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Soliton solution of the Kadomtse-Petviashvili equation by homotopy perturbation method

H. Mirgolbabaei[1], D. D. Ganji[2*], H. Taherian[2]

[1] Department of Mechanical engineering, Kajeh Nasir Toosi University of Technology, Tehran, Iran
[2] Department of Mechanical Engineering, Mazandaran University, P. O. Box 484, Babol, Iran

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Abstract. In this paper we present homotopy perturbation method to develop soliton solution of the nonlinear Kadomtsev-Petviashvili equation. One of the newest analytical methods to solve nonlinear equations is the application of homotopy perturbation techniques. The HPM deforms a difficult problem into a simple problem which can be easily solved. The results are compared with the exact solutions.

Keywords: Kadomtsev-Petviashvili equation, homotopy perturbation method, soliton solution

1 Introduction

One of the most useful problems in nonlinear evolution was distinctively formulated by Kortweg and de Vries defined in the form

$$u_t + 6\mu u u_x + u_{xxx} = 0,\ \mu = \pm 1 \tag{1}$$

The Kortweg de Vries (KDV) Eq. (1) initiated an explanation of the phenomenon of solitary waves in weakly dispersing media. The KDV equation represents the longtime evolution of wave phenomena[7] in which the steepest effect of the nonlinear term uu_x is counterbalanced by dispersion u_{xxx}. This equation motivated the study of solitary wave, a typical bell-shaped, plane wave[6] which translates in one space direction without changing its shape. The solitary waves arise in a wide variety of diverse physical applications such as the propagation of coherent optical pulses[6].

Solitons also appear in a number of areas in plasma physics such as hydromagnetic waves and ion acoustic waves. Refs.[5–9, 13, 23–26] presented useful surveys that discuss the general properties of solitons for nonlinear dispersive wave propagation.

In 1970, Kadomtsev and Petviashvili[24] generalized the KDV equation to two space variables and formulated the well-known Kadmotsev-Petviashvili equation to provide an explanation of the general weakly dispersive waves. The Kadmotsev-Petviashvili (KP) equation is given in the form

$$(u_t + 6\mu u u_x + u_{xxx}) + 3u_{yy} = 0\ \mu \pm 1 \tag{2}$$

or equivalently

$$u_{xt} + 6\mu u_x^2 + 6\mu u u_{xx} + u_{xxx} + 3u_{yy} = 0,\ \mu = \pm 1 \tag{3}$$

defined on the region $R = \Omega \times [t > 0]$, where Ω is a square defined by

$$\Omega = [-L_0 \leq x \leq L_1] \times [-L_0 \leq y \leq L_1] \tag{4}$$

* Corresponding author. *E-mail address*: ddg_davood@yahoo.com.

and $u = u(x, y, t)$ is a sufficiently often[5] differentiable function.

The initial condition associated with the KP Eq. (2) is assumed of the form

$$u(x, y, 0) = g(x, y) \tag{5}$$

The boundary conditions are assumed to be of the form

$$\begin{aligned} &u_x(-L_0, y, t) = u_y(x, -L_0, t) = 0 \; t > 0 \\ &u_x(L_1, y, t) = u_y(x, L_1, t) = 0 \; t > 0 \\ &u_x(x, y, t) = 0 \text{ at } x = -L_0, L_1 \text{ and } y = -L_0, L_1 \end{aligned} \tag{6}$$

A substantial amount of analytical and numerical work has been done on the KP equation for various structures of this equation. The KP equation has attracted a great deal of interest in recent years. One attractive feature of this equation is that explicit solution may include rational, multisoliton and certain periodic solution[13] in x and y. the inverse spectral method and motion invariants were used among other methods.

Grübaum[13] used elementary methods to derive some unknown solutions of KP equation. Latham[26] used the elementary methods of Grübaum[13] to provide explicit solutions of KP equation which are associated to rank-three commuting ordinary differential operators.

Recently, a numerical algorithm was implemented by Bratsos and Twizell[5] where an explicit finite difference scheme was effectively used to obtain a numerical solution and to study the soliton phenomenon. The approach introduced a reliable algorithm to handle the KP equation numerically.

Obviously, $u_x(x, y, t) = 0$ is a solution of Eq. (2), but the existence of nontrivial exact solution that demonstrate the soliton phenomenon is the question of physical interest.

The basic motivation of this work is to extend the work by Brastos and Twizell[5] and to approach the KP equation differently, but with less computational work.

The homotopy perturbation method (HPM) was first proposed by He[15, 18–22]. The method has been used by many authors in [1, 2, 10–12, 14, 28, 29, 33] and the references therein to handle a wide variety of scientific and engineering applications: linear and nonlinear, homogeneous and inhomogeneous as well. It was shown by many authors that this method provides improvements over existing numerical techniques. With the rapid development of nonlinear science, many different methods were proposed to solve various boundary-value problems (BVP)[4, 27], such as Homotopy perturbation method (HPM) and Variational iteration method (VIM)[3, 16, 17, 30–32]. These methods give successive approximations of high accuracy of the solution. In this paper, only a brief discussion of the Homotopy perturbation method will be emphasized, complete details of the method are found in many related works.

In this paper, homotopy perturbation method (HPM) is implemented to solve the nonlinear Kadomtsev-Petviashvili equation. The results are compared with the results obtained by exact solutions. The results reveal that the HPM is very effective, convenient and quite accurate when applied to nonlinear equations. Some examples are presented to show the ability of the method for nonlinear Kadomtsev-Petviashvili equation.

2 Analysis of he's homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential Equation:

$$A(u) - f(r) = 0 \; r \in \Omega \tag{7}$$

Considering the boundary conditions of:

$$B(u, \partial u/\partial n) = 0 \; r \in \Gamma \tag{8}$$

Where A is a general differential operator, B a boundary operator, $f(r)$ known analytical function and Γ is the boundary of the domain Ω.

The operator A can be, generally divided into two parts of L and N, where L is the linear part, while N is the nonlinear one. Eq. (7) can, therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0 \tag{9}$$

By the homotopy technique, we construct a homotopy as $v(r,p) : \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0,\ p \in [0,1],\ r \in \Omega \tag{10}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{11}$$

Where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (2) which satisfy the boundary conditions. Obviously, considering Eq. (10) and Eq. (11), we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{12}$$
$$H(v,l) = A(v) - f(r) = 0 \tag{13}$$

The changing process of P from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $U(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy.

According to HPM, we can first use the embedding parameter p as a “small parameter”, and assume that the solution of Eq. (10) and Eq. (11) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{14}$$

Setting $p = 1$ results in the approximate solution of Eq. (7):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{15}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which lessens the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques.

The series (15) is convergent for most cases. However, the convergence rate depends on the nonlinear operator $A(v)$. The following opinions are suggested by He:
(1) The second derivative of $N(v)$ with respect to v must be small because the parameter p may be relatively large, i.e. $p \to 1$.
(2) The norm of $L^{-1}\partial N/\partial v$ must be smaller than one so that the series converges.

3 Application of the HPM

To incorporate our discussion, two particular cases of the KP equation, which correspond to some physical processes, will be investigated and the single soliton will be examind. The two distinct examples were discussed thoroughly by Hirota[23] and Freeman[8], respectively.
Example 1. Consider the following KP equation

$$u_{xt} + 6U_x^2 + 6uu_{xx} + u_{xxxx} + 3u_{yy} = 0, \tag{16}$$

defined on the region $R = \Omega \times [t > 0]$, where Ω is a square defined by

$$\Omega = [-80 \le x \le 80] \times [-80 \le y \le 80] \tag{17}$$

and $u = u(x,y,t)$ is a sufficiently often[5] differentiable function.

The initial condition associated with the KP Eq. (16) is assumed of the form

$$u(x,y,0) = \frac{8e^{2x}}{(1+e^{2x})^2} \tag{18}$$

The boundary conditions are assumed to be of the form

$$\begin{aligned}
&u_x(-80, y, t) = u_y(x, -80, t) = 0,\ t > 0\\
&u_x(80, y, t) = u_y(x, 80, t) = 0,\ t > 0\\
&u_x(x, y, t) = 0 \text{ at } x = \pm 80, \text{ and } y = \pm 80
\end{aligned} \tag{19}$$

According to the HPM, we can construct a homotopy of Eq. (16) as follows:

$$(1 - p)(v_{xt} - u_{0,x,t}) + p(v_{xt} + 6v_x^2 + 6vv_{xx} + v_{xxxx} + 3v_{yy}) = 0 \tag{20}$$

and the initial approximations are as follows:

$$v_0(x, y, 0) = u(x, y, 0) \tag{21}$$

Substituting Eqs. (14) and (21) into Eq. (20) and rearranging based on powers of p-terms, we have:

$$\begin{aligned}
&(v_{1,xxxx} + v_{2,xt} + 3v_{1,yy} + 6v_1v_{0,xx} + 12v_{0,x}v_{1,x})p^2\\
&+ (v_{1,xt} + 6v_{0,x}^2 + v_{0,xxxx} + 3v_{0,yy} + 6v_0v_{0,xx})p + (v_{0,xt}) = 0
\end{aligned} \tag{22}$$

In order to obtain the unknowns $v_i, i = 1, 2, 3, \ldots$ we must construct and solve the following system which includes three equations with three unknowns:

$$\begin{aligned}
&v_{0,xt} = 0,\\
&v_{1,xt} + 6v_{0,x}^2 + v_{0,xxxx} + 3v_{0,yy} + 6v_0v_{0,xx} = 0\\
&v_{1,xxxx} + v_{2,xt} + 3v_{1,yy} + 6v_0v_{1,xx} + 6v_1v_{0,xx} + 12v_{0,x}v_{1,x} = 0
\end{aligned} \tag{23}$$

Therefore we obtain

$$\begin{aligned}
v_0(x, y, t) &= \frac{8e^{2x}}{(1 + e^{2x})^2}\\
v_1(x, y, t) &= \frac{64e^{2x}(-1 + e^{2x})t}{(1 + e^{2x})^3}\\
v_2(x, y, t) &= \frac{256t^2e^{2x}(1 - 4e^{2x} + e^{4x})}{(1 + e^{2x})^4}
\end{aligned} \tag{24}$$

In this manner the other components can be easily obtained.

The accuracy of the HPM for the combined KP equation is controllable, and absolute errors are very small with the present choice of t and x. These results are listed in Tab. 1; it is seen that the implemented method achieves a minimum accuracy of fourteen and maximum accuracy of twenty significant figures for Eq. (16), for the first three approximations. Both the exact results and the approximate solutions obtained for the first three approximations are plotted in Fig. 1. There are no visible differences in the two solutions.

Example 2. In this example, we consider the form discussed by Freeman[8]. In this case, $\mu = -1$, hence Eq. (2) becomes

$$u_{xt} - 6u_x^2 - 6uu_{xx} + u_{xxx} + 2u_{yy} = 0 \tag{25}$$

defined on the region R.

The initial condition associated with the KP Eq. (25) is assumed of the form

$$u(x, y, 0) = \frac{-8e^{2x+2y}}{(1 + e^{2x+2y})^2} \tag{26}$$

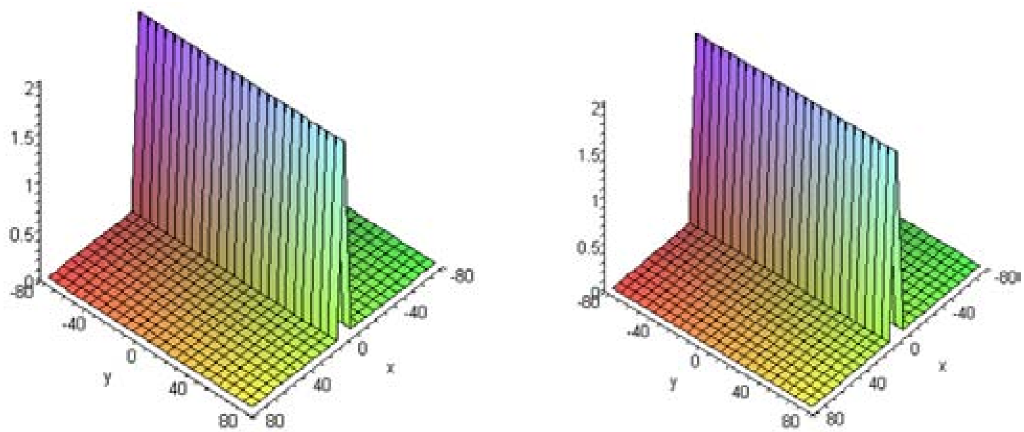
The boundary conditions are assumed to be of the form

Table 1. Numerical errors for several values of t where $x = 20$

t	$\lvert u_{exact} - u_{homotopy} \rvert$
0.02	$2.416023768e - 20$
0.04	$2.014659350e - 19$
0.06	$7.094548122e - 19$
0.08	$1.756469443e - 18$
0.1	$3.587001986e - 18$
0.2	$3.646897489e - 17$
0.3	$1.612055482e - 16$
0.4	$5.170257394e - 16$
0.5	$1.413789422e - 16$
1.00	$9.991986420e - 14$

Table 2. Numerical errors for several values of t where $x = 20, y = 20$.

t	$\lvert u_{exact} - u_{homotopy} \rvert$
0.02	$5.674181570e - 37$
0.04	$4.890786434e - 36$
0.06	$1.784160216e - 35$
0.08	$4.586549030e - 35$
0.1	$9.749084800e - 35$
0.2	$1.259735268e - 33$
0.3	$7.604392060e - 33$
0.4	$3.582939106e - 32$
0.5	$1.536481767e - 31$
1.00	$1.736254451e - 28$



(a)HPM (b)Exact

Fig. 1. The HPM result for $u(x, y, t)$, shown in(a), in comparison with the exact result[8], shown in (b) , for time $t = 0.01$.

$$
\begin{aligned}
&u_x(-80, y, t) = u_y(x, -80, t) = 0,\ t > 0 \\
&u_x(80, y, t) = u_y(x, 80, t) = 0,\ t > 0 \\
&u_x(x, y, t) = 0 \text{ at } x = \pm 80 \text{ and } y = \pm 80
\end{aligned} \tag{27}
$$

According to the HPM, we can construct a homotopy of Eq. (25) as follows:

$$
(1 - p)(v_{xt} - u_{0,x,t}) + p(v_{xt} - 6v_x^2 - 6vv_{xx} + v_{xxxx} + 3v_{yy}) = 0 \tag{28}
$$

and the initial approximations are as follows:

$$
v_0(x, y, 0) = u(x, y, 0) \tag{29}
$$

Substituting Eqs. (14) and (29) into Eq. (28) and rearranging based on powers of p-terms, we have:

$$
\begin{aligned}
&(v_{1,xxxx} + v_{2,xt} + 3v_{1,yy} - 6v_0 v_{1,xx} - 6v_1 v_{0,xx} - 12v_{0,x} v_{1,x})p^2 \\
&+ (v_{1,xt} - 6v_{0,x}^2 + v_{0,xxxx} + 3v_{0,yy} - 6v_0 v_{0,xx})p + (v_{0,xt}) = 0
\end{aligned} \tag{30}
$$

In order to obtain the unknowns v_i, $i = 1, 2, 3, \ldots$ we must construct and solve the following system which includes three equations with three unknowns:

$$
\begin{aligned}
&v_{0,xt} = 0 \\
&v_{1,xt} - 6v_{0,x}^2 + v_{0,xxxx} + 3v_{0,yy} - 6v_0 v_{0,xx} = 0 \\
&v_{1,xxxx} + v_{2,xt} + 3v_{1,yy} - 6v_0 v_{1,xx} - 6v_0 v_{0,xx} - 12v_{0,x} v_{1,x} = 0
\end{aligned} \tag{31}
$$

Therefore we obtain

$$
\begin{aligned}
v_0(x,y,t) &= \frac{-8e^{2x+2y}}{(1+e^{2x+2y})^2} \\
v_1(x,y,t) &= -\frac{112e^{2x+2y}(-1+e^{2x+2y})t}{(1+e^{2x+2y})^3} \\
v_2(x,y,t) &= \frac{748t^2e^{2x+2y}(-1+4e^{2x+2y}-e^{4x+4y})}{(1+e^{2x+2y})^4}
\end{aligned}
\tag{32}
$$

In this manner the other components can be easily obtained.

The accuracy of the HPM is as well as Exam. 1 and absolute errors are very small with the present choice of t, x and y. These results are listed in Tab. 2. The implemented method achieves a minimum accuracy of twenty eight and maximum accuracy of thirty seven significant figures for Eq. (25), for the first three approximations. Both the exact results and the approximate solutions obtained for the first three approximations are plotted in Fig. 2.

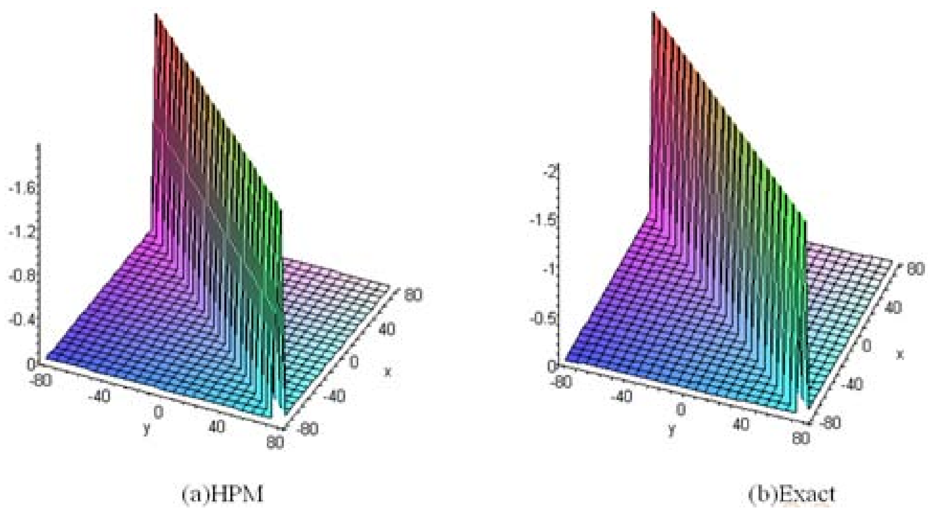


Fig. 2. The HPM result for $u(x, y, t)$, shown in(a), in comparison with the exact result[8], shown in (b) , for time $t = 0.01$.

4 Conclusion

We have presented a method to develop a numerical approximation to the nonlinear evolution KP equation. It is worth pointing out that the HPM presents a rapid convergence for the solutions. The obtained solutions are compared with the exact solution. examples show that the results of the present method are in excellent agreement with exact ones. The HPM has got many merits and much more advantages than other methods for example the Adomian's decomposition method. This method is to overcome the difficulties arising in calculation of Adomian polynomials. Also the HPM does not require small parameters in the equation, so that the limitations of the traditional perturbation methods can be eliminated, and also the calculations in the HPM are simple and straightforward. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability.

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