1}e^{-tau} + a_{n2}e^{-2\tau} + \cdots + a_{nn}e^{-n\tau}, \quad \text{for} \quad n \geq 2, \tag{28}$$

and the solution of Eq. (7) is given as follows:

$$\theta(\tau) = e^{-\tau}\ \left[1 + \varepsilon\ C_1 + \varepsilon^2\ C_2 + \cdots + \varepsilon^n\ C_n\right], \tag{29}$$

where the coefficient, a_{nl}, appearing in the expression (29) is given by the following expressions:

$$a_{nl} = -\frac{l+1}{l}\left[a_{n-1,l-1} + \sum_{i=1}^{\frac{n}{2}-1}\sum_{j_{\min}}^{j_{\max}} a_{i,j}\ a_{n-i-1,l-j-1}\right], \text{ if } n \text{ is pair} \tag{30}$$

$$a_{nl} = -\frac{l+1}{l}\left[a_{n-1,l-1} + \sum_{p=1}^{\frac{n-3}{2}}\sum_{j_{\min}}^{j_{\max}} a_{i,j}a_{n-i-1,l-j-1} + \frac{1}{2}\sum_{j_{\min}}^{j_{\max}} a_{\frac{n-1}{2},j}a_{\frac{n-1}{2},l-j-1}\right], \text{ if } n \text{ is impair}, 0 \leq l \leq n \tag{31}$$

with

$$j_{\min} = \text{Max}(0,\ l + i - n),\ j_{\max} = \text{Min}(i,\ l - 1). \tag{32}$$

It should be noted that the exact analytical solution for Eq. (7) is given by the following expression:

$$Ln\ (\theta) + \varepsilon\ (1 - \theta) = -\ \tau \tag{33}$$

3 Discussion of results

Fig. 1 displays the time wise variation of the solutions $\theta(\tau)$ obtained by the homotopy perturbation method (Eq. (29)) and the exact analytical solution (Eq. (33)), for different values of the parameter, ε, $(0 \leq \varepsilon \leq 1)$. The analysis of such figure shows clearly the complete agreement between the two solutions. As it can be seen from this figure, as the parameter, ε, decreases, the solution, $\theta(\tau)$, decreases rapidly and reaches the value 0. It should be noted that the non linear Eq. (7) is the energy equation that governs the cooling of a body, with initial dimensionless temeprature $\theta = 1$, placed in an environment with a dimensionless temperature $\theta = 0$. As time progress, the body cools and its temperature decreases and attains the equilibrum temperature, $\theta = 0$, which is equal to the environment, when a steady state is reached.

It should be also noted that the HPM method presents high accuracy not only for lower values, $\varepsilon(\sim 0.2)$ but also for $\varepsilon = 1$.

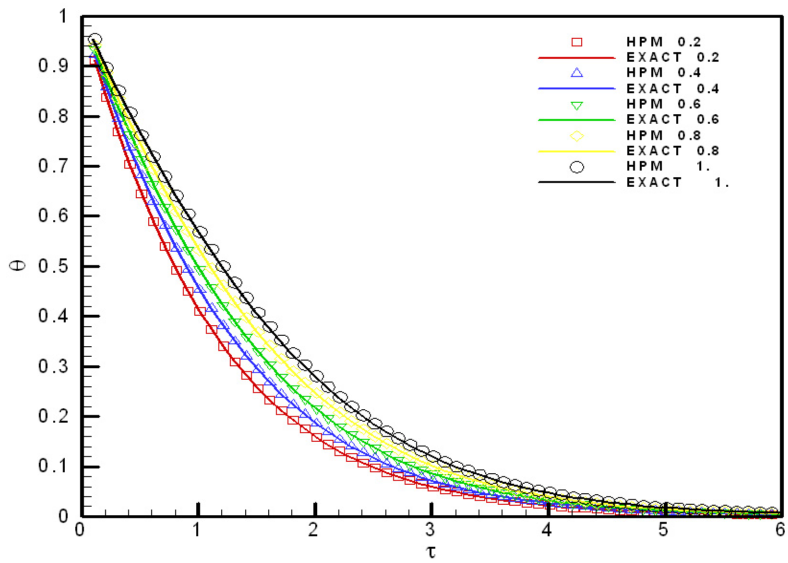

Fig. 1. The time wise variation of dimensionless temperature, θ, for different values of the parameter, ε(HPM correspond to symbols and exact analytical solution correspond to solid lines).

4 Conclusion

Homotopy perturbation method (HPM) is used to solve one dimensional non homogeneous parabolic partial differential equation and non linear differential equation. An exact analytical solution to non homogeneous parabolic partial differential equation and an approximate explicit solution for a non linear differential equation were obtained. The results obtained by HPM for the non linear differential equation were compared with those results obtained by the exact analytical solution. The comparison shows a complete agreement between results not only for lower values of parameter, ε(∼0.2), but also for, ε, close to one. The study also shows the capability of this new method for solving engineering problem because it needs less computations efforts and is easier than others.

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