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# The extended Jacobi Elliptic Functions expansion method and new exact solutions for the Zakharov equations *

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**Abstract.** We extended the Jacobi elliptic function expansion method by constructing four new Jacobian elliptic functions, and apply this method to Zakharov equations for illustration, abundant new doubly periodic solutions are obtained, these solutions are degenerated to the solitary wave solutions and the triangle function solutions in the limit cases when the modulus of the Jacobian elliptic functions m→1 or 0, which shows that the new method is more powerful to seek the exact solutions of the nonlinear partial differential equations in mathematical physics.

**Keywords:** Zakharov equations, extended Jacobian Elliptic Functions expansion method, doubly periodic solutions, exact solutions

## 1 Introduction

In the interaction of laser-plasma the system of Zakharov equation (SZE) plays an important role[18, 19]. This system attracted many scientists' wide interest and attention. In one dimension, the formation, evolution and interaction of the Langmuir solution differ from solutions of the KdV equation. In multi-dimensions, the Langmuir solution will collapse. Since 1980s, the effects including magnetic field have been considered, and the system of Zakharov equation includes more general form and rich contents. For example, SZE with Landau damping effect was given in [8, 14]. Under some conditions its inverse scattering transformation has been found. In [8], the longitudinal and transverse oscillating and magnetic field effect was examined, the solution properties and collapse in multi-dimensions have been revealed.

More recently, some authors considered the exact and explicit solutions of the system of Zakharov equations by different methods in [3, 4, 10, 15, 16, 20–22, 24, 25]. In this paper, we consider the system of Zakharov equations by constructing four new types of Jacobian elliptic functions, and abundant new families of exact solutions are obtained.

This paper is arranged as follows. In section 2, we briefly describe the new extended Jacobi elliptic functions expansion method. In section 3, several families of solutions to the Zakharov equations are obtained, which are degenerated to new solitary wave solutions and new periodic wave solutions in the limit case. In section 4, some conclusions are given.

## 2 Summary of the extended jacobi elliptic functions expansion method

In this section, we propose a general method, namely, the extended Jacobi elliptic functions expansion method, given nonlinear partial differential equation, for instance, in two variables $x$ and $t$, as follows:

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$$P(u, u_t, u_x, u_{xx}, \cdots) = 0 \tag{1}$$

We seek the following formal solutions of the given system by a new intermediate transformation:

$$u(\xi) = \sum_{i=1}^{n} a_i F^i(\xi) + \sum_{i=1}^{n} b_i F^{i-1}(\xi)E(\xi) + \sum_{i=1}^{n} c_i F^{i-1}(\xi)G(\xi) + \sum_{i=1}^{n} d_i F^{i-1}(\xi)H(\xi) + a_0 \tag{2}$$

where $a_0$, $a_i$, $b_i$, $c_i, d_i$ $(i = 1, 2, \cdots, n)$ are constants to be determined later. $\xi = \xi(x,t)$ are arbitrary functions with the variables $x$ and $t$, The parameter $n$ can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq. (1). And $E(\xi)$, $F(\xi)$, $G(\xi)$, $H(\xi)$ are an arbitrary array of the four function $e = e(\xi)$, $f = f(\xi)$, $g = g(\xi)$ and $h = h(\xi)$, the selection obey the principle which makes the calculation more simple. Here we get

$$\begin{cases} e = \dfrac{1}{p + qsn[\xi,m] + rcn[\xi,m] + ldn[\xi,m]}, \; f = \dfrac{sn[\xi,m]}{p + qsn[\xi,m] + rcn[\xi,m] + ldn[\xi,m]} \\ g = \dfrac{cn[\xi,m]}{p + qsn[\xi,m] + rcn[\xi,m] + ldn[\xi,m]}, \; h = \dfrac{dn[\xi,m]}{p + qsn[\xi,m] + rcn[\xi,m] + ldn[\xi,m]} \end{cases} \tag{3}$$

where $p, q, r, l$ are arbitrary constants, the four function $e, f, g, h$ satisfy the following restricted relation

$$\begin{cases} e^{'} = -qgh + rfh + lm^2 fg, f^{'} = pgh + reh + leg, g^{'} = -pfh - qeh + l(m^2-1)ef, \\ h^{'} = -m^2 pfg - r(m^2-1)ef - qeg, g^2 = e^2 - f^2, h^2 = e^2 - m^2 f^2 \end{cases} \tag{4}$$

where "$'$" denotes $\frac{d}{d\xi}$. $m$ is the modulus of the Jacobi elliptic function ($0 \le m \le 1$), and $e, f, g, h$ satisfy the following six relation:

$$\text{Family } 1: \quad \text{when } l = p = 0, \text{ we have } qf + rg = 1. \tag{5}$$
$$\text{Family } 2: \quad \text{when } l = r = 0, \text{ we have } pe + qf = 1. \tag{6}$$
$$\text{Family } 3: \quad \text{when } l = q = 0, \text{ we have } pe + rg = 1. \tag{7}$$
$$\text{Family } 4: \quad \text{when } p = r = 0, \text{ we have } lh + qf = 1. \tag{8}$$
$$\text{Family } 5: \quad \text{when } p = q = 0, \text{ we have } lh + rg = 1. \tag{9}$$
$$\text{Family } 6: \quad \text{when } q = r = 0, \text{ we have } lh + pe = 1. \tag{10}$$

Here we can select F($\xi$) = g($\xi$) in Eq. (7) and Eq. (9), F($\xi$) = f($\xi$) in Eq. (5), Eq. (6), Eq. (8) and $F(\xi) = e(\xi)$ in Eq. (10), Substituting Eq. (2) and Eq. (4) along with Eq. (5) $\sim$ Eq. (10) into Eq. (1) separately yields six families of polynomial equations for $E(\xi)$, $F(\xi)$, $G(\xi)$, $H(\xi)$. Setting the coefficients of $F^i(\xi)$, $F^i(\xi)E(\xi)$, $F^i(\xi)G(\xi)$, $F^i(\xi)H(\xi)$, $F^i(\xi)E(\xi)G(\xi)$, $F^i(\xi)E(\xi)H(\xi)$, $F^i(\xi)G(\xi)H(\xi)$ $(i = 0, 1, 2, \cdots)$ to zero yields a set of nonlinear algebraic equations (NAEs) in $a_0$, $a_i$, $b_i$, $c_i$, $d_i$ $(i = 1,\ 2,\ \cdots,\ n)$ and $\xi(x,t)$, solving the NAEs by Mathematica and Wu elimination, we can obtain many exact solutions of Eq. (1) according to Eq. (2) $\sim$ Eq. (4) and Eq. (5) $\sim$ Eq. (10).

Obviously, if we choose the special value of $p, q, r, l$ in Eq. (3), then we can get the result of [14-21]. For example, when we choose $p = 1$ in Eq. (6), Eq. (7) and Eq. (10), then we can get all the results in [14, 15], when we choose $m \to 1, r = 1$ in Eq. (5) and Eq. (7), we can get the results in [17-19], when we choose three of them to be zero, we can get the results in [16, 20-21].

*Remark 1.* The restricted equations Eq. (4), Eq. (5) $\sim$ Eq. (10) and the solutions of Eq. (3) are new, they contain the results in [7] $\sim$ [16] completely, while the form turns much simplified. Their practical value turns stronger.

*Remark 2.* Here the value of $i$ can be extended to $i = -n, \cdots, n$, so the method contain plenty of Jacobi function expansion methods[5, 9, 17, 23], noticed that $sn\xi \to \tanh\xi$, $cn\xi \to \sec h\xi$, $dn\xi \to \sec h\xi$ when the modulus $m \to 1$ and $sn\xi \to \sin\xi$, $cn\xi \to \cos\xi, dn\xi \to 1$ when the modulus $m \to 0$, we can obtain the corresponding solitary wave solutions and triangle function solutions.

In the following, we will use this method to solve the zakharov equations.

## 3 Exact solutions to the zakharov equations

We consider the following system of Zakharov Eq. (1) $\sim$ Eq. (14).

$$\begin{cases} u_{tt} - c_s^2 u_{xx} - \beta(|v|^2)_{xx} = 0 & (11a) \\ iv_t + \alpha v_{xx} - \delta uv = 0 & (11b) \end{cases} \tag{11}$$

where $u = u(x,t)$ is the perturbed number density of the ion (in the low-frequency response), $v = v(x,t)$ is the slow variation amplitude of the electric field intensity, $c_s$ is the thermal transportation velocity of the electron-ion, $\alpha \neq 0,\ \beta \neq 0,\ \delta \neq 0,\ c_s$ are constants. Eq. (11a) and Eq. (11b) are one of the fundamental models governing dynamics of nonlinear waves, and describe the interactions between high- and low- frequency waves. The physically most important example involves the interaction between Langmuir and ion-acoustic waves in plasma.

Since $v(x,t)$ is a complex function, thus we introduce a gauge transformation:

$$\begin{cases} u = u(x,t) = u(\xi) & (12a) \\ v = v(x,t) = v(\xi) = \phi(\xi)\exp[i(sx - \omega t)] & (12b) \\ \xi = k(x - ct) + \xi_0 & (12c) \end{cases} \tag{12}$$

where $\phi(\xi)$ is a real-valued function, $s, \omega, k, c$ are four real constants to be determined, and $\xi_0$ is an arbitrary constant.

Substituting (12) into (11), we have:

$$\begin{cases} k^2(c^2 - c_s^2)u'' - \beta k^2(\phi^2)'' = 0 & (13a) \\ \alpha k^2\phi'' + (\omega - \alpha s^2)\phi - \delta u\phi + i(2\alpha sk - kc)\phi' = 0 & (13b) \end{cases} \tag{13}$$

Integrating Eq. (13a) twice with respect to $u$ and put the integration constants to zero, we obtain

$$u = \frac{\beta}{c^2 - c_s^2}\phi^2 \tag{14}$$

Let $c = 2\alpha s$ and substituting (14) into (13b), we obtain:

$$\alpha k^2\phi'' + (\omega - \alpha s^2)\phi - \frac{\delta\beta}{c^2 - c_s^2}\phi^3 = 0 \tag{15}$$

By the homogenous balance principle we have $n$= 1, thus we assume that the Liènard Eq. (15) have the following solutions:

$$\phi = c_0 + c_1 e + c_2 f + c_3 g + c_4 h \tag{16}$$

where $\phi = \phi(\xi),\ e = e(\xi),\ f = f(\xi),\ g = g(\xi),\ h = h(\xi)$ and $e,\ f,\ g,\ h$ satisfy ((4) $\sim$ (10)). Substituting (4) and ((5) $\sim$ (10)) along with (12) and (16) into (15) and setting the coefficients of $F^i(\xi),\ F^i(\xi)E(\xi),\ F^i(\xi)G(\xi),\ F^i(\xi)H(\xi), F^i(\xi)E(\xi)G(\xi),\ F^i(\xi)E(\xi)H(\xi),\ F^i(\xi)G(\xi)H(\xi)$, (i = 0, 1, 2, $\cdots$) to zero yields nonlinear algebraic equations (NAEs) with respect to the unknown $k,\ \omega,\ s,\ c,\ c_i,\ (i = 0,1,2,3,4), p,\ q,\ r,\ l$.

We could determine the following solutions of Zakharov Eq. (11).

*Family 1*

*Case 1.* $\omega = (k^2(1-2m^2)+s^2)\alpha,\ c_0 = c_2 = c_3 = c_4 = 0,\ c_1 = \pm kr\sqrt{\frac{2\alpha(1-m^2)(c^2-c_s^2)}{\beta\delta}},\ q = 0$

*Case 2.* $\omega = (k^2(1-\frac{m^2}{2})+s^2)\alpha,\ c_0 = c_2 = c_3 = c_4 = 0,\ c_1 = \pm kmr\sqrt{\frac{\alpha(c_s^2-c^2)}{2\beta\delta}},\ q = \varepsilon ir$

*Case 3.* $\omega = (k^2(1+m^2)+s^2)\alpha,\ c_0 = c_1 = c_2 = c_3 = 0,\ c_4 = \pm kr\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}},\ q = 0$

*Case 4.* $\omega = (k^2(1-\frac{m^2}{2})+s^2)\alpha,\ c_0 = c_1 = c_2 = c_4 = 0,\ c_3 = \pm krm\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ q = \varepsilon ir\sqrt{1-m^2}$

*Case 5.* $\omega = (k^2(1-\frac{m^2}{2})+s^2)\alpha,\ c_0 = c_2 = c_3 = 0,\ c_1 = \pm k\sqrt{\frac{\alpha(q^2+(1-m^2)r^2)(c^2-c_s^2)}{2\beta\delta}},$
$c_4 = \varepsilon k\sqrt{\frac{\alpha(q^2+r^2)(c^2-c_s^2)}{2\beta\delta}}$

where $c = 2\alpha s,\ \varepsilon = \pm 1,\ i = \sqrt{-1},\ p,\ q,\ r,\ l,\ k,\ s$ are arbitrary constants, so do the following situations.

Therefore, from Eq. (3), Eq. (5) $\sim$ Eq. (10), Eq. (12), Eq. (14), Eq. (16) and Cases 1–5, we obtain the solutions to the Eq. (11):

$$\begin{cases} u_1 = \dfrac{2\alpha k^2(1-m^2)}{\delta} nc^2[\xi_1, m] \\ v_1 = \pm k\sqrt{\dfrac{2\alpha(1-m^2)(c^2-c_s^2)}{\beta\delta}} nc[\xi_1, m]\exp[i(sx-(k^2(1-2m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_2 = \dfrac{\alpha k^2 m^2}{2\delta}\dfrac{1}{(\varepsilon isn[\xi_2, m]+cn[\xi_2, m])^2} \\ v_2 = \pm km\sqrt{\dfrac{\alpha(c_s^2-c^2)}{2\beta\delta}}\dfrac{\exp[i(sx-(k^2(1-\frac{m^2}{2})+s^2)\alpha t)]}{\varepsilon isn[\xi_2, m]+cn[\xi_2, m]} \end{cases}$$

$$\begin{cases} u_3 = \dfrac{2\alpha k^2}{\delta} dc^2[\xi_3, m] \\ v_3 = \pm k\sqrt{\dfrac{2\alpha(c^2-c_s^2)}{\beta\delta}} dc[\xi_3, m]\exp[i(sx-(k^2(1+m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_4 = \dfrac{\alpha k^2 m^2}{2\delta}\dfrac{cn^2[\xi_4, m]}{(\varepsilon i\sqrt{1-m^2}sn[\xi_4, m]+cn[\xi_4, m])^2} \\ v_4 = \pm km\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{cn[\xi_4, m]\exp[i(sx-(k^2(1-\frac{m^2}{2})+s^2)\alpha t)]}{\varepsilon i\sqrt{1-m^2}sn[\xi_4, m]+cn[\xi_4, m]} \end{cases}$$

$$\begin{cases} u_5 = \dfrac{\alpha k^2}{2\delta}\dfrac{(\pm\sqrt{q^2+(1-m^2)r^2}+\varepsilon\sqrt{q^2+r^2}dn[\xi_5, m])^2}{(qsn[\xi_5, m]+rcn[\xi_5, m])^2} \\ v_5 = k\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{(\pm\sqrt{q^2+(1-m^2)r^2}+\varepsilon\sqrt{q^2+r^2}dn[\xi_5, m])}{qsn[\xi_5, m]+rcn[\xi_5, m]}\cdot \\ \exp[i(sx-(k^2(1-\frac{m^2}{2})+s^2)\alpha t)] \end{cases}$$

where $\xi_i = k(x-2\alpha st)+\xi_0,\ (i = 1, \cdots, 5)$.

*Family 2*

*Case 6.* $\omega = (k^2(1-2m^2)+s^2)\alpha,\ c_0 = c_1 = c_2 = c_4 = 0,\ c_3 = \pm kmp\sqrt{\frac{-2\alpha(c^2-c_s^2)}{\beta\delta}},\ q = 0$

*Case 7.* $\omega = -\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha,\ c_0 = c_1 = c_2 = c_4 = 0, c_3 = \pm kp\sqrt{\frac{\alpha(1-m^2)(c^2-c_s^2)}{2\beta\delta}},\ q = \varepsilon p$

*Case 8.* $\omega = -\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha,\ c_0 = c_1 = c_2 = 0,\ c_3 = \pm k\sqrt{\frac{\alpha(q^2-m^2p^2)(c^2-c_s^2)}{2\beta\delta}},$
$c_4 = \varepsilon k\sqrt{\frac{\alpha(q^2-p^2)(c^2-c_s^2)}{2\beta\delta}}$

*Case 9.* $\omega = (k^2(1+m^2)+s^2)\alpha,\ c_0 = c_1 = c_3 = c_4 = 0,\ c_2 = \pm kmp\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}},\ q = 0$

We can derive the following solutions:

$$\begin{cases} u_6 = \frac{-2\alpha\beta k^2m^2}{\delta}cn^2[\xi_6, m] \\ v_6 = \pm km\sqrt{\frac{-2\alpha(c^2-c_s^2)}{\beta\delta}}cn[\xi_6, m]\exp[i(sx-(k^2(1-2m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_7 = \frac{\alpha k^2(1-m^2)}{2\delta}\frac{cn^2[\xi_7, m]}{(1+\varepsilon sn[\xi_7, m])^2} \\ v_7 = \pm k\sqrt{\frac{\alpha(1-m^2)(c^2-c_s^2)}{2\beta\delta}}\frac{cn[\xi_7, m]\exp[i(sx+\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha t)]}{1+\varepsilon sn[\xi_7, m]} \end{cases}$$

$$\begin{cases} u_8 = \frac{\alpha k^2}{2\delta}\frac{(\pm k\sqrt{q^2-m^2p^2}cn[\xi_8, m]+\varepsilon\sqrt{q^2-p^2}dn[\xi_8, m])^2}{(p+qsn[\xi_8, m])^2} \\ v_8 = k\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}}\frac{(\pm k\sqrt{q^2-m^2p^2}cn[\xi_8, m]+\varepsilon\sqrt{q^2-p^2}dn[\xi_8, m])\exp[i(sx+\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha t)]}{p+qsn[\xi_8, m]} \end{cases}$$

$$\begin{cases} u_9 = \frac{2\alpha k^2m^2}{\delta}sn^2[\xi_9, m] \\ v_9 = \pm km\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}}sn[\xi_9, m]\exp[i(sx-(k^2(1+m^2)+s^2)\alpha t)] \end{cases}$$

where $\xi_i = k(x-2\alpha st)+\xi_0,\ (i = 6, \cdots, 9)$.

*Family 3*

*Case 10.* $\omega = (k^2(-\frac{1}{2}+m^2)+s^2)\alpha,\ c_0 = c_1 = c_3 = 0,\ c_2 = \pm k\sqrt{\frac{\alpha(r^2+m^2(p^2-r^2))(c^2-c_s^2)}{2\beta\delta}},$
$c_4 = \varepsilon k\sqrt{\frac{-\alpha(p^2-r^2)(c^2-c_s^2)}{2\beta\delta}}$

*Case 11.* $\omega = (k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0 = c_1 = c_3 = c_4 = 0,\ c_2 = \pm kp\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ r = \varepsilon p$

*Case 12.* $\omega = (k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0 = c_1 = c_2 = c_3 = 0,\ c_4 = \pm kp\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta(m^2-1)}},\ r = \frac{\varepsilon mp}{\sqrt{m^2-1}}$

*Case 13.* $\omega = -\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha,\ c_0 = c_1 = c_2 = 0,\ c_3 = \pm kmp\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}},$
$c_4 = \varepsilon kp\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}},\ r = 0$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{10}=\dfrac{\alpha k^2}{2\delta}\dfrac{(\pm\sqrt{r^2+m^2(p^2-r^2)}sn[\xi_{10},m]+\varepsilon\sqrt{r^2-p^2})dn[\xi_{10},m])^2}{(p+rcn[\xi_{10},m])^2} \\ v_{10}=k\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{(\pm\sqrt{r^2+m^2(p^2-r^2)}sn[\xi_{10},m]+\varepsilon\sqrt{r^2-p^2})dn[\xi_{10},m])}{p+rcn[\xi_{10},m]} \\ \exp[i(sx-(k^2(-\dfrac{1}{2}+m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{11}=\dfrac{\alpha k^2}{2\delta}\dfrac{sn^2[\xi_{11},m]}{(1+\varepsilon cn[\xi_{11},m])^2} \\ v_{11}=\pm k\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{sn[\xi_{11},m]}{1+\varepsilon cn[\xi_{11},m]}\exp[i(sx-(k^2(m^2-\frac{1}{2})+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{12}=\dfrac{\alpha k^2}{2\delta}\dfrac{dn^2[\xi_{12},m]}{(\sqrt{m^2-1}+\varepsilon mcn[\xi_{12},m])^2} \\ v_{12}=\pm k\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{dn[\xi_{12},m]}{\sqrt{m^2-1}+\varepsilon mcn[\xi_{12},m]}\exp[i(sx-(k^2(m^2-\dfrac{1}{2})+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{13}=\dfrac{-\alpha k^2}{2\delta}(\pm mcn[\xi_{13},m]+\varepsilon dn[\xi_{13},m])^2 \\ v_{13}=k\sqrt{\dfrac{-\alpha(c^2-c_s^2)}{2\beta\delta}}(\pm mcn[\xi_{13},m]+\varepsilon dn[\xi_{13},m])\exp[i(sx+\dfrac{1}{2}(k^2(1+m^2)-2s^2)\alpha t)] \end{cases}$$

where $\xi_i=k(x-2\alpha st)+\xi_0,(i=10,\cdots,13)$.

*Family 4*

*Case 14.* $\omega=(k^2(m^2-2)+s^2)\alpha, c_0=c_2=c_3=c_4=0, c_1=\pm kl\sqrt{\frac{2\alpha(m^2-1)(c^2-c_s^2)}{\beta\delta}}, q=0$

*Case 15.* $\omega=(k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0=c_2=c_3=c_4=0,\ c_1=\pm kl\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}},\ q=\varepsilon iml$

*Case 16.* $\omega=(k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0=c_1=c_2=c_4=0,\ c_3=\pm kl\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ q=\varepsilon l\sqrt{1-m^2}$

*Case 17.* $\omega=(k^2(1-\frac{1}{2}m^2)+s^2)\alpha,\ c_0=c_1=c_3=0,\ c_2=\pm kml\sqrt{\frac{\alpha(m^2-1)(c^2-c_s^2)}{2\beta\delta}},$
$c_3=\varepsilon kml\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ q=0$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{14}=\dfrac{2\alpha k^2(m^2-1)}{\delta}nd^2[\xi_{14},m] \\ v_{14}=\pm k\sqrt{\dfrac{2\alpha(m^2-1)(c^2-c_s^2)}{\beta\delta}}nd[\xi_{14},m]\exp[i(sx-(k^2(m^2-2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{15}=\dfrac{-\alpha k^2}{2\delta}\dfrac{1}{(\varepsilon imsn[\xi_{15},m]+dn[\xi_{15},m])^2} \\ v_{15}=\pm k\sqrt{\dfrac{-\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{1}{\varepsilon imsn[\xi_{15},m]+dn[\xi_{15},m]}\exp[i(sx-(k^2(m^2-\frac{1}{2})+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{16} = \dfrac{\beta}{c^2 - c_s^2} \pm k\sqrt{\dfrac{\alpha k^2 (c^2 - c_s^2)}{2\beta\delta}} \dfrac{cn^2[\xi_{16}, m]}{(\varepsilon\sqrt{1-m^2} sn[\xi_{16}, m] + dn[\xi_{16}, m])^2} \\ v_{16} = \pm k\sqrt{\dfrac{\alpha(c^2 - c_s^2)}{2\beta\delta}} \dfrac{cn[\xi_{16}, m] \exp[i(sx - (k^2(m^2 - \frac{1}{2}) + s^2)\alpha t)]}{\varepsilon\sqrt{1-m^2} sn[\xi_{16}, m] + dn[\xi_{16}, m]} \end{cases}$$

$$\begin{cases} u_{17} = \dfrac{\alpha k^2 m^2}{2\delta}(\pm\sqrt{m^2-1} sd[\xi_{17}, m] + \varepsilon cd[\xi_{17}, m])^2 \\ v_{17} = km\sqrt{\dfrac{\alpha(c^2 - c_s^2)}{2\beta\delta}}(\pm\sqrt{m^2-1} sd[\xi_{17}, m] + \varepsilon cd[\xi_{17}, m]) \exp[i(sx - (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha t)] \end{cases}$$

where $\xi_i = k(x - 2\alpha st) + \xi_0$, $(i = 14, \cdots, 17)$.

*Family 5*

*Case 18.* $\omega = -\frac{1}{2}(k^2(m^2+1) - 2s^2)\alpha$, $c_0 = c_2 = c_3 = c_4 = 0$, $c_1 = \pm kl(1-m^2)\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}}$, $r = \varepsilon ml$

*Case 19.* $\omega = (k^2(1-2m^2) + s^2)\alpha$, $c_0 = c_1 = c_3 = c_4 = 0$, $c_2 = \pm kml\sqrt{\frac{2\alpha(m^2-1)(c^2-c_s^2)}{\beta\delta}}$, $r = 0$

*Case 20.* $\omega = -\frac{1}{2}(k^2(m^2+1) - 2s^2)\alpha$, $c_0 = c_1 = c_3 = c_4 = 0$, $c_2 = \pm kl(1-m^2)\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}}$, $r = \varepsilon l$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{18} = \dfrac{-\alpha k^2(1-m^2)^2}{2\delta} \dfrac{1}{(\varepsilon m cn[\xi_{18}, m] + dn[\xi_{18}, m])^2} \\ v_{18} = \pm k(1-m^2)\sqrt{\dfrac{-\alpha(c^2 - c_s^2)}{2\beta\delta}} \dfrac{\exp[i(sx + \frac{1}{2}(k^2(m^2+1) - 2s^2)\alpha t)]}{\varepsilon m cn[\xi_{18}, m] + dn[\xi_{18}, m]} \end{cases}$$

$$\begin{cases} u_{19} = \dfrac{2\alpha k^2 m^2(m^2-1)}{\delta} sd^2[\xi_{19}, m] \\ v_{19} = \pm km\sqrt{\dfrac{2\alpha(m^2-1)(c^2 - c_s^2)}{\beta\delta}} sd[\xi_{19}, m] \exp[i(sx - (k^2(1-2m^2) + s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{20} = \dfrac{\alpha k^2(1-m^2)^2}{2\delta} \dfrac{sn^2[\xi_{20}, m]}{(\varepsilon cn[\xi_{20}, m] + dn[\xi_{20}, m])^2} \\ v_{20} = \pm k(1-m^2)\sqrt{\dfrac{\alpha(c^2 - c_s^2)}{2\beta\delta}} \dfrac{sn[\xi_{20}, m] \exp[i(sx + \frac{1}{2}(k^2(m^2+1) - 2s^2)\alpha t)]}{\varepsilon cn[\xi_{20}, m] + dn[\xi_{20}, m]} \end{cases}$$

where $\xi_i = k(x - 2\alpha st) + \xi_0$, $(i = 18, \cdots, 20)$.

*Family 6*

*Case 21.* $\omega = (k^2(m^2-2) + s^2)\alpha$, $c_0 = c_1 = c_2 = c_3 = 0$, $c_4 = \pm kp\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}}$, $l = 0$

*Case 22.* $\omega = (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha$, $c_0 = c_1 = c_2 = c_4 = 0$, $c_3 = \pm km^2 p\sqrt{\frac{\alpha(c^2-c_s^2)}{2(1-m^2)\beta\delta}}$, $l = \frac{\varepsilon p}{\sqrt{1-m^2}}$

*Case 23.* $\omega = (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha$, $c_0 = c_1 = c_3 = c_4 = 0$, $c_2 = \pm km^2 p\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}}$, $l = \varepsilon p$

*Case 24.* $\omega = (k^2(1-\frac{1}{2}m^2)+s^2)\alpha,\ c_0 = c_1 = c_4 = 0,\ c_2 = km\sqrt{\frac{\alpha(p^2+(m^2-1)l^2)(c^2-c_s^2)}{2\beta\delta}},$
$c_3 = \varepsilon km\sqrt{\frac{-\alpha(p^2-l^2)(c^2-c_s^2)}{2\beta\delta}}$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{21} = \frac{2\alpha k^2}{\delta}dn^2[\xi_{21},m] \\ v_{21} = \pm k\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}}dn[\xi_{21},m]\exp[i(sx-(k^2(m^2-2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{22} = \frac{\alpha k^2m^4}{2\delta}\frac{dn^2[\xi_{22},m]}{(\sqrt{1-m^2}+\varepsilon dn[\xi_{22},m])^2} \\ v_{22} = \pm km^2\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}}\frac{dn[\xi_{22},m]\exp[i(sx-(k^2(1-\frac{1}{2}m^2)+s^2)\alpha t)]}{\sqrt{1-m^2}+\varepsilon dn[\xi_{22},m]} \end{cases}$$

$$\begin{cases} u_{23} = \frac{\alpha k^2m^4}{2\delta}\frac{sn^2[\xi_{23},m]}{(1+\varepsilon dn[\xi_{23},m])^2} \\ v_{23} = \pm km^2\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}}\frac{sn[\xi_{23},m]}{1+\varepsilon dn[\xi_{23},m]}\exp[i(sx-(k^2(1-\frac{1}{2}m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{24} = \frac{\alpha k^2m^2}{2\delta}\frac{(\sqrt{p^2+(m^2-1)l^2}sn[\xi_{24},m]+\varepsilon\sqrt{l^2-p^2}cn[\xi_{24},m])^2}{(p+ldn[\xi_{24},m])^2} \\ v_{24} = km\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}}\frac{(\sqrt{p^2+(m^2-1)l^2}sn[\xi_{24},m]+\varepsilon\sqrt{l^2-p^2}cn[\xi_{24},m])}{p+ldn[\xi_{24},m]}\cdot \\ \exp\left[i\left(sx-(k^2(1-\frac{1}{2}m^2)+s^2)\alpha t\right)\right] \end{cases}$$

where $\xi_i = k(x-2\alpha st)+\xi_0,\ (i=21,\cdots,24)$.

*Remark 3.* Here $u_7, v_7$; $u_{11}, v_{11}$; $u_{12}, v_{12}$; $u_{23}, v_{23}$ contain the result of (13) (27); (18) (32); (19); (23) (37) in Ref. [16], if we let $q = \pm mp$; $q = r,\ p = 1$ or $p = r,\ q = 1$ in $u_8,\ v_8$, we can get the result of (14)(15)(29) in Ref. [16], if we let $p = r = 1$ or $r = 1, p = r$ in $u_{10},\ v_{10}$, we can get the result of (32) (34) in Ref. [16], if we let $p = 1,\ l = r$ or $p = r, l = 1$ in $u_{24}, v_{24}$ we can get the result of (25) (39) in Ref. [16]. $u_1, v_1$; $u_{14}, v_{14}$; $u_{19}, v_{19}$ contain the result of $u_{12}, v_{12}$; $u_{18}, v_{18}$; $u_{19}, v_{19}$; in Ref. [17], if we let $l = 0$ in $u_{24}, v_{24}$ and $r = 0$ in $u_{10},\ v_{10}$ we can get the result of (15-18) in Ref. [17].

*Remark 4.* It is notable that the other types of solutions we obtained here to systems (11) are not shown in the previous literature. They are degenerated to the corresponding solitary wave solutions and triangle function solutions in the limit cases when $m \to 1$ or $m \to 0$.

## 4 Conclusion

In this paper, we succeed to propose an approach for finding new exact solutions for nonlinear evolution equations by constructing the four new types of Jacobian elliptic functions. By using this method and computerized symbolic computation, we have found thirteen new types of exact solutions for the zakharov Eq. (11). More importantly, our method is much simple and powerful to find new solutions to various kinds of nonlinear evolution equations, such as KdV equation, mKdV equation, Boussinesq equation, etc. We believe that this method should play an important role for finding exact solutions in the mathematical physics.

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