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The extended Jacobi Elliptic Functions expansion method and new exact solutions for the Zakharov equations *

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Abstract. We extended the Jacobi elliptic function expansion method by constructing four new Jacobian elliptic functions, and apply this method to Zakharov equations for illustration, abundant new doubly periodic solutions are obtained, these solutions are degenerated to the solitary wave solutions and the triangle function solutions in the limit cases when the modulus of the Jacobian elliptic functions m→1 or 0, which shows that the new method is more powerful to seek the exact solutions of the nonlinear partial differential equations in mathematical physics.

Keywords: Zakharov equations, extended Jacobian Elliptic Functions expansion method, doubly periodic solutions, exact solutions

1 Introduction

In the interaction of laser-plasma the system of Zakharov equation (SZE) plays an important role[18, 19]. This system attracted many scientists' wide interest and attention. In one dimension, the formation, evolution and interaction of the Langmuir solution differ from solutions of the KdV equation. In multi-dimensions, the Langmuir solution will collapse. Since 1980s, the effects including magnetic field have been considered, and the system of Zakharov equation includes more general form and rich contents. For example, SZE with Landau damping effect was given in [8, 14]. Under some conditions its inverse scattering transformation has been found. In [8], the longitudinal and transverse oscillating and magnetic field effect was examined, the solution properties and collapse in multi-dimensions have been revealed.

More recently, some authors considered the exact and explicit solutions of the system of Zakharov equations by different methods in [3, 4, 10, 15, 16, 20–22, 24, 25]. In this paper, we consider the system of Zakharov equations by constructing four new types of Jacobian elliptic functions, and abundant new families of exact solutions are obtained.

This paper is arranged as follows. In section 2, we briefly describe the new extended Jacobi elliptic functions expansion method. In section 3, several families of solutions to the Zakharov equations are obtained, which are degenerated to new solitary wave solutions and new periodic wave solutions in the limit case. In section 4, some conclusions are given.

2 Summary of the extended jacobi elliptic functions expansion method

In this section, we propose a general method, namely, the extended Jacobi elliptic functions expansion method, given nonlinear partial differential equation, for instance, in two variables x and t, as follows:

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$$P(u, u_t, u_x, u_{xx}, \cdots) = 0 \tag{1}$$

We seek the following formal solutions of the given system by a new intermediate transformation:

$$u(\xi) = \sum_{i=1}^{n} a_i F^i(\xi) + \sum_{i=1}^{n} b_i F^{i-1}(\xi)E(\xi) + \sum_{i=1}^{n} c_i F^{i-1}(\xi)G(\xi) + \sum_{i=1}^{n} d_i F^{i-1}(\xi)H(\xi) + a_0 \tag{2}$$

where $a_0,\, a_i,\, b_i,\, c_i, d_i\ \ (i = 1, 2, \cdots, n)$ are constants to be determined later. $\xi = \xi(x, t)$ are arbitrary functions with the variables x and t, The parameter n can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq. (1). And $E(\xi),\, F(\xi),\, G(\xi),\, H(\xi)$ are an arbitrary array of the four function $e = e(\xi),\ f = f(\xi),\ g = g(\xi)$ and $h = h(\xi)$, the selection obey the principle which makes the calculation more simple. Here we get

$$\begin{cases} e = \dfrac{1}{p + qsn[\xi, m] + rcn[\xi, m] + ldn[\xi, m]},\ f = \dfrac{sn[\xi, m]}{p + qsn[\xi, m] + rcn[\xi, m] + ldn[\xi, m]} \\ g = \dfrac{cn[\xi, m]}{p + qsn[\xi, m] + rcn[\xi, m] + ldn[\xi, m]},\ h = \dfrac{dn[\xi, m]}{p + qsn[\xi, m] + rcn[\xi, m] + ldn[\xi, m]} \end{cases} \tag{3}$$

where p, q, r, l are arbitrary constants, the four function e, f, g, h satisfy the following restricted relation

$$\begin{cases} e' = -qgh + rfh + lm^2 fg, f' = pgh + reh + leg, g' = -pfh - qeh + l(m^2 - 1)ef, \\ h' = -m^2 pfg - r(m^2 - 1)ef - qeg, g^2 = e^2 - f^2, h^2 = e^2 - m^2 f^2 \end{cases} \tag{4}$$

where "$'$" denotes $\dfrac{d}{d\xi}$. m is the modulus of the Jacobi elliptic function ($0 \le m \le 1$), and e, f, g, h satisfy the following six relation:

$$\text{Family } 1: \quad \text{when } l = p = 0, \text{ we have } qf + rg = 1. \tag{5}$$
$$\text{Family } 2: \quad \text{when } l = r = 0, \text{ we have } pe + qf = 1. \tag{6}$$
$$\text{Family } 3: \quad \text{when } l = q = 0, \text{ we have } pe + rg = 1. \tag{7}$$
$$\text{Family } 4: \quad \text{when } p = r = 0, \text{ we have } lh + qf = 1. \tag{8}$$
$$\text{Family } 5: \quad \text{when } p = q = 0, \text{ we have } lh + rg = 1. \tag{9}$$
$$\text{Family } 6: \quad \text{when } q = r = 0, \text{ we have } lh + pe = 1. \tag{10}$$

Here we can select F(ξ) = g(ξ) in Eq. (7) and Eq. (9), F(ξ) = f(ξ) in Eq. (5), Eq. (6), Eq. (8) and $F(\xi) = e(\xi)$ in Eq. (10), Substituting Eq. (2) and Eq. (4) along with Eq. (5) $\sim$ Eq. (10) into Eq. (1) separately yields six families of polynomial equations for $E(\xi),\, F(\xi),\, G(\xi),\, H(\xi)$. Setting the coefficients of $F^i(\xi),\, F^i(\xi)E(\xi)$, $F^i(\xi)G(\xi),\, F^i(\xi)H(\xi),\, F^i(\xi)E(\xi)G(\xi),\, F^i(\xi)E(\xi)H(\xi), F^i(\xi)G(\xi)H(\xi)\ (i = 0, 1, 2, \cdots)$ to zero yields a set of nonlinear algebraic equations (NAEs) in $a_0,\, a_i,\, b_i,\, c_i,\, d_i\ (i = 1,\, 2,\, \cdots,\, n)$ and $\xi(x, t)$, solving the NAEs by Mathematica and Wu elimination, we can obtain many exact solutions of Eq. (1) according to Eq. (2) $\sim$ Eq. (4) and Eq. (5) $\sim$ Eq. (10).

Obviously, if we choose the special value of p, q, r, l in Eq. (3), then we can get the result of [14-21]. For example, when we choose $p = 1$ in Eq. (6), Eq. (7) and Eq. (10), then we can get all the results in [14, 15], when we choose $m \to 1, r = 1$ in Eq. (5) and Eq. (7), we can get the results in [17-19], when we choose three of them to be zero, we can get the results in [16, 20-21].

Remark 1. The restricted equations Eq. (4), Eq. (5) $\sim$ Eq. (10) and the solutions of Eq. (3) are new, they contain the results in [7] $\sim$ [16] completely, while the form turns much simplified. Their practical value turns stronger.

Remark 2. Here the value of i can be extended to $i = -n, \cdots, n$, so the method contain plenty of Jacobi function expansion methods[5, 9, 17, 23], noticed that $sn\xi \to \tanh\xi,\ cn\xi \to \sec h\xi,\ dn\xi \to \sec h\xi$ when the modulus $m \to 1$ and $sn\xi \to \sin\xi,\ cn\xi \to \cos\xi, dn\xi \to 1$ when the modulus $m \to 0$, we can obtain the corresponding solitary wave solutions and triangle function solutions.

In the following, we will use this method to solve the zakharov equations.

3 Exact solutions to the zakharov equations

We consider the following system of Zakharov Eq. (1) $\sim$ Eq. (14).

$$\begin{cases} u_{tt} - c_s^2 u_{xx} - \beta(|v|^2)_{xx} = 0 & (11a) \\ iv_t + \alpha v_{xx} - \delta uv = 0 & (11b) \end{cases} \tag{11}$$

where $u = u(x,t)$ is the perturbed number density of the ion (in the low-frequency response), $v = v(x,t)$ is the slow variation amplitude of the electric field intensity, c_s is the thermal transportation velocity of the electron-ion, $\alpha \neq 0,\ \beta \neq 0,\ \delta \neq 0,\ c_s$ are constants. Eq. (11a) and Eq. (11b) are one of the fundamental models governing dynamics of nonlinear waves, and describe the interactions between high- and low- frequency waves. The physically most important example involves the interaction between Langmuir and ion-acoustic waves in plasma.

Since $v(x,t)$ is a complex function, thus we introduce a gauge transformation:

$$\begin{cases} u = u(x,t) = u(\xi) & (12a) \\ v = v(x,t) = v(\xi) = \phi(\xi)\exp[i(sx - \omega t)] & (12b) \\ \xi = k(x - ct) + \xi_0 & (12c) \end{cases} \tag{12}$$

where $\phi(\xi)$ is a real-valued function, s, ω, k, c are four real constants to be determined, and ξ_0 is an arbitrary constant.

Substituting (12) into (11), we have:

$$\begin{cases} k^2(c^2 - c_s^2)u'' - \beta k^2(\phi^2)'' = 0 & (13a) \\ \alpha k^2\phi'' + (\omega - \alpha s^2)\phi - \delta u\phi + i(2\alpha sk - kc)\phi' = 0 & (13b) \end{cases} \tag{13}$$

Integrating Eq. (13a) twice with respect to u and put the integration constants to zero, we obtain

$$u = \frac{\beta}{c^2 - c_s^2}\phi^2 \tag{14}$$

Let $c = 2\alpha s$ and substituting (14) into (13b), we obtain:

$$\alpha k^2\phi'' + (\omega - \alpha s^2)\phi - \frac{\delta\beta}{c^2 - c_s^2}\phi^3 = 0 \tag{15}$$

By the homogenous balance principle we have n= 1, thus we assume that the Liènard Eq. (15) have the following solutions:

$$\phi = c_0 + c_1 e + c_2 f + c_3 g + c_4 h \tag{16}$$

where $\phi = \phi(\xi),\ e = e(\xi),\ f = f(\xi),\ g = g(\xi),\ h = h(\xi)$ and $e,\ f,\ g,\ h$ satisfy ((4) $\sim$ (10)). Substituting (4) and ((5) $\sim$ (10)) along with (12) and (16) into (15) and setting the coefficients of $F^i(\xi),\ F^i(\xi)E(\xi),\ F^i(\xi)G(\xi),\ F^i(\xi)H(\xi), F^i(\xi)E(\xi)G(\xi),\ F^i(\xi)E(\xi)H(\xi),\ F^i(\xi)G(\xi)H(\xi)$, (i = 0, 1, 2, $\cdots$) to zero yields nonlinear algebraic equations (NAEs) with respect to the unknown $k,\ \omega,\ s,\ c,\ c_i,\ (i = 0, 1, 2, 3, 4), p,\ q,\ r,\ l$.

We could determine the following solutions of Zakharov Eq. (11).

Family 1

Case 1. $\omega = (k^2(1 - 2m^2) + s^2)\alpha,\ c_0 = c_2 = c_3 = c_4 = 0,\ c_1 = \pm kr\sqrt{\frac{2\alpha(1-m^2)(c^2-c_s^2)}{\beta\delta}},\ q = 0$

Case 2. $\omega = (k^2(1 - \frac{m^2}{2}) + s^2)\alpha,\ c_0 = c_2 = c_3 = c_4 = 0,\ c_1 = \pm kmr\sqrt{\frac{\alpha(c_s^2-c^2)}{2\beta\delta}},\ q = \varepsilon ir$

Case 3. $\omega = (k^2(1+m^2)+s^2)\alpha,\ c_0 = c_1 = c_2 = c_3 = 0,\ c_4 = \pm kr\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}},\ q = 0$

Case 4. $\omega = (k^2(1-\frac{m^2}{2})+s^2)\alpha,\ c_0 = c_1 = c_2 = c_4 = 0,\ c_3 = \pm krm\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ q = \varepsilon ir\sqrt{1-m^2}$

Case 5.
$$\omega = (k^2(1-\frac{m^2}{2})+s^2)\alpha,\ c_0 = c_2 = c_3 = 0,\ c_1 = \pm k\sqrt{\frac{\alpha(q^2+(1-m^2)r^2)(c^2-c_s^2)}{2\beta\delta}},$$
$$c_4 = \varepsilon k\sqrt{\frac{\alpha(q^2+r^2)(c^2-c_s^2)}{2\beta\delta}}$$

where $c = 2\alpha s,\ \varepsilon = \pm 1,\ i = \sqrt{-1},\ p,\ q,\ r,\ l,\ k,\ s$ are arbitrary constants, so do the following situations.

Therefore, from Eq. (3), Eq. (5) $\sim$ Eq. (10), Eq. (12), Eq. (14), Eq. (16) and Cases 1–5, we obtain the solutions to the Eq. (11):

$$\begin{cases} u_1 = \dfrac{2\alpha k^2(1-m^2)}{\delta} nc^2[\xi_1, m] \\ v_1 = \pm k\sqrt{\dfrac{2\alpha(1-m^2)(c^2-c_s^2)}{\beta\delta}} nc[\xi_1, m]\exp[i(sx-(k^2(1-2m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_2 = \dfrac{\alpha k^2 m^2}{2\delta}\dfrac{1}{(\varepsilon i sn[\xi_2, m]+cn[\xi_2, m])^2} \\ v_2 = \pm km\sqrt{\dfrac{\alpha(c_s^2-c^2)}{2\beta\delta}}\dfrac{\exp[i(sx-(k^2(1-\frac{m^2}{2})+s^2)\alpha t)]}{\varepsilon i sn[\xi_2, m]+cn[\xi_2, m]} \end{cases}$$

$$\begin{cases} u_3 = \dfrac{2\alpha k^2}{\delta} dc^2[\xi_3, m] \\ v_3 = \pm k\sqrt{\dfrac{2\alpha(c^2-c_s^2)}{\beta\delta}} dc[\xi_3, m]\exp[i(sx-(k^2(1+m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_4 = \dfrac{\alpha k^2 m^2}{2\delta}\dfrac{cn^2[\xi_4, m]}{(\varepsilon i\sqrt{1-m^2}sn[\xi_4, m]+cn[\xi_4, m])^2} \\ v_4 = \pm km\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{cn[\xi_4, m]\exp[i(sx-(k^2(1-\frac{m^2}{2})+s^2)\alpha t)]}{\varepsilon i\sqrt{1-m^2}sn[\xi_4, m]+cn[\xi_4, m]} \end{cases}$$

$$\begin{cases} u_5 = \dfrac{\alpha k^2}{2\delta}\dfrac{(\pm\sqrt{q^2+(1-m^2)r^2}+\varepsilon\sqrt{q^2+r^2}dn[\xi_5, m])^2}{(q sn[\xi_5, m]+r cn[\xi_5, m])^2} \\ v_5 = k\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{(\pm\sqrt{q^2+(1-m^2)r^2}+\varepsilon\sqrt{q^2+r^2}dn[\xi_5, m])}{q sn[\xi_5, m]+r cn[\xi_5, m]}\cdot \\ \exp[i(sx-(k^2(1-\frac{m^2}{2})+s^2)\alpha t)] \end{cases}$$

where $\xi_i = k(x-2\alpha st)+\xi_0,\ (i = 1, \cdots, 5)$.

Family 2

Case 6. $\omega = (k^2(1-2m^2)+s^2)\alpha,\ c_0 = c_1 = c_2 = c_4 = 0,\ c_3 = \pm kmp\sqrt{\frac{-2\alpha(c^2-c_s^2)}{\beta\delta}},\ q = 0$

Case 7. $\omega = -\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha,\ c_0 = c_1 = c_2 = c_4 = 0, c_3 = \pm kp\sqrt{\frac{\alpha(1-m^2)(c^2-c_s^2)}{2\beta\delta}},\ q = \varepsilon p$

Case 8. $\omega = -\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha,\ c_0=c_1=c_2=0,\ c_3=\pm k\sqrt{\frac{\alpha(q^2-m^2p^2)(c^2-c_s^2)}{2\beta\delta}},$
$c_4=\varepsilon k\sqrt{\frac{\alpha(q^2-p^2)(c^2-c_s^2)}{2\beta\delta}}$

Case 9. $\omega=(k^2(1+m^2)+s^2)\alpha,\ c_0=c_1=c_3=c_4=0,\ c_2=\pm kmp\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}},\ q=0$

We can derive the following solutions:

$$\begin{cases} u_6=\dfrac{-2\alpha\beta k^2m^2}{\delta}cn^2[\xi_6,m] \\ v_6=\pm km\sqrt{\dfrac{-2\alpha(c^2-c_s^2)}{\beta\delta}}cn[\xi_6,m]\exp[i(sx-(k^2(1-2m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_7=\dfrac{\alpha k^2(1-m^2)}{2\delta}\dfrac{cn^2[\xi_7,m]}{(1+\varepsilon sn[\xi_7,m])^2} \\ v_7=\pm k\sqrt{\dfrac{\alpha(1-m^2)(c^2-c_s^2)}{2\beta\delta}}\dfrac{cn[\xi_7,m]\exp[i(sx+\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha t)]}{1+\varepsilon sn[\xi_7,m]} \end{cases}$$

$$\begin{cases} u_8=\dfrac{\alpha k^2}{2\delta}\dfrac{(\pm k\sqrt{q^2-m^2p^2}cn[\xi_8,m]+\varepsilon\sqrt{q^2-p^2}dn[\xi_8,m])^2}{(p+qsn[\xi_8,m])^2} \\ v_8=k\sqrt{\dfrac{\alpha(c^2-c_s^2)}{2\beta\delta}}\dfrac{(\pm k\sqrt{q^2-m^2p^2}cn[\xi_8,m]+\varepsilon\sqrt{q^2-p^2}dn[\xi_8,m])\exp[i(sx+\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha t)]}{p+qsn[\xi_8,m]} \end{cases}$$

$$\begin{cases} u_9=\dfrac{2\alpha k^2m^2}{\delta}sn^2[\xi_9,m] \\ v_9=\pm km\sqrt{\dfrac{2\alpha(c^2-c_s^2)}{\beta\delta}}sn[\xi_9,m]\exp[i(sx-(k^2(1+m^2)+s^2)\alpha t)] \end{cases}$$

where $\xi_i=k(x-2\alpha st)+\xi_0,\ (i=6,\cdots,9)$.

Family 3

Case 10. $\omega=(k^2(-\frac{1}{2}+m^2)+s^2)\alpha,\ c_0=c_1=c_3=0,\ c_2=\pm k\sqrt{\frac{\alpha(r^2+m^2(p^2-r^2))(c^2-c_s^2)}{2\beta\delta}},$
$c_4=\varepsilon k\sqrt{\frac{-\alpha(p^2-r^2)(c^2-c_s^2)}{2\beta\delta}}$

Case 11. $\omega=(k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0=c_1=c_3=c_4=0,\ c_2=\pm kp\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ r=\varepsilon p$

Case 12. $\omega=(k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0=c_1=c_2=c_3=0,\ c_4=\pm kp\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta(m^2-1)}},\ r=\frac{\varepsilon mp}{\sqrt{m^2-1}}$

Case 13. $\omega=-\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha,\ c_0=c_1=c_2=0,\ c_3=\pm kmp\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}},$
$c_4=\varepsilon kp\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}},\ r=0$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{10} = \frac{\alpha k^2}{2\delta} \frac{(\pm\sqrt{r^2+m^2(p^2-r^2)}sn[\xi_{10},m] + \varepsilon\sqrt{r^2-p^2})dn[\xi_{10},m])^2}{(p+rcn[\xi_{10},m])^2} \\ v_{10} = k\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}} \frac{(\pm\sqrt{r^2+m^2(p^2-r^2)}sn[\xi_{10},m] + \varepsilon\sqrt{r^2-p^2})dn[\xi_{10},m])}{p+rcn[\xi_{10},m]} \\ \exp[i(sx-(k^2(-\frac{1}{2}+m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{11} = \frac{\alpha k^2}{2\delta} \frac{sn^2[\xi_{11},m]}{(1+\varepsilon cn[\xi_{11},m])^2} \\ v_{11} = \pm k\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}} \frac{sn[\xi_{11},m]}{1+\varepsilon cn[\xi_{11},m]} \exp[i(sx-(k^2(m^2-\frac{1}{2})+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{12} = \frac{\alpha k^2}{2\delta} \frac{dn^2[\xi_{12},m]}{(\sqrt{m^2-1}+\varepsilon mcn[\xi_{12},m])^2} \\ v_{12} = \pm k\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}} \frac{dn[\xi_{12},m]}{\sqrt{m^2-1}+\varepsilon mcn[\xi_{12},m]} \exp[i(sx-(k^2(m^2-\frac{1}{2})+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{13} = \frac{-\alpha k^2}{2\delta}(\pm mcn[\xi_{13},m]+\varepsilon dn[\xi_{13},m])^2 \\ v_{13} = k\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}}(\pm mcn[\xi_{13},m]+\varepsilon dn[\xi_{13},m]) \exp[i(sx+\frac{1}{2}(k^2(1+m^2)-2s^2)\alpha t)] \end{cases}$$

where $\xi_i = k(x-2\alpha st)+\xi_0, (i=10,\cdots,13)$.

Family 4

Case 14. $\omega = (k^2(m^2-2)+s^2)\alpha, c_0=c_2=c_3=c_4=0, c_1=\pm kl\sqrt{\frac{2\alpha(m^2-1)(c^2-c_s^2)}{\beta\delta}}, q=0$

Case 15. $\omega = (k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0=c_2=c_3=c_4=0,\ c_1=\pm kl\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}},\ q=\varepsilon iml$

Case 16. $\omega = (k^2(m^2-\frac{1}{2})+s^2)\alpha,\ c_0=c_1=c_2=c_4=0,\ c_3=\pm kl\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ q=\varepsilon l\sqrt{1-m^2}$

Case 17. $\omega = (k^2(1-\frac{1}{2}m^2)+s^2)\alpha,\ c_0=c_1=c_3=0,\ c_2=\pm kml\sqrt{\frac{\alpha(m^2-1)(c^2-c_s^2)}{2\beta\delta}},$
$c_3=\varepsilon kml\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}},\ q=0$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{14} = \frac{2\alpha k^2(m^2-1)}{\delta} nd^2[\xi_{14},m] \\ v_{14} = \pm k\sqrt{\frac{2\alpha(m^2-1)(c^2-c_s^2)}{\beta\delta}} nd[\xi_{14},m] \exp[i(sx-(k^2(m^2-2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{15} = \frac{-\alpha k^2}{2\delta} \frac{1}{(\varepsilon imsn[\xi_{15},m]+dn[\xi_{15},m])^2} \\ v_{15} = \pm k\sqrt{\frac{-\alpha(c^2-c_s^2)}{2\beta\delta}} \frac{1}{\varepsilon imsn[\xi_{15},m]+dn[\xi_{15},m]} \exp[i(sx-(k^2(m^2-\frac{1}{2})+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{16} = \dfrac{\beta}{c^2 - c_s^2} \pm k\sqrt{\dfrac{\alpha k^2(c^2 - c_s^2)}{2\beta\delta}} \dfrac{cn^2[\xi_{16}, m]}{(\varepsilon\sqrt{1 - m^2}sn[\xi_{16}, m] + dn[\xi_{16}, m])^2} \\ v_{16} = \pm k\sqrt{\dfrac{\alpha(c^2 - c_s^2)}{2\beta\delta}} \dfrac{cn[\xi_{16}, m]\exp[i(sx - (k^2(m^2 - \frac{1}{2}) + s^2)\alpha t)]}{\varepsilon\sqrt{1 - m^2}sn[\xi_{16}, m] + dn[\xi_{16}, m]} \end{cases}$$

$$\begin{cases} u_{17} = \dfrac{\alpha k^2 m^2}{2\delta}(\pm\sqrt{m^2 - 1}sd[\xi_{17}, m] + \varepsilon cd[\xi_{17}, m])^2 \\ v_{17} = km\sqrt{\dfrac{\alpha(c^2 - c_s^2)}{2\beta\delta}}(\pm\sqrt{m^2 - 1}sd[\xi_{17}, m] + \varepsilon cd[\xi_{17}, m])\exp[i(sx - (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha t)] \end{cases}$$

where $\xi_i = k(x - 2\alpha st) + \xi_0$, $(i = 14, \cdots, 17)$.

Family 5

Case 18. $\omega = -\frac{1}{2}(k^2(m^2 + 1) - 2s^2)\alpha$, $c_0 = c_2 = c_3 = c_4 = 0$, $c_1 = \pm kl(1 - m^2)\sqrt{\frac{-\alpha(c^2 - c_s^2)}{2\beta\delta}}$, $r = \varepsilon ml$

Case 19. $\omega = (k^2(1 - 2m^2) + s^2)\alpha$, $c_0 = c_1 = c_3 = c_4 = 0$, $c_2 = \pm kml\sqrt{\frac{2\alpha(m^2 - 1)(c^2 - c_s^2)}{\beta\delta}}$, $r = 0$

Case 20. $\omega = -\frac{1}{2}(k^2(m^2 + 1) - 2s^2)\alpha$, $c_0 = c_1 = c_3 = c_4 = 0$, $c_2 = \pm kl(1 - m^2)\sqrt{\frac{\alpha(c^2 - c_s^2)}{2\beta\delta}}$, $r = \varepsilon l$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{18} = \dfrac{-\alpha k^2(1 - m^2)^2}{2\delta} \dfrac{1}{(\varepsilon mcn[\xi_{18}, m] + dn[\xi_{18}, m])^2} \\ v_{18} = \pm k(1 - m^2)\sqrt{\dfrac{-\alpha(c^2 - c_s^2)}{2\beta\delta}} \dfrac{\exp[i(sx + \frac{1}{2}(k^2(m^2 + 1) - 2s^2)\alpha t)]}{\varepsilon mcn[\xi_{18}, m] + dn[\xi_{18}, m]} \end{cases}$$

$$\begin{cases} u_{19} = \dfrac{2\alpha k^2 m^2(m^2 - 1)}{\delta}sd^2[\xi_{19}, m] \\ v_{19} = \pm km\sqrt{\dfrac{2\alpha(m^2 - 1)(c^2 - c_s^2)}{\beta\delta}}sd[\xi_{19}, m]\exp[i(sx - (k^2(1 - 2m^2) + s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{20} = \dfrac{\alpha k^2(1 - m^2)^2}{2\delta} \dfrac{sn^2[\xi_{20}, m]}{(\varepsilon cn[\xi_{20}, m] + dn[\xi_{20}, m])^2} \\ v_{20} = \pm k(1 - m^2)\sqrt{\dfrac{\alpha(c^2 - c_s^2)}{2\beta\delta}} \dfrac{sn[\xi_{20}, m]\exp[i(sx + \frac{1}{2}(k^2(m^2 + 1) - 2s^2)\alpha t)]}{\varepsilon cn[\xi_{20}, m] + dn[\xi_{20}, m]} \end{cases}$$

where $\xi_i = k(x - 2\alpha st) + \xi_0$, $(i = 18, \cdots, 20)$.

Family 6

Case 21. $\omega = (k^2(m^2 - 2) + s^2)\alpha$, $c_0 = c_1 = c_2 = c_3 = 0$, $c_4 = \pm kp\sqrt{\frac{2\alpha(c^2 - c_s^2)}{\beta\delta}}$, $l = 0$

Case 22. $\omega = (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha$, $c_0 = c_1 = c_2 = c_4 = 0$, $c_3 = \pm km^2p\sqrt{\frac{\alpha(c^2 - c_s^2)}{2(1 - m^2)\beta\delta}}$, $l = \frac{\varepsilon p}{\sqrt{1 - m^2}}$

Case 23. $\omega = (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha$, $c_0 = c_1 = c_3 = c_4 = 0$, $c_2 = \pm km^2p\sqrt{\frac{\alpha(c^2 - c_s^2)}{2\beta\delta}}$, $l = \varepsilon p$

Case 24. $\omega = (k^2(1-\frac{1}{2}m^2)+s^2)\alpha,\ c_0 = c_1 = c_4 = 0,\ c_2 = km\sqrt{\frac{\alpha(p^2+(m^2-1)l^2)(c^2-c_s^2)}{2\beta\delta}},$
$c_3 = \varepsilon km\sqrt{\frac{-\alpha(p^2-l^2)(c^2-c_s^2)}{2\beta\delta}}$

We derive the following solutions of Eq. (11).

$$\begin{cases} u_{21} = \frac{2\alpha k^2}{\delta} dn^2[\xi_{21}, m] \\ v_{21} = \pm k\sqrt{\frac{2\alpha(c^2-c_s^2)}{\beta\delta}} dn[\xi_{21}, m] \exp[i(sx - (k^2(m^2-2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{22} = \frac{\alpha k^2 m^4}{2\delta} \frac{dn^2[\xi_{22}, m]}{(\sqrt{1-m^2}+\varepsilon dn[\xi_{22}, m])^2} \\ v_{22} = \pm km^2\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}} \frac{dn[\xi_{22}, m]\exp[i(sx-(k^2(1-\frac{1}{2}m^2)+s^2)\alpha t)]}{\sqrt{1-m^2}+\varepsilon dn[\xi_{22}, m]} \end{cases}$$

$$\begin{cases} u_{23} = \frac{\alpha k^2 m^4}{2\delta} \frac{sn^2[\xi_{23}, m]}{(1+\varepsilon dn[\xi_{23}, m])^2} \\ v_{23} = \pm km^2\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}} \frac{sn[\xi_{23}, m]}{1+\varepsilon dn[\xi_{23}, m]} \exp[i(sx-(k^2(1-\frac{1}{2}m^2)+s^2)\alpha t)] \end{cases}$$

$$\begin{cases} u_{24} = \frac{\alpha k^2 m^2}{2\delta} \frac{(\sqrt{p^2+(m^2-1)l^2}\,sn[\xi_{24}, m]+\varepsilon\sqrt{l^2-p^2}\,cn[\xi_{24}, m])^2}{(p+l dn[\xi_{24}, m])^2} \\ v_{24} = km\sqrt{\frac{\alpha(c^2-c_s^2)}{2\beta\delta}} \frac{(\sqrt{p^2+(m^2-1)l^2}\,sn[\xi_{24}, m]+\varepsilon\sqrt{l^2-p^2}\,cn[\xi_{24}, m])}{p+l dn[\xi_{24}, m]} \cdot \\ \exp\left[i\left(sx-(k^2(1-\frac{1}{2}m^2)+s^2)\alpha t\right)\right] \end{cases}$$

where $\xi_i = k(x-2\alpha st)+\xi_0,\ (i = 21, \cdots, 24)$.

Remark 3. Here u_7, v_7; u_{11}, v_{11}; u_{12}, v_{12}; u_{23}, v_{23} contain the result of (13) (27); (18) (32); (19); (23) (37) in Ref. [16], if we let $q = \pm mp$; $q = r,\ p = 1$ or $p = r,\ q = 1$ in $u_8,\ v_8$, we can get the result of (14)(15)(29) in Ref. [16], if we let $p = r = 1$ or $r = 1, p = r$ in $u_{10},\ v_{10}$, we can get the result of (32) (34) in Ref. [16], if we let $p = 1,\ l = r$ or $p = r, l = 1$ in u_{24}, v_{24} we can get the result of (25) (39) in Ref. [16]. u_1, v_1; u_{14}, v_{14}; u_{19}, v_{19} contain the result of u_{12}, v_{12}; u_{18}, v_{18}; u_{19}, v_{19}; in Ref. [17], if we let $l = 0$ in u_{24}, v_{24} and $r = 0$ in $u_{10},\ v_{10}$ we can get the result of (15-18) in Ref. [17].

Remark 4. It is notable that the other types of solutions we obtained here to systems (11) are not shown in the previous literature. They are degenerated to the corresponding solitary wave solutions and triangle function solutions in the limit cases when $m \to 1$ or $m \to 0$.

4 Conclusion

In this paper, we succeed to propose an approach for finding new exact solutions for nonlinear evolution equations by constructing the four new types of Jacobian elliptic functions. By using this method and computerized symbolic computation, we have found thirteen new types of exact solutions for the zakharov Eq. (11). More importantly, our method is much simple and powerful to find new solutions to various kinds of nonlinear evolution equations, such as KdV equation, mKdV equation, Boussinesq equation, etc. We believe that this method should play an important role for finding exact solutions in the mathematical physics.

References

[1] R. Conte, M. Musette. Link between solitary waves and projective Riccati equations. *Phys. A: Math. Gen.*, 1992, **25**(21): 5609–5623.

[2] Z. Fu, S. Liu, et al. New jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations. *Phys. Lett . A*, 2001, **290**(1-2): 72–76.

[3] D. Huang, H. hang. New exact traveling waves solutions to the combined kdv-mkdv and generalized zakharov equations. *Reports on Math. Phys.*, 2006, **57**: 257–270.

[4] D. Huang, H. Zhang. Extended hyperbolic function method and new exact solitary wave solutions of Zakharov equations. *Acta Phys. Sin.*, 2004, **53**(8): 2434–2438.

[5] S. Lai, X. Lv, M. Shuai. The jacobi elliptic function solutions to a generalized Benjamin-Bona-Mahony Equation. *Math. and Comp. Model*, 2008.

[6] S. Liu, Z. Fu, et al. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. *Phys. Lett. A*, 2001, **289**(1-2): 69–74.

[7] S. Liu, Z. Fu, et al. Jacobi elliptic function expansion solution to the variable coefficient nonlinear equations. *Acta Phys. Sin.*, 2002, **51**(9): 1923–1927.

[8] V. Makankov. Dynamics of classical solitons (in nonintegrable systems. *Phys Letter. Sect. C Phys. Rep*, 1978, **35**: 1–128.

[9] Y. Peng. Exact traveling wave solutions for the Zakharov-Kuznetsov equation. *Appl. Math. Comp.*, 2008, **199**: 397–405.

[10] Y. Shang, Y. Huang, W. Yuan. The extended hyperbolic functions method and new exact solutions to the Zakharov equations. *Math. and Comp.*, 2008, **200**: 110–122.

[11] Y. Shi, P. Guo, et al. Expansion method for modified jacobi elliptic function and its application. *Acta Phys . Sin.*, 2004, **53**(3): 3265–3270.

[12] Y. Shi, K. L, et al. The solitary wave solutions to kdv-burgers equation. *Acta Phys. Sin.*, 2001, **50**(11): 2074–2077.

[13] Y. Shi, K. L, et al. Explicit and exact solutions of the combined Kdv equation. *Acta Phys.Sin.*, 2003, **52**(2): 267–271.

[14] S. Thornhill, D. Haar. Langmuir turbulence and modulation instability. *Phys Lett. Sect. C Phys. Rep*, 1978, **43**: 43–99.

[15] M. Wang, Y. Wang, J. Zhang. The periodic wave solutions for two systems of nonlinear wave equations. *Chin. Phys*, 2003, **12**(12): 1341–1348.

[16] G. Wu, M. Zhang, et al. The extended expansion method for jacobi elliptic function and new exact periodic solutions of Zakharov equations. *Acta Phys. Sin.*, 2007, **56**(9): 5054–5060.

[17] Y. Xiao, H. Xuen, H. Zhang. Extended Jacobi Elliptic Function Expansion Solution to the Zakharov Equation. *Journal of hebei university of technology*, 2004, **33**(3): 10–14.

[18] V. Zakharov. Collapse of Langmuir waves. *Sov. Phys. JETP*, 1972, **35**: 908–914.

[19] V. Zakharov, V. Syankh. The nature of the self-focusing singularity. *Sov. Phys. JETP*, **41**(1976): 465–468.

[20] H. Zhang. New exact traveling wave solutions of the generalized zakharov equations. *Reports on Mathematical Physics*, 2007, **60**: 97–107.

[21] J. Zhang, M. Wang. Complex Tanh-function expansion method and exact solutions to two systems of nonlinear wave equations. *Commun. Theor. Phys.*, 2004, **42**(4): 491.

[22] J. Zhang, Y. Wang, et al. Exact solutions for two classes of nonlinear wave equations. *Acta. Phys. Sin.*, 2003, **52**(7): 1574.

[23] L. Zhang, L. Zhang, C. Li. Some new exact solutions of Jacobian elliptic function about the generalized Boussinesq equation and Boussinesq-Burgers equation. *Chin. Phys. Soc.*, 2008, **17**(2): 403–411.

[24] W. Zhang, Q. Chang, E. Fan. Methods of judging shape of solitary wave and solution formulae for some evolution equations with nonlinear terms of high order. *J. Math. Anal.*, 2003, **287**: 1–18.

[25] C. Zhao, Z. Sheng. Explicit traveling wave solutions for Zakharov equations. *Acta Phys. Sin.*, 2004, **53**(6): 1629–1634.