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Global stability of a predator-prey model with stage structure for the predator*

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Abstract. This paper studies the asymptotic behavior of a predator-prey model with stage structure for the predator. By using the theory of competitive systems, compound matrices and stability of periodic orbits, sufficient conditions are derived for the global stability of a positive equilibrium of the model.

Keywords: permanence, global stability, predator-prey model, competitive system, stage structure

1 Introduction

Stage structure models have received much attention in recent years[1, 2, 6, 9, 10]. This is not only because they are much more simple than the models governed by partial differential equations but also they can exhibit phenomena similar to those of partial differential models[2]. The single species model with stage structure was studied by Aiello and Freedman[1]. Two species models with stage structure were investigated by many authors (see, for example, Wang and Chen[10], Wang[9], and Magnusson[6]). In [9, 10], predator-prey models with stage structure were established under the assumptions that the predator is divided into two groups, one immature and the other mature, and that only the mature predators can attack prey and have reproductive ability, while the immature predator does not attack prey and has no reproductive ability. In this paper, we consider the following predator-prey model with stage structure:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(r - ax(t) - \frac{bx(t)y_2(t)}{1+mx^2(t)}\right), \\ \dot{y}_1(t) &= \frac{kbx^2(t)y_2(t)}{1+mx^2(t)} - (D+v_1)\,y_1(t), \\ \dot{y}_2(t) &= Dy_1(t) - v_2y_2(t). \end{aligned} \tag{1}$$

where $x(t)$ is the density of the prey at time $t, y_1(t), y_2(t)$ are the densities of the immature and mature predators at time t. All parameters are positive constants. $\frac{bx^2}{1+mx^2}$ is the response function of the mature predator, r is the intrinsic growth rate of the prey, $v_1(v_2)$ is the death rate of the immature (mature) predator, constant k denotes the coefficient in converting the prey into a new immature predator, constant D denotes the rate of the immature predator becoming the mature predator.

The paper is organized as follows. In the next section, we prove that system (1) is competitive and permanent. In Section 3, the local asymptotical stability of a positive equilibrium of system (1) is studied. The stability of periodic orbits of system (1) is obtained by using the criterion given by Muldowney in [7], and sufficient conditions are derived for the global stability of the positive equilibrium of system (1) by using the above results and the same method in proving Theorem 2 in [5]. Numerical simulations are presented to illustrate the main results in Section 4.

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2 Permanence

In this section, we first prove that system (1) is competitive.

Theorem 1. *System (1) is competitive in* $D = \{(x, y_1, y_2) \in: x \geq 0, y_1 \geq 0, y_2 \geq 0\}$.

Proof. The Jacobian matrix of system (1) is

$$J = \begin{pmatrix} r - 2ax - \dfrac{2bxy_2}{(1+mx^2)^2} & 0 & \dfrac{-bx^2}{1+mx^2} \\ \dfrac{2kbxy_2}{(1+mx^2)^2} & -(D+v_1) & \dfrac{kbx^2}{1+mx^2} \\ 0 & D & -v_2 \end{pmatrix}$$

Choosing the matrix H as

$$H = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

it follows that

$$HJH = \begin{pmatrix} r - 2ax - \dfrac{2bxy_2}{(1+mx^2)^2} & 0 & \dfrac{-bx^2}{1+mx^2} \\ \dfrac{-2kbxy_2}{(1+mx^2)^2} & -(D+v_1) & -\dfrac{kbx^2}{1+mx^2} \\ 0 & -D & -v_2 \end{pmatrix}$$

Hence, system (1) is competitive in D.

We now discuss the permanence of system (1). It is easy to show that system (1) always has two equilibria $(0,0,0), (\frac{r}{a},0,0)$. Further, if $v_2(D+v_1) < \frac{kbDr^2}{a^2+mr^2}$, system (1) has a unique positive equilibrium $E^* = (x^*, y_1^*, y_2^*)$, where

$$x^* = \sqrt{\frac{v_2(D+v_1)}{kbD - mv_2(D+v_1)}}, \quad y_1^* = \frac{v_2(r-ax^*)(1+mx^{*2})}{Dbx^*}, \quad y_2^* = \frac{(r-ax^*)(1+mx^{*2})}{bx^*}.$$

In order to discuss the permanence of system (1), we first prove the following result.

Theorem 2. *Let*

$$v_2(D+v_1) > \frac{kbDr^2}{a^2+mr^2}. \tag{2}$$

Then any positive solution of system (1) satisfies

$$\lim_{t\to+\infty} x(t) = \frac{r}{a}, \quad \lim_{t\to+\infty} y_1(t) = 0, \quad \lim_{t\to+\infty} y_2(t) = 0.$$

Proof. Note that system (1) is competitive in D. From [8], we see that the omega limit set of an orbit of system (1) is a closed orbit if it contains no equilibrium, and that every closed orbit must contain one equilibrium at least. Since there is no positive equilibrium for system (1) due to (2), the omega limit set of system (1) intersects with the boundary of R^3_+.

The Jacobian matrix of system (1) at the point $(\frac{r}{a},0,0)$ is

$$J = \begin{pmatrix} -r & 0 & \dfrac{-br^2}{a^2+mr^2} \\ 0 & -(D+v_1) & \dfrac{kbr^2}{a^2+mr^2} \\ 0 & D & -v_2 \end{pmatrix}$$

Consequently, the characteristic equation is

$$(\lambda + r)\left[\lambda^2 + (D + v_1 + v_2)\lambda + v_2(D + v_1) - \frac{kbr^2}{a^2 + mr^2}\right] = 0,$$

and the characteristic roots are

$$\lambda_1 = -r, \lambda_{2,3} = \frac{-(D + v_1 + v_2) \pm \sqrt{(D + v_1 + v_2)^2 - 4\left[v_2(D + v_1) - \dfrac{kbDr^2}{a^2 + mr^2}\right]}}{2}$$

Clearly, the real parts of $\lambda_{2,3}$ are negative due to (2). Thus, $(\frac{r}{a}, 0, 0)$ is asymptotically stable.

The Jacobian matrix of system (1) at the trivial equilibrium $(0, 0, 0)$ is

$$J = \begin{pmatrix} r & 0 & 0 \\ 0 & -(D + v_1) & 0 \\ 0 & D & -v_2 \end{pmatrix}$$

Consequently, the characteristic equation is

$$(\lambda - r)[\lambda^2 + (D + v_1 + v_2)\lambda + v_2(D + v_1)] = 0$$

and the characteristic roots are

$$\lambda_1 = r > 0, \lambda_2 = -(D + v_1 + v_2), \lambda_3 = -v_2(D + v_1)$$

Thus, $(0, 0, 0)$ is unstable. It follows from the above discussion that the omega limit set of any positive orbit of system (1) is $(\frac{r}{a}, 0, 0)$. This completes the proof.

Theorem 3. *Let*

$$v_2(D + v_1) < \frac{kbDr^2}{a^2 + mr^2}. \tag{3}$$

Then system (1) is permanent.

Proof. We begin by verifying weak persistence of system (1). If it is not weakly persistent, it follows from the proof of Theorem 2 that there is a positive orbit $(x(t), y_1(t), y_2(t))$ of system (1) such that

$$\lim_{t\to+\infty} x(t) = \frac{r}{a}, \quad \lim_{t\to+\infty} y_1(t) = 0, \quad \lim_{t\to+\infty} y_2(t) = 0.$$

Choose $\varepsilon > 0$ small enough such that

$$v_2(D + v_1) < \frac{kbD(r - \varepsilon)^2}{a^2 + m(r - \varepsilon)^2}. \tag{4}$$

Let

$$\begin{aligned} \dot{z_1}(t) &= \frac{kb(r - \varepsilon)^2}{a^2 + m(r - \varepsilon)^2} z_2 - (D + v_1)z_1, \\ \dot{z_2}(t) &= Dz_1(t) - v_2 z_2(t) \end{aligned} \tag{5}$$

Then choose $t_0 > 0$ large enough such that if $t > t_0$, we have

$$\begin{aligned} \dot{y_1}(t) &> \frac{kb(r - \varepsilon)^2}{a^2 + m(r - \varepsilon)^2} y_2 - (D + v_1)y_1, \\ \dot{y_2}(t) &= Dy_1(t) - v_2 y_2(t) \end{aligned} \tag{6}$$

The Jacobian matrix of system (5) is

$$J_\varepsilon = \begin{pmatrix} -D - v_1 & \dfrac{kb(r-\varepsilon)^2}{a^2 + m(r-\varepsilon)^2} \\ D & -v_2 \end{pmatrix}$$

Since J_ε admits positive off-diagonal elements, the Perron-Frobenius theorem implies that there is a positive eigenvector $v = (v_1, v_2)$ for the maximum root α of J_ε.

The characteristic equation is

$$\lambda^2 + (D + v_1 + v_2)\lambda + [v_2(D + v_1) - \frac{kbD(r-\varepsilon)^2}{a^2 + m(r-\varepsilon)^2}] = 0$$

Let α, β be the characteristic roots of J_ε.

$$\alpha\beta = v_2(D + v_1) - \frac{kbD(r-\varepsilon)^2}{a^2 + m(r-\varepsilon)^2} < 0$$

Since α is the maximum root of J_ε, $\alpha > 0$.

Let $z(t) = (z_1(t), z_2(t))$ be a solution of system (5) through (lv_1, lv_2) at $t = t_0$, where $l > 0$ satisfies $lv_1 < y_1(t_0), lv_2 < y_2(t_0)$.

We know that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} le^{\alpha t} v_1 \\ le^{\alpha t} v_2 \end{pmatrix}$$

$z_i(t)$ is strictly increasing and $z_i(t) \to +\infty$ as $t \to +\infty,\ i = 1, 2$. Consequently, $y_i(t) \to +\infty$ as $t \to +\infty,\ i = 1, 2$, contradicting $\lim\limits_{t\to+\infty} y_i(t) = 0,\ i = 1, 2$. Thus, no positive orbit of system (1) tends to $(\frac{r}{a}, 0, 0)$ as $t \to +\infty$. This shows that system (1) is weakly persistent. An application of the techniques of paper [4] concludes the permanence of system (1). This completes the proof.

3 Global stability of the positive equilibrium

In this section, we always assume that (3) holds true. Theorem 3 shows that if (3) holds, there exist positive constants $M_0,\ M_i,\ \underline{x},\ \underline{y_i},\ i = 1, 2$ such that any solution $(x(t), y_1(t), y_2(t))$ of system (1) with nonnegative initial data satisfies

$$\underline{x} \le \lim_{t\to+\infty} \inf x(t) \le \lim_{t\to+\infty} \sup x(t) \le M_0$$
$$\underline{y_i} \le \lim_{t\to+\infty} \inf y_i(t) \le \lim_{t\to+\infty} \sup y_i(t) \le M_i,\quad i = 1, 2.$$

Denote

$$\Omega = \{(x, y_1, y_2) \in R^3_+ : 0 < x, y_1, y_2 \le M\},$$

where $M = \max\{M_0, M_1, M_2\}$.

Theorem 4. *The positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ of system (1) is asymptotically stable provided the following assumption holds:*

$(H1)\ \ a^2 - mr^2 > 0, \quad I < v_2(D + v_1) < \dfrac{kbDr^2}{a^2 + mr^2},$

where $I = \max\{\frac{kbDr^2}{1+mr^2},\ \frac{2ma^2kbD-1}{2m^2a^2}\}$.

Proof. The characteristic equation of system (1) at (x^*, y_1^*, y_2^*) is of the form

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0,$$

where $p_0 = \frac{2bx^*y_2^*v_2(D+v_1)}{(1+mx^{*2})^2}, p_1 = \frac{(2max^{*3}-mrx^{*2}+r)(D+v_1+v_2)}{1+mx^{*2}}, p_2 = D + v_1 + v_2 + \frac{2max^{*3}-mrx^{*2}+r}{1+mx^{*2}}$.

It is easy to know that if $a^2 - mr^2 > 0, v_2(D + v_1) > \frac{2ma^2kbD-1}{2m^2a^2}$, then $2max^{*3} + r - mrx^{*2} > 4m^2a^3v_2(D + v_1)x^*$, and if $v_2(D + v_1) > \frac{kbDr^2}{1+mr^2}$, then $x^* > r$.

Thus, $p_0 > 0, p_2 > 0$.

$$\begin{aligned} p_1p_2 - p_0 =& \frac{(2max^{*3} - mrx^{*2} + r)(D + v_1 + v_2)}{1 + mx^{*2}}(D + v_1 + v_2 + \frac{2max^{*3} - mrx^{*2} + r}{1 + mx^{*2}}) \\ &- \frac{2bx^*y_2^*v_2(D + v_1)}{(1 + mx^{*2})^2} \\ >& \frac{2v_2(D + v_1)}{(1 + mx^{*2})^2}\{[(D + v_1 + v_2)(1 + mx^{*2}) + 4m^2a^3v_2(D + v_1)x^*](D + v_1 + v_2)2m^2a^3x^* \\ &- (r - ax^*)(1 + mx^{*2})\} \\ >& \frac{2v_2(D + v_1)}{(1 + mx^{*2})^2}\{[2m^2a^3(D + v_1 + v_2)^2 + a](1 + mx^{*2})(x^* - r) + 8m^4a^6v_2(D + v_1)x^{*2}\} \\ >& 0 \end{aligned}$$

By the Routh-Hurwitz criterion, all characteristic roots have negative real parts. Hence, the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$ is asymptotically stable.

Now we give our main result.

Theorem 5. *Let (H1) hold. The positive equilibrium is globally asymptotically stable provided one of the two assumptions holds:*

$(H2)\ \ D + v_1 > r, and\ \ \underline{x} > \dfrac{r}{2a};$

$(H3)\ \ D + v_1 < r, and\ \ \underline{x} > \dfrac{r + D + v_1}{2a}.$

To prove Theorem 5 we need to use some results about competitive systems and the stability of periodic orbits.

Let $F \subset R^n$ be an open set, and $\bar{x} \to f(\bar{x}) \in R^n$ be a C^1 function defined in F. We consider the autonomous system in R^n given by

$$\bar{x}' = f(\bar{x}) \tag{7}$$

Suppose system (7) has a periodic solution $p(t)$ with minimal period $\omega > 0$ and orbit $\Gamma : \{p(t) : 0 \leq t \leq \omega\}$. We recall some definitions that we will use later.

Definition 1. [4] *The orbit Γ is orbitally stable if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $\bar{x}(t)$, for which the distance of $\bar{x}(0)$ from Γ is less than δ, remains at a distance less than ε from Γ, for all $t \geq 0$.*

Definition 2. [4] *The orbit Γ is asymptotically orbitally stable, if it is orbitally stable and the distance of $\bar{x}(t)$ from Γ also tends to zero as t goes to infinity.*

Definition 3. *System (7) is said to have the property of stability of periodic orbits, if and only if the orbit of any periodic solution, if it exists, is asymptotically orbitally stable.*

The following theorem is important in proving the global stability of the positive equilibrium.

Theorem 6. *Assume that $n = 3$, and F is convex and bounded. Suppose system (7) is competitive, permanent and has the property of stability of periodic orbits. If $\bar{x}_0$ is the only equilibrium point in int$F$, and if it is locally asymptotically stable, then it is globally asymptotically stable in int$F$.*

The proof of this theorem is the same as that of Theorems 2.1 and 4.2 in [5]. Notice that system (1) is competitive, permanent and E^* is locally asymptotically stable if (H1) holds true. Further, using Theorem 6, Theorem 3.2 can be proved if we show that system (1) has the property of stability of periodic orbits. In the following, we shall prove system (1) has this property.

Theorem 7. *Assume condition (H2) or (H3) holds true. Then system (1) has the property of the stability of periodic orbits.*

Proof. Let $p(t) = (x(t), y_1(t), y_2(t))$ be a periodic solution of system (1) whose orbit Γ is contained in intΩ. In accordance with the criterion given by Muldowney in [7], for the asymptotic orbital stability of a periodic orbit of a general autonomous system, it is sufficient to prove that the linear non-autonomous system

$$\dot{W}(t) = (DF^{[2]}(p(t)))W(t) \tag{8}$$

is asymptotically stable, where $DF^{[2]}$ is the second additive compound matrix of the Jacobian DF (see Appendix for the definition of the additive compound matrix of dimension three).

The Jacobian of system (1) is given by

$$DF = \begin{pmatrix} r - 2ax - \dfrac{2bxy_2}{(1+mx^2)^2} & 0 & \dfrac{-bx^2}{1+mx^2} \\ \dfrac{2kbxy_2}{(1+mx^2)^2} & -(D+v_1) & \dfrac{kbx^2}{1+mx^2} \\ 0 & D & -v_2 \end{pmatrix}$$

The second additive compound matrix of the Jacobian DF is

$$DF^{[2]} = \begin{pmatrix} r - 2ax - \dfrac{2bxy_2}{(1+mx^2)^2} - (D+v_1) & \dfrac{kbx^2}{1+mx^2} & \dfrac{bx^2}{1+mx^2} \\ D & r - 2ax - \dfrac{2bxy_2}{(1+mx^2)^2} - v_2 & 0 \\ 0 & \dfrac{2kbxy_2}{(1+mx^2)^2} & -(D+v_1+v_2) \end{pmatrix}$$

For the solution $p(t)$, system (8) becomes

$$\begin{aligned} \dot{w}_1(t) &= -(-r + 2ax + \frac{2bxy_2}{(1+mx^2)^2} + D + v_1)w_1 + \frac{kbx^2}{1+mx^2}w_2 + \frac{bx^2}{1+mx^2}w_3, \\ \dot{w}_2(t) &= Dw_1 + (r - 2ax - \frac{2bxy_2}{(1+mx^2)^2} - v_2)w_2, \\ \dot{w}_3(t) &= \frac{2kby_2x}{(1+mx^2)^2}w_2 - (D+v_1+v_2)w_3 \end{aligned} \tag{9}$$

In order to prove that system (9) is asymptotically stable, we shall use the following Lyapunov function:

$$V(w_1(t), w_2(t), w_3(t), x(t), y_1(t), y_2(t)) = \|(w_1(t), \frac{y_1(t)}{y_2(t)}w_2(t), \frac{y_1(t)}{ky_2(t)}w_3(t))\|.$$

where $\|\cdot\|$ is the norm in R^3 defined by

$$\|(w_1, w_2, w_3)\| = \sup\{|w_1|, |w_2| + |w_3|\}.$$

From Theorem 3, we obtain that the orbit of $p(t)$ remains at a positive distance from the boundary of Ω. Therefore

$$y_1(t) \geq \eta, y_2(t) \geq \eta, \quad \eta = \min\{\underline{y_1}, \underline{y_2}\},$$

for all large t. Hence, along the orbit $p(t)$,

$$\begin{aligned} V(w_1,w_2,w_3;x,y_1,y_2) =&\sup\{|w_1|,\frac{y_1}{y_2}|w_2|+\frac{y_1}{ky_2}|w_3|\} \\ \geq&\frac{\eta}{M}\sup\{|w_1|,|w_2|+\frac{|w_3|}{k}\} \\ \geq&\frac{\eta}{M}\|(w_1,w_2,\frac{w_3}{k})\|. \end{aligned} \tag{10}$$

Along a solution $(w_1(t), w_2(t), w_3(t))$ of system (9), V becomes

$$V(t)=\sup\left\{|w_1(t)|,\frac{y_1(t)}{y_2(t)}(|w_2(t)|+\frac{|w_3(t)|}{k})\right\}$$

Then we have the following inequalities:

$$\begin{aligned} D_+|w_1(t)| \leq& -\left[-r+2ax+\frac{2bxy_2}{(1+mx^2)^2}+D+v_1\right]|w_1(t)| \\ &+\frac{kbx^2}{1+mx^2}|w_2(t)|+\frac{bx^2}{1+mx^2}|w_3(t)| \\ =& -\left[-r+2ax+\frac{2bxy_2}{(1+mx^2)^2}+D+v_1\right]|w_1(t)| \\ &+\frac{kbx^2y_2}{y_1(1+mx^2)}\left[\frac{y_1}{y_2}\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right)\right], \end{aligned} \tag{11}$$

$$D_+|w_2(t)| \leq -\left[-r+2ax+\frac{2bxy_2}{(1+mx^2)^2}+v_2\right]|w_2(t)|+D|w_1(t)|, \tag{12}$$

and

$$D_+|w_3(t)| \leq -(D+v_1+v_2)|w_3(t)|+\frac{2bxy_2}{(1+mx^2)^2}|w_2(t)|. \tag{13}$$

From (12) and (13), we get

$$\begin{aligned} D_+(|w_2(t)|+\frac{|w_3(t)|}{k}) \leq& D|w_1(t)|-\left[(-r+2ax+v_2)|w_2(t)|+(D+v_1+v_2)\frac{|w_3(t)|}{k}\right] \\ \leq& D|w_1(t)|-G\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right), \end{aligned} \tag{14}$$

where $G=\min\{-r+2ax+v_2, D+v_1+v_2\}$. Therefore

$$\begin{aligned} &D_+\left[\frac{y_1(t)}{y_2(t)}\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right)\right] \\ =&\left(\frac{\dot{y}_1}{y_1}-\frac{\dot{y}_2}{y_2}\right)\frac{y_1}{y_2}\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right)+\frac{y_1(t)}{y_2(t)}D_+\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right) \\ \leq&\left(\frac{\dot{y}_1}{y_1}-\frac{\dot{y}_2}{y_2}\right)\frac{y_1}{y_2}\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right)+\frac{y_1(t)}{y_2(t)}\left[D|w_1(t)|-G\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right)\right] \\ =&D\frac{y_1(t)}{y_2(t)}|w_1(t)|+\left(\frac{\dot{y}_1}{y_1}-\frac{\dot{y}_2}{y_2}-G\right)\frac{y_1}{y_2}\left(|w_2(t)|+\frac{|w_3(t)|}{k}\right) \end{aligned} \tag{15}$$

According to (11) and (15), if $|w_1(t)| \geq \frac{y_1(t)}{y_2(t)}(|w_2(t)|+\frac{|w_3(t)|}{k})$, we have that

$$\begin{aligned} D_+V(t) =& D_+|w_1(t)| \\ \leq& \left\{- \left[-r + 2ax + \frac{2bxy_2}{(1+mx^2)^2} + D + v_1\right] + \frac{kbx^2y_2}{y_1(1+mx^2)}\right\}|w_1(t)| = h_1(t)V(t), \end{aligned} \tag{16}$$

where $h_1(t) = -\left[-r + 2ax + \frac{2bxy_2}{(1+mx^2)^2} + D + v_1\right] + \frac{kbx^2y_2}{y_1(1+mx^2)}$.

If $|w_1(t)| \leq \frac{y_1(t)}{y_2(t)}(|w_2(t)| + \frac{|w_3(t)|}{k})$,

$$\begin{aligned} D_+V(t) =& D_+\left[\frac{y_1(t)}{y_2(t)}\left(|w_2(t)| + \frac{|w_3(t)|}{k}\right)\right] \\ \leq& \left[D\frac{y_1(t)}{y_2(t)} + \left(\frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - G\right)\right]\frac{y_1}{y_2}\left(|w_2(t)| + \frac{|w_3(t)|}{k}\right) = h_2(t)V(t), \end{aligned} \tag{17}$$

where $h_2(t) = D\frac{y_1}{y_2} + \left(\frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - G\right)$.

From (16) and (17) it follows that

$$D_+V(t) \leq \sup\{h_1(t), h_2(t)\}V(t). \tag{18}$$

From the last two equations of system (1), we have $\frac{kbx^2y_2}{y_1(1+mx^2)} = \frac{\dot{y}_1}{y_1} + (D + v_1)$, then

$$\begin{aligned} h_1(t) =& -\left[-r + 2ax + \frac{2bxy_2}{(1+mx^2)^2} + D + v_1\right] + \frac{\dot{y}_1}{y_1} + (D + v_1) \\ =& r - 2ax - \frac{2bxy_2}{(1+mx^2)^2} + \frac{\dot{y}_1}{y_1} \end{aligned}$$

If (H2) holds true, then $-(D + v_1) < r - 2ax < 0$. We get

$$h_2(t) = D\frac{y_1}{y_2} + \frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - (-r + 2ax + v_2) = \frac{y_1}{y_1} + r - 2ax$$

Hence

$$\begin{aligned} \sup\{h_1(t), h_2(t)\} =& \sup\left\{r - 2ax - \frac{2bxy_2}{(1+mx^2)^2} + \frac{\dot{y}_1}{y_1}, \frac{\dot{y}_1}{y_1} + r - 2ax\right\} \\ =& \frac{\dot{y}_1}{y_1} + r - 2ax \leq -\mu + \frac{\dot{y}_1}{y_1}, \end{aligned} \tag{19}$$

where $\mu > 0$ such that $r - 2ax \leq -\mu < 0$.

If (H3) holds true, then $r - 2ax \leq -(D + v_1)$. We get

$$h_2(t) = D\frac{y_1}{y_2} + \frac{\dot{y}_1}{y_1} - \frac{\dot{y}_2}{y_2} - (D + v_1 + v_2) = -(D + v_1) + \frac{\dot{y}_1}{y_1}$$

Hence

$$\begin{aligned} \sup\{h_1(t), h_2(t)\} =& \sup\left\{r - 2ax - \frac{2bxy_2}{(1+mx^2)^2} + \frac{\dot{y}_1}{y_1}, \frac{\dot{y}_1}{y_1} - (D + v_1)\right\} \\ \leq& \frac{\dot{y}_1}{y_1} - (D + v_1). \end{aligned} \tag{20}$$

Let $\bar{\mu} = \min\{\mu, D + v_1\}$, then from (19) and (20), we have

$$\sup\{h_1(t), h_2(t)\} \leq \frac{\dot{y}_1}{y_1} - \bar{\mu} \tag{21}$$

From (18) and (21),

$$D_+V(t) \leq (\frac{\dot{y}_1}{y_1} - \bar{\mu})V$$

Using the Gronwall inequality, we obtain

$$V(t) \leq \frac{V(0)}{y_1(0)} y_1(t) e^{-\bar{\mu}t} \leq \frac{V(0)}{y_1(0)} M e^{-\bar{\mu}t}$$

which implies that $V(t) \to 0$ as $t \to +\infty$. By system (10) it turns out that

$$(w_1(t), w_2(t), w_3(t)) \to 0, \quad as \quad t \to +\infty$$

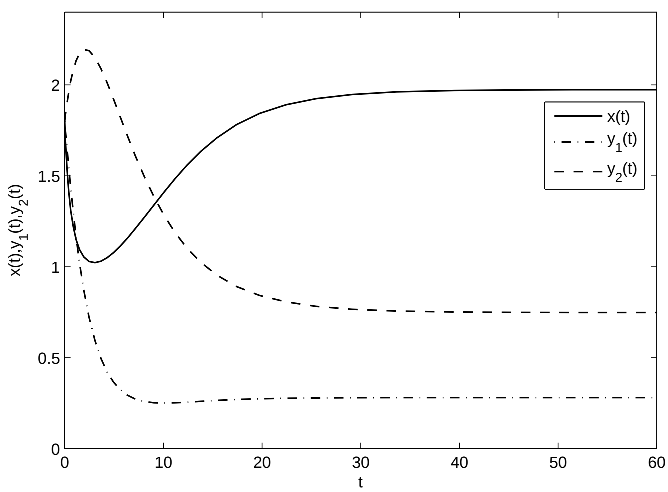

Fig. 1. The solution found by numerical integration of system (1) with $a = 0.4, b = 0.5, k = 0.1, D = 0.4, v_1 = 0.1, v_2 = 0.15, r = 1.5, m = 0.01$ and the initial data $(1.8, 1.8, 18)$

This implies that the linear system (9) is asymptotically stable. Then system (1) has the property of the stability of periodic orbits. This completes the proof. As we noted before, this result proves Theorem 5.

4 Numerical simulation

In this section, we show the feasibility of the conditions of Theorem 5.
Let $V(t) = kx + y_1 + y_2$, then

$$\dot{V}(t) \leq -vV + \frac{k(r+v)^2}{4a}$$

where $v = \min\{v_1, v_2\}$. Hence, $\lim_{t\to+\infty} V(t) \leq \frac{k(v+r)^2}{4av}$. The solutions of system (1) have the ultimately upper bound $M = \frac{k(v+r)^2}{4av}$. Then from the first equation of system (1), we know that

$$\dot{x}(t) \geq x(t)\left(r - ax(t) - \frac{bM^2}{1+mx^2(t)}\right) \geq x(t)\left(r - ax(t) - bM^2\right)$$

Hence $\lim_{t\to+\infty} x(t) \geq \underline{x}, \;\; \underline{x} \triangleq \frac{r-bM^2}{a}$.

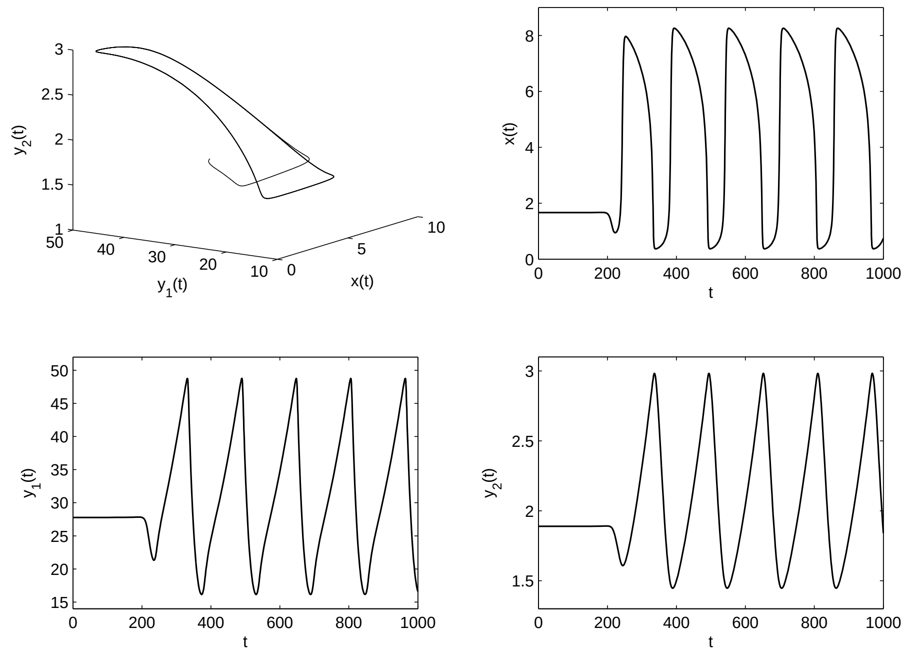

Fig. 2. The solution found by numerical integration of system (1) with $a = 0.1, b = 1, k = 1, D = 0.0034, v_1 = 0.0466, v_2 = 0.05, r = 1, m = 1$ and the initial data $(1.66667, 27.7778, 1.88889)$

Example 1. In system (1), let $a = 0.4, b = 0.5, k = 0.1, D = 0.4, v_1 = 0.1, v_2 = 0.15, r = 1.5, m = 0.01$. It is easy to show that system (1) admits a unique positive equilibrium $E^*(1.97386,\ 0.275279, 0.734078)$ and the conditions of Theorem 5 are satisfied. Hence, the positive equilibrium E^* is globally asymptotically stable. Numerical simulation illustrates this conclusion (See, Fig. 1).
Example 2. In system (1), let $a = 0.1, b = 1, k = 1, D = 0.0034, v_1 = 0.0466, v_2 = 0.05, r = 1, m = 1$. By calculation, we know that the condition (H3) holds but (H1) does not hold in Theorem 5. Numerical simulation shows that system (1) admits a periodic solution (See, Fig. 2).

We would like to mention here that if (H2) or (H3) holds but (H1) does not hold, a Hopf bifurcation may occur, system (1) may exhibit more complex dynamics. We leave this for future consideration.

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Appendix

Assume

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then $A^{[k]}$ are defined as follows:

$$A^{[1]} = A, \quad A^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}, \quad A^{[3]} = a_{11} + a_{22} + a_{33}.$$