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Algorithms for positive solutions of a nonlinear elliptic equations

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Abstract. In a recent result (see Jaffar Ali and shivaji [1]), it was shown via the method of sub-super solutions that a semipositone problem with a sign changing weight has at least one positive solution. In this paper we want to investigate that solution numerically.

Keywords: algorithms for positive solutions, sub and super-solutions numerical method

1 Introduction

Let Ω be an open bounded region with boundary $\partial\Omega$ in class C^2 in R^n for $n \geq 1$. We seek numericall solution to the boundary problem

$$\begin{cases} -\Delta u(x) = \lambda(g(x)[u(1-u)^p] - ch(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

where $p > 0, c > 0$, and $\lambda > 0$ are parameters. Here $g : \bar{\Omega} \to R$ is a C^α function while $h : \Omega \to R$ is a nonnegative C^α function with $\|h\|_\infty = 1$.

In [1], the author proved the existence of a positive solution when $g(x)$ is a sign changing function. They also deduce a result for the case when $g(x) \geq 0$. We briefly summarize the method of proof in the work:

Let $\lambda_1 > 0$ be the principal eigenvalue and $\phi_1 > 0$ with $\|\phi_1\|_\infty = 1$ the corresponding eigenfunction of $-\Delta$ with the Dirichlet boundary conditions. It is well known that $\frac{\partial \phi_1}{\partial \upsilon} < 0$ on $\partial\Omega$ where υ is the unit outward normal. Hence there exist $\delta > 0, \sigma \in (0,1]$ and $m > 0$ such that

$$\begin{cases} |\nabla\phi_1|^2 - \lambda_1\phi_1^2 \geq m & on & \overline{\Omega}_\delta \\ \phi_1 \geq \sigma & on & \Omega - \Omega_\delta \end{cases} \tag{2}$$

where $\Omega_\delta := \{x \in \Omega |\ d(x, \partial\Omega) < \delta\}$.

In this paper we assume that the weight g takes negative value in Ω_δ but require g to be strictly positive in $\Omega - \Omega_\delta$. Define $\gamma = \min_{\Omega-\Omega_\delta} g(x), \mu := \min_{\bar{\Omega}_\delta} g(x)$, and we assume that

$$|\mu| < \frac{m\gamma}{\lambda_1}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \tag{3}$$

Further let $0 < x_1 < x_2 < \frac{\gamma}{2\lambda_1}$ be the positive roots of $q(x) = -\mu$, where

$$q(x) := x\left[1 - \frac{2\lambda_1}{\gamma}x\right]^{\frac{1}{p}}\left(\frac{p+1}{p}\right)2m.$$

Using $w = k_0\phi^2$, as a subsolution where

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$$k_0 = \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]^{\frac{1}{p}}$$

and $v \equiv 1$ as a supersolution, the following results are proved in [1]:

Theorem 1. *Suppose Eq. (3) holds, $\frac{1}{x_2} < \lambda < \frac{1}{x_1}$ and $c \le c_0(\lambda)$, where*

$$c_0(\lambda) := \min\left\{ \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\frac{2m}{\lambda}\left(1 - \frac{2\lambda_1}{\lambda\gamma}\right)^{\frac{1}{p}} + \frac{\mu p}{p+1}\right], \frac{p\gamma\sigma^2}{(p+1)^{(p+1)/p}}\left[1 - \frac{2\lambda_1}{\lambda\gamma}\right]^{p+1/p}\right\}.$$

Then Eq. (1) has at least one positive solution u such that $\|u\|_\infty < 1$.

Note that when $c > 0$, Eq. (1) is a semipositone problem and it is well known in the literature that the study of positive solutions is mathematically challenging (see [3, 5]). Here we also include the additional challenging of dealing with a sign changing weight function g.

Corollary 1. *If $g(x) \ge 0$ on $\bar{\Omega}_\delta$ and $c = 0$, then for any $\lambda \ge \frac{2\lambda_1}{\gamma}$ Eq. (1) has a positive solution.*

2 Numerical results

It is well-known (see [2, 6]) that there must always exists a solution for problems such as Eq. (1) between a sub-solution $\underline{v}$ and a super-solution $\overline{u}$ such that $\underline{v} \le \overline{u}$ for all $x \in \Omega$.

Consider the boundary value problem

$$\begin{cases} \Delta u(x) + f(x, u(x)) = 0 & on \ \ \Omega \\ u(x) = 0 & on \ \ \partial\Omega. \end{cases} \tag{4}$$

Let $\overline{u}$, $\underline{v} \in C^2(\bar{\Omega})$ satisfy $\overline{u} \ge \underline{v}$ as well as

$$\begin{aligned} \Delta\bar{u}(x) + f(x, \bar{u}(x)) \le 0 \quad on \quad \Omega, \quad \bar{u} \ge 0 \quad on \quad \partial\Omega \\ \Delta\underline{v}(x) + f(x, \underline{v}(x)) \ge 0 \quad on \quad \Omega, \quad \underline{v} \le 0 \quad on \quad \partial\Omega. \end{aligned}$$

Choose a number $K > 0$ such that $K + \frac{\partial f(x,u)}{\partial u} > 0 \quad \forall\, (x, u) \in \bar{\Omega} \times [\underline{v}, \overline{u}]$ and such that the operator $(\Delta - c)$ with Dirichlet boundary condition has its spectrum strictly contained in the open left-half complex plane. Then the mapping

$$T : \phi \to w, \quad w = T\phi, \quad \phi \in C^2(\bar{\Omega}), \quad \phi(x) \in [\underline{v}, \overline{u}], \quad \forall x \in \bar{\Omega} \tag{5}$$

where $w(x)$ is the unique solution of the BVP

$$\begin{cases} \Delta w(x) - Kw(x) = -[K\phi(x) + f(x, \phi(x))] & on \ \ \Omega \\ w(x) = 0 & on \ \ \partial\Omega, \end{cases} \tag{6}$$

is monotone, i.e. for any ϕ_1, ϕ_2 satisfying (5) and $\phi_1 \le \phi_2$, we have $T\phi_1, T\phi_2$ satisfies (5), and $T\phi_1 \le T\phi_2$ on Ω.

Consequently, by letting $f_K(x, u) = Ku + f(x, u)$, the iterations

$$\begin{cases} u_0(x) = \overline{u}(x) \\ (\Delta - K)u_{n+1}(x) = -f_K(x, u_n(x)) & on \ \ \Omega, \\ & n = 0, 1, 2, ... \\ u_{n+1}(x) = 0 & on \ \ \partial\Omega, \end{cases} \tag{7}$$

and

$$\begin{cases} v_0(x) = \underline{v}(x) \\ (\Delta - K)v_{n+1}(x) = -f_K(x, v_n(x)) & on\ \Omega, \\ & n = 0, 1, 2, ... \\ v_{n+1}(x) = 0 & on\ \partial\Omega, \end{cases} \tag{8}$$

yield iteration u_n and v_n satisfying

$$\underline{v} = v_0 \le v_1 \le \cdots \le v_n \le \cdots \le u_n \le \cdots \le u_1 \le u_0 = \overline{u},$$

so that the limits

$$u_\infty(x) = \lim_{n\to\infty} u_n(x), \qquad v_\infty(x) = \lim_{n\to\infty} v_n(x)$$

exists in $C^2(\bar{\Omega})$. We have
(i) $v_\infty(x) \le u_\infty(x)$ on $\bar{\Omega}$
(ii) u_∞ and v_∞ are, respectively, stable from above and below;
(iii) if $u_\infty \not\equiv v_\infty$ and both u_∞ and v_∞ are asymptotically stable, then there exists an unstable solution $\phi \in C^2(\bar{\Omega})$ such that $v_\infty \le \phi \le u_\infty$.

We use following algorithm (see [4])
(1) Find a subsolution v_0 and a supersolution u_0. Choose a number $K > 0$.
(2) Solve the boundary value problem

$$\begin{cases} -\Delta w_{n+1}(x) - Kw_{n+1}(x) = -f_K(x, w_n(x)) & on\ \Omega \\ w_{n+1}(x) = 0 & on\ \partial\Omega. \end{cases} \tag{9}$$

for $w_n = v_n$ and $w_n = u_n$, respectively.
(3) If $\|w_{n+1} - w_n\| < \epsilon$, output and stop. Else go to step 2.

For the PDEs $\Delta u + f(u) = 0$ on the square region $\Omega = (0, 1) \times (0, 1)$ with zero Dirichlet boundary condition, it is well known that eigenvalues and eigenfunctions of $-\Delta$ are

$$\lambda_{m,n} = (m^2 + n^2)\pi^2$$

and

$$\Phi_{m,n} = 2\sin(m\pi x)\sin(n\pi x)$$

m and n range over all positive integers. So we get $\phi_1 = \sin(i\pi x)\sin(i\pi y)$ which $\|\phi\|_\infty = 1$ in Eq. (2). According to the following table, we obtain $m = 1.3$, $\delta = 0.2$ and $\sigma = 0.34$.

Table 1. Approximate value of $|\nabla\phi_1|^2 - \lambda_1\phi_1^2$

$x\backslash y$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	1.3042	2.5010	4.0453	5.3470	5.9091	5.5168	4.3200	2.7757	1.4739
0.2	2.5010	1.0258	-0.5631	-1.6588	-1.8426	-1.0445	0.4308	2.0197	3.1153
0.3	4.0453	-0.5631	-5.8149	-9.7041	-10.7452	-8.5404	-3.9320	1.3198	5.2090
0.4	5.3470	-1.6588	-9.7041	-15.7160	-17.3981	-14.1078	-7.1020	0.9434	6.9552
0.5	5.9091	-1.8426	-10.7452	-17.3981	-19.2601	-15.6201	-7.8683	1.0342	7.6871
0.6	5.5168	-1.0445	-8.5404	-14.1078	-15.6201	-12.4996	-5.9383	1.5576	7.1250
0.7	4.3200	0.4308	-3.9320	-7.1020	-7.8683	-5.9383	-2.0491	2.3137	5.4837
0.8	2.7757	2.0197	1.3198	0.9434	1.0342	1.5576	2.3137	3.0136	3.3900
0.9	1.4739	3.1153	5.2090	6.9552	7.6871	7.1250	5.4837	3.3900	1.6437

Letting $g(x, y) = 1.21 - xy$ and $p = 7$, we obtain $\gamma = 0.39$, $\mu = 0.01$ and $x_1 = 0.358 \times 10^{-2}, x_2 = 0.987 \times 10^{-2}$. According to Theorem 1, problem (1) has at least one positive solution for $\lambda \in (101, 278)$ and $c < c_0(\lambda)$. For brevity we express just some of those numerical results:

Also letting $g(x, y) = xy$ and $p = 7$, we obtain $\gamma = 0.09$, $\mu = 0.01$. According to Corollary 1, problem (1) has at least one positive solution for $\lambda \ge 438.64$ and $c = 0$. For brevity we express just some of those numerical results:

Table 2. Approximation of u for $\lambda = 110$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.5766	0.8376	0.8406	0.8153	0.4966
0.3	0.8376	0.9965	0.9980	0.9929	0.7367
0.5	0.8406	0.9980	0.9997	0.9909	0.6673
0.7	0.8153	0.9929	0.9909	0.9527	0.5088
0.9	0.4966	0.7367	0.6673	0.5088	0.1984

Table 3. Approximation of u for $\lambda = 150$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.7195	0.8942	0.8923	0.8823	0.6486
0.3	0.8942	0.9987	0.9992	0.9975	0.8331
0.5	0.8923	0.9992	0.9999	0.9965	0.7649
0.7	0.8823	0.9975	0.9965	0.9787	0.6026
0.9	0.6486	0.8331	0.7649	0.6026	0.2457

Table 4. Approximation of u for $\lambda = 200$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.8176	0.9258	0.9235	0.9187	0.7708
0.3	0.9258	0.9995	0.9997	0.9990	0.8877
0.5	0.9235	0.9997	1.0000	0.9986	0.8382
0.7	0.9187	0.9990	0.9986	0.9906	0.6965
0.9	0.7708	0.8877	0.8382	0.6965	0.2995

Table 5. Approximation of u for $\lambda = 250$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.8669	0.9427	0.9407	0.9376	0.8356
0.3	0.9427	0.9997	0.9998	0.9995	0.9151
0.5	0.9407	0.9998	1.0000	0.9993	0.8791
0.7	0.9376	0.9995	0.9993	0.9953	0.7672
0.9	0.8356	0.9151	0.8791	0.7672	0.3491

Table 6. Approximation of u for $\lambda = 450$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.1855	0.4803	0.6221	0.6813	0.4629
0.3	0.4803	0.9551	0.9927	0.9958	0.8943
0.5	0.6221	0.9927	0.9999	0.9999	0.9403
0.7	0.6813	0.9958	0.9998	0.9998	0.9602
0.9	0.4629	0.8943	0.9403	0.9567	0.9236

Table 7. Approximation of u for $\lambda = 1000$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.3080	0.7052	0.8399	0.8883	0.7785
0.3	0.7052	0.9945	0.9993	0.9997	0.9608
0.5	0.8399	0.9993	1.0000	1.0000	0.9749
0.7	0.8883	0.9997	1.0000	1.0000	0.9815
0.9	0.7785	0.9608	0.9749	0.9815	0.9691

Table 8. Approximation of u for $\lambda = 10000$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.9139	0.9814	0.9878	0.9909	0.9852
0.3	0.9814	1.0000	1.0000	1.0000	0.9964
0.5	0.9878	1.0000	1.0000	1.0000	0.9976
0.7	0.9909	1.0000	1.0000	1.0000	0.9982
0.9	0.9852	0.9964	0.9976	0.9982	0.9971

Table 9. Approximation of u for $\lambda = 1 \times 10^6$

x / y	0.1	0.3	0.5	0.7	0.9
0.1	0.993	0.9998	0.999	0.9999	0.9999
0.3	0.9998	1.0000	1.0000	1.0000	1.0000
0.5	0.9999	1.0000	1.0000	1.0000	1.0000
0.7	0.9999	1.0000	1.0000	1.0000	1.0000
0.9	0.9999	1.0000	1.0000	1.0000	1.0000

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