On the results of a class of Laplacian problems with Neumann boundary conditions

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Abstract. In this paper we study the following Neumann boundary value problem

\[
\begin{align*}
-u''(x) &= f(\lambda, u(x)), & x \in (0,1), \\
u'(0) &= 0 = u'(1)
\end{align*}
\]

where \(\lambda > 0\) is a parameter and \(f(\lambda, u(x)) = u^2(x) - \lambda\). We study the existence and nonexistence of solutions of this problem with respect to a parameter \(\rho\) (i.e. \(u(0) = \rho\)) in all \(\mathbb{R}\). By using a quadrature method, we obtain our results. Also we provide some details about the graph of the solutions that are obtained.

Keywords: existence solutions, interior critical points, quadrature method, Neumann boundary condition, Laplacian problem.

1 Introduction

We study the nonlinear two point boundary value problem

\[
\begin{align*}
-u''(x) &= f(\lambda, u(x)), & x \in (0,1), \\
u'(0) &= 0 = u'(1)
\end{align*}
\]

where \(\lambda > 0\) and the right hand side of (1) is even superlinear second members, i.e. \(f(\lambda, u(x)) = u^2(x) - \lambda\). In [3] problem (1) with Dirichlet boundary value conditions have been studied by Ammar khodja for the case Laplacian problem and in [1] the problem (1) with Dirichlet boundary value conditions have been extended by Addou to the general quasilinear case \(p\)-Laplacian with \(p > 1\). Also, Castro and Shivaji have studied for nonnegative solution curves with Dirichlet boundary value conditions and semipositone problems in the case \(p = 2\) in [2]. In [4] for semipositone problems, existence and multiplicity results have been established for the case \(p = 2\) with Neumann boundary value conditions.

This paper is organized as follows. In Section 2, we first state some notations and remarks and next our main result and finally in Section 3, we provide the proof of our main result that contains several lemmas and claims.

2 Notations and main result

By a solution of (1)-(2) we mean a function \(u \in C^1([0,1])\) for which \(u' \in C^1([0,1])\) and both the equation and the boundary value conditions are satisfied. Let us define for each integer number \(k \geq 0\) the following sets. Note that in defined sets, we suppose that \(u \in C^1([0,1])\) admits exactly \(k\) interior critical points in \([0,1]\) and \(u\) is symmetric about its interior critical points and \(u'(0) = 0 = u'(1)\) and \(u(0) = \rho\).

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\(A_k = \{ u : u' < 0 \text{ and } u' > 0 \text{ at the beginning and end of } (0,1), \text{ respectively} \},\)
\(B_k = \{ u : u' < 0 \text{ at the beginning and end of } (0,1) \},\)
\(C_k = \{ u : u' > 0 \text{ and } u' < 0 \text{ at the beginning and end of } (0,1), \text{ respectively} \},\)
\(D_k = \{ u : u' > 0 \text{ at the beginning and end of } (0,1) \} \). 

We note that in \(B_k, D_k, k \in \{ k = 2n : n = 0,1,2,3, \cdots \} \) and in \(A_k, C_k, k \in \{ k = 2n + 1 : n = 0,1,2,3, \cdots \} \). Throughout this paper we denote by \(\rho\), the value of the solution at zero (i.e. \(u(0) = \rho\)). Also we denote by \(S_{\rho}\), the values of \(\lambda\) in \((0, \infty)\) that the problem (1)-(2) has a solution \(u\) with \(u(0) = \rho\) at \(\lambda\). We now state some remarks in the following paper:

Remark 1. Suppose \(u\) is a solution of (1)-(2) at \(\lambda\), then
\[
\int_0^1 \{u(x)\}^2 dx = \lambda.
\]

In fact, by integrating of both sides of the equation (1) on \((0,1)\), one can obtain
\[
- \int_0^1 u''(x) dx = \int_0^1 \{u(x)\}^2 - \lambda dx,
\]
then
\[
-u'(x)|_0^1 = \int_0^1 \{u(x)\}^2 - \lambda dx,
\]
by applying the Neumann conditions \((u'(0) = 0 = u'(1))\), the left side of last statement is equal to zero.

It is require to note that if \(\lambda < 0\) and \(u\) be a solution of (1)-(2), then from the equation (1), one can conclude that \(u''\) must be negative throughout the interval \((0,1)\). With attention to this point and the condition (2), one can conclude that \(u\) must be decreasing and increasing at the beginning and end of the interval \((0,1)\). But the continuity of \(u\) implies that \(u\) is convex in some parts of the interval \((0,1)\), which this matter contradicts \(u'' < 0\). Thus one can state the Remark 2 below:

Remark 2. For any \(\lambda < 0\), the problem (1)-(2) has no solution.

Remark 3. For any \(\lambda \geq 0\), the problem (1)-(2) has always a trivial solution \(u \equiv \{ \lambda \}^{\frac{1}{2}}\).

Remark 4. (cf. [4], Lemma 2.1) Every solution \(u\) of (1)-(2) is symmetric about any interior critical points such that for any point \(x_0 \in (0,1)\) where \(u'(x_0) = 0\), we have \(u(x_0+z) = u(x_0-z)\) for all \(z \in [0, \min\{x_0, 1-x_0\}]\).

Remark 5. If \(u\) is a solution of (1)-(2), then \(u(1-x)\) (that may be the same \(u\) or difference one) is also a solution of (1)-(2).

Now, we present main result in the Theorem 1.

Theorem 1. Let \(n = 0,1,2, \ldots \) and \(\rho \in \mathbb{R}\). Then the problem (1)-(2) has:
(i) exactly one solution \(u\) which either belong to \(A_k\) or \(B_k\), with \(u(0) = \rho\) at any \(\lambda\) where \(\frac{\rho^2}{4} < \lambda < \rho^2\), if \(\rho \in (0, \infty)\). In this case if \(\frac{\rho^2}{4} < \lambda < \frac{\rho^2}{3}\), \(\lambda = \frac{\rho^2}{2}\), and \(\frac{\rho^2}{3} < \lambda < \rho^2\), the solution is sing-changing, nonnegative and positive, respectively, and the corresponding solution is defined by
\[
\int_{\rho}^{u(x)} \left\{ \frac{\rho^3}{3} - \frac{s^3}{3} + \lambda(s - \rho) \right\}^{-\frac{1}{2}} ds = -2^{1/2} x, \quad x \in (0, \frac{1}{k+1}).
\]
(ii) exactly one solution \(u\) which either belong to \(C_k\) or \(D_k\), with \(u(0) = \rho\) at any \(\lambda\) where \(\rho^2 < \lambda\), such that \(u\) is positive, nonnegative and sign-changing if \(\rho > 0\), \(\rho = 0\) and \(\rho < 0\), respectively, and the corresponding solution is defined by
\[
\int_{\rho}^{u(x)} \left\{ \frac{\rho^3}{3} - \frac{s^3}{3} + \lambda(s - \rho) \right\}^{-\frac{1}{2}} ds = 2^{1/2} x, \quad x \in (0, \frac{1}{k+1}).
\]
(iii) only a constant solution \(u \equiv \rho^2\) at \(\lambda = \rho^2\).
(iv) no solution \(u\) with \(u(0) = \rho\) at \(\lambda\) where \(0 < \lambda < \rho^2\), if \(\rho \in (-\infty, 0)\) and \(0 < \lambda \leq \frac{\rho^2}{4}\), if \(\rho \in (0, \infty)\).
3 Proof

Let \( u \) be a nontrivial solution of (1)-(2) with \( u(0) = \rho \). Now multiplying (1) throughout by \( u' \) and integrating over \((0, x)\), we obtain

\[ [u'(x)]^2 = 2\left[-\frac{u^3(x)}{3} + \lambda u(x) + C\right], \]

where \( C \) is a constant. Applying the condition \( u(0) = \rho \) and \( u'(0) = 0 \), we have the energy relation,

\[ [u'(x)]^2 = 2\left[\frac{\rho^3}{3} - \frac{u^3(x)}{3} + \lambda(u(x) - \rho)\right], \quad x \in (0, 1). \quad (3) \]

Now, we define a function and then try to find the variations of this function. We define the function,

\[ s \mapsto M(\rho, \lambda, s) := \frac{\rho^3}{3} - \frac{s^3}{3} + \lambda(s - \rho), \quad \text{on} \quad \mathbb{R}, \quad (4) \]

that \( \lambda > 0 \) and \( \rho \in \mathbb{R} \) are parameters. The following lemma collects the variations of this function that follows immediately and we omit its proof.

**Lemma 1.** For all \( \lambda \in \mathbb{R}^+ \) and \( \rho \in \mathbb{R} \),

(a) \( M(\rho, \lambda, \rho) = 0 \) and \( \lim_{s \to \pm \infty} M(\rho, \lambda, s) = \mp \infty \).

(b) \( M(\rho, \lambda, \cdot) \) is convex on \((\infty, 0)\) and concave on \((0, \infty)\).

(c) \( M(\rho, \lambda, \cdot) \) is decreasing on \((-\infty, -\sqrt{\lambda}) \cup (\sqrt{\lambda}, \infty)\) and increasing on \((-\sqrt{\lambda}, \sqrt{\lambda})\), and if \( \rho > 0 \),

\[
\max_{s \in \mathbb{R}^+} M(\rho, \lambda, s) = M(\rho, \lambda, \sqrt{\lambda}) = \frac{2}{3} \lambda \sqrt{\lambda} + \frac{\rho^3}{3} - \lambda \rho \begin{cases} = 0, & \text{for } \lambda = \rho^2, \\ > 0, & \text{for } \lambda \neq \rho^2, \end{cases}
\]

and

\[
\min_{s \in \mathbb{R}^-} M(\rho, \lambda, s) = M(\rho, \lambda, -\sqrt{\lambda}) = -\frac{2}{3} \lambda \sqrt{\lambda} + \frac{\rho^3}{3} - \lambda \rho \begin{cases} > 0, & \text{for } 0 < \lambda < \frac{\rho^2}{4}, \\ = 0, & \text{for } \lambda = \frac{\rho^2}{4}, \\ < 0, & \text{for } \frac{\rho^2}{4} < \lambda, \end{cases}
\]

and if \( \rho < 0 \),

\[
\max_{s \in \mathbb{R}^+} M(\rho, \lambda, s) = M(\rho, \lambda, \sqrt{\lambda}) = \frac{2}{3} \lambda \sqrt{\lambda} + \frac{\rho^3}{3} - \lambda \rho \begin{cases} < 0, & \text{for } 0 < \lambda < \frac{\rho^2}{4}, \\ = 0, & \text{for } \lambda = \frac{\rho^2}{4}, \\ > 0, & \text{for } \frac{\rho^2}{4} < \lambda, \end{cases}
\]

and

\[
\min_{s \in \mathbb{R}^-} M(\rho, \lambda, s) = M(\rho, \lambda, -\sqrt{\lambda}) = -\frac{2}{3} \lambda \sqrt{\lambda} + \frac{\rho^3}{3} - \lambda \rho \begin{cases} = 0, & \text{for } \lambda = \rho^2, \\ < 0, & \text{for } \lambda \neq \rho^2, \end{cases}
\]

and if \( \rho = 0 \), then for any \( \lambda > 0 \),

\[
\max_{s \in \mathbb{R}^+} M(\rho, \lambda, s) = M(\rho, \lambda, \sqrt{\lambda}) = \frac{2}{3} \lambda \sqrt{\lambda} > 0,
\]

\[
\min_{s \in \mathbb{R}^-} M(\rho, \lambda, s) = M(\rho, \lambda, -\sqrt{\lambda}) = -\frac{2}{3} \lambda \sqrt{\lambda} < 0.
\]

(d) The \( y \) – intercept of the graph of \( M(\rho, \lambda, \cdot) \), i.e.

\[
M(\rho, \lambda, 0) = \frac{\rho^3}{3} - \lambda \rho \begin{cases} \text{for } 0 < \lambda < \frac{\rho^2}{3} \text{ and } & \rho > 0 \text{ is positive,} \\ \rho < 0 \text{ is negative,} \end{cases}
\]

\[
\begin{cases} \text{for } \lambda = \frac{\rho^2}{3} & \text{is zero,} \\ \text{for } \frac{\rho^2}{3} < \lambda \text{ and } & \rho > 0 \text{ is negative,} \\ \rho < 0 \text{ is positive,} \\ \text{for } \lambda > 0 \text{ and } & \rho = 0 \text{ is zero.} \end{cases}
\]
(e) $0 < \lambda < \frac{\rho^2}{4}$, if and only if $M(\rho, \lambda, s)$ has only one zero. Also $\frac{\rho^2}{4} \leq \lambda$, if and only if $M(\rho, \lambda, s)$ has two zeros $\rho_0$ and $\rho_00$ in addition to $\rho$ which are equal to

$$\rho_0 = \frac{1}{2}(-\rho + \sqrt{12\lambda - 3\rho^2}), \quad \rho_00 = \frac{1}{2}(\rho + \sqrt{12\lambda - 3\rho^2}),$$

for $\lambda = \frac{\rho^2}{4}$ and for \begin{align*}
\begin{cases}
    \rho > 0 : \rho_0 = \rho_00 = -\frac{\rho}{2} < 0, \\
    \rho < 0 : \rho_0 = \rho_00 = -\frac{\rho}{2} > 0, 
\end{cases}
\end{align*}

for $\frac{\rho^2}{4} < \lambda < \frac{\rho^2}{3}$ and for \begin{align*}
\begin{cases}
    \rho > 0 : \rho_00 < \rho_0 < 0, \\
    \rho < 0 : 0 < \rho_00 < \rho_0, 
\end{cases}
\end{align*}

for $\lambda = \frac{\rho^2}{3}$ and for \begin{align*}
\begin{cases}
    \rho > 0 : \rho_0 = 0, \rho_00 = -\rho < 0, \\
    \rho < 0 : \rho_00 = 0, \rho_0 = -\rho > 0, 
\end{cases}
\end{align*}

for $\frac{\rho^2}{3} < \lambda < \rho^2$ and for \begin{align*}
\begin{cases}
    \rho > 0 : \rho_00 < 0 < \rho_0 < \rho, \\
    \rho < 0 : \rho < \rho_00 < 0 < \rho_0, 
\end{cases}
\end{align*}

for $\lambda = \rho^2$ and for \begin{align*}
\begin{cases}
    \rho > 0 : \rho_0 = \rho > 0, \rho_00 = -2\rho < 0, \\
    \rho < 0 : \rho_0 = -2\rho > 0, \rho_00 = \rho < 0, 
\end{cases}
\end{align*}

for $\lambda > \rho^2$ and for \begin{align*}
\begin{cases}
    \rho > 0 : \rho_00 < 0 < \rho < \rho_0, \\
    \rho < 0 : \rho_00 < \rho < 0 < \rho_0, 
\end{cases}
\end{align*}

for $\lambda > 0$ and $\rho = 0 : \rho_0 = \sqrt{3\lambda} > 0, \quad \rho_00 = -\sqrt{3\lambda} < 0$.

**Lemma 2.** Let $I = [a, b]$ be a bounded interval such that for any $\rho \in \mathbb{R}$ and $\lambda > 0$ satisfies the conditions below:

- $s \in (a, b) \Rightarrow M(\rho, \lambda, s) > 0$,
- $M(\rho, \lambda, a) = 0 = M(\rho, \lambda, b)$,
- $a = \rho$ or $b = \rho$,

in this case, then:

(a) \begin{align*}
\rho < 0 \Rightarrow I &= \begin{cases}
    \emptyset, & \text{iff } \lambda \in (0, \rho^2), \\
    [\rho_0, \rho], & \text{iff } \lambda \in [\rho^2, \infty),
\end{cases}
\end{align*}

(b) \begin{align*}
\rho > 0 \Rightarrow I &= \begin{cases}
    \emptyset, & \text{iff } \lambda \in (0, \frac{\rho^2}{4}), \\
    [\rho_0, \rho], & \text{iff } \lambda \in [\frac{\rho^2}{4}, \rho^2), \\
    \{\rho\}, & \text{iff } \lambda = \rho^2, \\
    [\rho, \rho_0], & \text{iff } \lambda \in (\rho^2, \infty),
\end{cases}
\end{align*}

where $\rho_0 = \frac{1}{2}(-\rho + \sqrt{12\lambda - 3\rho^2})$, as described in the Lemma 1 (e).

$$\rho = 0 \Rightarrow I = [0, \sqrt{3\lambda}] \text{ for any } \lambda > 0.$$

**Proof.** By applying the Lemma 1, one can conclude the Lemma 2. $\triangle$

**Lemma 3.** Let $u$ be a nontrivial solution to (1)-(2) at $\lambda$ with $u(0) = \rho$ and $k = 0, 1, 2, 3, \ldots$ interior critical points where if $k \neq 0$, $x_0$ is the first interior critical point, then

(a)
Let $M \triangleq \frac{\rho - \rho_0}{\rho_0}$.

Since (4), one can conclude that $M$ and if $\rho \neq 0$.

It is clear that the interval $u \in [0, 1]$.

If $k \neq 0$, the interior critical points of $u$ are $x_0 = \frac{1}{k+1}, x_1 = 2x_0 = \frac{2}{k+1}, x_2 = 3x_0 = \frac{3}{k+1}, \ldots, x_{k-1} = kx_0 = \frac{k}{k+1}$ and $(\frac{1}{k+1}, \rho_0)$ is the coordinate of the first interior critical point of $u$.

Proof. (a) Let $x_0$ be the first interior critical point of $u$ with $k \neq 0$ interior critical points.

Claim 1 For any $x \in (0, x_0)$, $u$ must be between $u(0)$ and $u(x_0)$. Now we show that $u(x_0) = \rho_0$. We know that $u'(x_0) = 0$, hence from (3) and (4), one can conclude that $M(\rho, \lambda, u(x_0)) = 0$, also from the Lemma 1(a), $M(\rho, \lambda, u(0)) = 0$. On the other hand $M(\rho, \lambda, u(x)) > 0$ for any $x \in (0, x_0)$. In fact if there exists a real number $x_{00} \in (0, x_0)$ such that $M(\rho, \lambda, u(x_{00})) = 0$ then from (3), one conclude that $u'(x_{00}) = 0$, i.e. $x_{00} \in (0, x_0)$ is an interior critical point of $u$ and this contradicts the fact that $x_0$ is the first interior critical point of $u$ in the interval $(0, 1)$.

It is clear that the interval $u|_{[0, x_0]} = I$ satisfies the conditions of Lemma 2. Thus, from the Lemma 2, it follows that

$$ u|_{[0, x_0]} = \begin{cases} [\rho_0, \rho] \text{ or } [\rho, \rho_0], & \text{if } \rho > 0, \\ [0, \sqrt{3}\lambda], & \text{if } \rho = 0, \\ [\rho, \rho_0], & \text{if } \rho < 0. \end{cases} $$

On the other hand by Remark 4, $u$ is symmetric about any interior critical point thus for any $x \in [0, 1]$, the statement of this part holds. Hence the proof of part (a) follows.

(b) $u$ must be strictly increasing or decreasing on the interval $(0, x_0)$. If $u$ is decreasing on $(0, x_0)$, then $\max_{x \in [0, x_0]} u(x) = u(0)$ and $\min_{x \in [0, x_0]} u(x) = u(x_0)$ and by the Remark 4 and the fact that $u(x_0) = \rho_0$, one can conclude that $\max_{x \in [0, x_0]} u(x) = \rho = u(0) = u(x_1) = u(x_2) = \cdots$ and $\min_{x \in [0, x_0]} u(x) = \rho_0 = u(x_0) = u(x_2) = u(x_4) = \cdots$. On the other hand by the first statement of this lemma and the fact that $u$ attains its maximum and minimum values at $x = 0$ and $x = x_0$, respectively, it follows that $||u||_{\infty} = u(0) = \rho$ and $\min_{x \in [0, 1]} u(x) = \rho_0 = u(x_0)$. Hence (5) and (6) hold. If $u$ is increasing on $(0, x_0)$, by similar argument, one can conclude that (7) and (8) hold. The proof of part (b) follows.

(c) Suppose $(x_0, u(x_0))$ be the coordinate of the first interior critical point of $u$. Since $u$ has $k \neq 0$ interior critical point in $(0, 1)$, where $x_0$ is the first interior critical point hence by Remark 4, one can conclude that $2x_0, 3x_0, \ldots, kx_0$ are the rest interior critical points of $u$ and $kx_0 + x_0 = 1$. Thus $x_0 = \frac{1}{k+1}$, and $x_1 = 2x_0 = \frac{2}{k+1}, x_2 = 3x_0 = \frac{3}{k+1}, \ldots, x_{k-1} = kx_0 = \frac{k}{k+1}$ are the interior critical points of $u$. Also as described in the proof of part (a), $u(x_0) = \rho_0$. Thus $(\frac{1}{k+1}, \rho_0)$ is the coordinate of the first interior critical point of $u$. The proof of part (c) follows.

Lemma 4. Let $u$ be a nontrivial solution of (1)-(2) with $u(0) = \rho$ and $k = 0, 1, 2, \cdots$ interior critical points at $\lambda \in S_\rho$, then:

(a)
At first we have
\[
S_\rho = \begin{cases} 
(\frac{\rho^2}{4}, \rho^2) \cup (\rho^2, \infty), & \text{if } \rho > 0, \\
(0, \infty), & \text{if } \rho = 0, \\
(\rho^2, \infty), & \text{if } \rho < 0.
\end{cases}
\]

(b) The corresponding solution is defined by
\[
\int_{\rho}^{u(x)} \{M(\rho, \lambda, s)\}^{-\frac{1}{2}} ds = \kappa 2^{1/2} x, \quad x \in (0, x_0),
\]
where
\[
\kappa = \begin{cases} 
-1, & \text{if } \lambda \in (\frac{\rho^2}{4}, \rho^2) \text{ and } \rho > 0, \\
+1, & \text{if } \lambda \in (\rho^2, \infty) \text{ and } \rho \in \mathbb{R}.
\end{cases}
\]

Proof. (a) The case \( \rho > 0 \): By the Lemma 3(a), \( u(x) \in [\rho_0, \rho] \) or \([\rho, \rho_0]\) for any \( x \in [0, 1] \), and it, by Lemma 1(e), (3) and (4), yields that \( \lambda \) must belong to the set \([\frac{\rho^2}{4}, \infty]\). Now we show that \( \lambda \) does not belong to \([\frac{\rho^2}{4}, \rho^2]\).

In fact, if \( \lambda = \frac{\rho^2}{4} \), then \( \rho_0 = -\frac{\rho}{2} \) (by Lemma 1(e)). Now by integrating (3) on \((0, x)_x\), where \( x \in [0, x_0] \), we obtain
\[
\int_{\rho}^{u(x)} \{M(\rho, \rho^2/4, s)\}^{-\frac{1}{2}} ds = 2^{1/2} x, \quad x \in (0, x_0).
\]

By substituting \( x = x_0 \) in (9) and using the facts that \( x_0 = \frac{1}{k+1} \) (by Lemma 3(c)) and \( u(x_0) = -\frac{\rho}{2} \) (by Lemma 3(b)), we get
\[
\int_{\rho}^{u(x_0)} \{M(\rho, \rho^2/4, s)\}^{-\frac{1}{2}} ds = \frac{2^{1/2}}{k+1}.
\]

Claim 2 The integral \( \int_{\rho}^{\frac{\rho^2}{4}} \{M(\rho, \rho^2/4, s)\}^{-\frac{1}{2}} ds \) is divergent.

Proof. At first we have
\[
\lim_{s \to (-\frac{\rho^2}{2})} (s + \frac{\rho^2}{4}) \{M(\rho, \rho^2/4, s)\}^{-\frac{1}{2}} = \lim_{s \to (-\frac{\rho^2}{2})} \frac{1}{|s - \rho|^{1/2}} = \sqrt{\frac{2}{3\rho}} \neq 0, \infty,
\]

And since the integral \( \int_{\rho}^{\frac{\rho^2}{4}} |s + \frac{\rho^2}{4}|^{-\frac{1}{2}} ds \) is divergent. Thus one can conclude that the divergence of the integral \( \int_{\rho}^{\frac{\rho^2}{4}} \{M(\rho, \rho^2/4, s)\}^{-\frac{1}{2}} ds \) is a consequence of that of the integral \( \int_{\rho}^{\frac{\rho^2}{4}} |s + \frac{\rho^2}{4}|^{-\frac{1}{2}} ds \).

According to (10), this is a contradiction, because the left hand side is infinity but the right hand side is not.

Also if \( \lambda = \rho^2 \), then \( \rho_0 = \rho \) (by Lemma 1(e)), hence by Lemma 3(a), \( u \equiv \rho \) and this a contradiction, because the solution \( u \) is nontrivial. Thus we conclude that \( S_\rho = (\frac{\rho^2}{4}, \rho^2) \cup (\rho^2, \infty) \).

The case \( \rho = 0 \): In this case by the Lemma 3(a), \( u(x) \in [\rho_0, 0] \) where \( \rho_0 = \sqrt{3}\lambda \) and therefore it, by Lemma 1(e) and (3) and (4), yields that \( \lambda \) must belong to \([0, \infty)\), i.e. \( S_\rho = (0, \infty) \).

The case \( \rho < 0 \): In this case by the Lemma 3(a), \( u(x) \in [\rho, \rho_0] \) for any \( x \in [0, 1] \) and therefore it, by Lemma 1(e), (3) and (4), yields that \( \lambda \) must belong to the set \([\rho^2, \infty]\). But \( \lambda \neq \rho^2 \). In fact, if \( \lambda = \rho^2 \), then \( \rho_0 = -2\rho \) (by Lemma 1(e)). Now by integrating (3) on \((0, x)_x\), where \( x \in [0, x_0] \), we obtain
\[
\int_{\rho}^{u(x)} \{M(\rho, \rho^2, s)\}^{-\frac{1}{2}} ds = 2^{1/2} x, \quad x \in (0, x_0).
\]

By substituting \( x = x_0 \) in (11) and using the facts that \( x_0 = \frac{1}{k+1} \) (by Lemma 3(c)) and \( u(x_0) = -2\rho \) (by Lemma 3(b)), we get
\[
\int_{\rho}^{2\rho} \{M(\rho, \rho^2, s)\}^{-\frac{1}{2}} ds = \frac{2^{1/2}}{k+1}.
\]

Claim 3 The integral \( \int_{\rho}^{2\rho} \{M(\rho, \rho^2, s)\}^{-\frac{1}{2}} ds \) is divergent.
At first we have
\[
\lim_{s \to \rho^+} (s - \rho)\{M(\rho^2, \rho, s)\}^{-\frac{1}{2}} = \lim_{s \to \rho^+} \frac{1}{\{s + 2\rho\}^{1/2}} = \frac{1}{\sqrt{3\rho}} \neq 0, \infty,
\]
And since the integral \( \int_{\rho}^{-2\rho} (s - \rho)^{-1} ds \) is divergent. Thus one can conclude that the divergence of the integral \( \int_{\rho}^{-2\rho} \{M(\rho, \rho^2, s)\}^{-\frac{1}{2}} ds \) is a consequence of that of the integral \( \int_{\rho}^{-2\rho} (s - \rho)^{-1} ds \). △

According to (12), this is a contradiction, because the left hand side is infinity but the right hand side is not. Thus we conclude that \( S_\rho = (\rho^2, \infty) \). Here the proof of Lemma 4(a) is complete.

(b) By Remark 4, to study solution of (1)-(2) on \((0, 1)\), it suffices to study that of in \([0, x_0]\) where \( x_0 \) is the first interior critical point. If \( \lambda \in (\frac{\rho^2}{4}, \rho^2) \) and \( \rho > 0 \) then by Lemma 1(e), \( \rho_0 < \rho \) and so, by Lemma 3(b), \( u(x_0) < u(0) \). Therefore \( u \) must be decreasing on \([0, x_0]\). Hence from (3), we have

\[
u'(x) = -2\frac{1}{2} \{M(\rho, \lambda, u(x))\}^{\frac{1}{2}}, \quad x \in (0, x_0).
\]

Also if \( \lambda \in (\rho^2, \infty) \) and \( \rho \in \mathbb{R} \) then by Lemma 1(e), \( \rho_0 > \rho \) and so, by Lemma 3(b), \( u(x_0) > u(0) \). Therefore \( u \) must be increasing on \([0, x_0]\). Hence from (3), we have

\[
u'(x) = 2\frac{1}{2} \{M(\rho, \lambda, u(x))\}^{\frac{1}{2}}, \quad x \in (0, x_0).
\]

Now, integrating (13) and (14) on \((0, x)\) where \( x \in (0, x_0) \), one can obtain

\[
\int_{\rho}^{u(x)} \{M(\rho, \lambda, s)\}^{-\frac{1}{2}} ds = \kappa 2^{1/2}x, \quad x \in (0, x_0),
\]

where

\[
\kappa = \begin{cases} 
- \lambda, & \text{if } \lambda \in (\frac{\rho^2}{4}, \rho^2) \text{ and } \rho > 0, \\
+ \lambda, & \text{if } \lambda \in (\rho^2, \infty) \text{ and } \rho \in \mathbb{R},
\end{cases}
\]

by substituting \( x = x_0 \) in (15) and using the fact that \( u(x_0) = \rho_0 \) (by Lemma 3(b)), we get

\[
\int_{\Omega} \{M(\rho, \lambda, s)\}^{-\frac{1}{2}} ds = 2^{1/2}x_0,
\]

where

\[
\Omega = \begin{cases} 
(\rho_0, \rho), & \text{if } \lambda \in (\frac{\rho^2}{4}, \rho^2) \text{ and } \rho > 0, \\
(\rho, \rho_0), & \text{if } \lambda \in (\rho^2, \infty) \text{ and } \rho \in \mathbb{R}.
\end{cases}
\]

We now show that the integral in (16) is convergent.

Claim 4 \( \int_{\Omega} \{M(\rho, \lambda, s)\}^{-\frac{1}{2}} ds \in (0, \infty). \)

Proof. At first we have

\[
\lim_{s \to s_0} |s - \rho|^{\frac{1}{2}} \{M(\rho, \lambda, s)\}^{-\frac{1}{2}} = \lim_{s \to s_0} \frac{\sqrt{3}}{|s^2 - \rho^2 + 3\lambda - \rho^2|^{1/2}} = \frac{1}{\sqrt{|\lambda - \rho^2|^{1/2}}} \neq 0, \infty,
\]

where

\[
s_0 = \begin{cases} 
\rho^-, & \text{if } \rho^2 < \lambda < \rho^2 \text{ and } \rho > 0, \\
\rho^+, & \text{if } \rho^2 < \lambda \text{ and } \rho \in \mathbb{R},
\end{cases}
\]

and

\[
\lim_{s \to s_0} |s - \rho_0|^{\frac{1}{2}} \{M(\rho, \lambda, s)\}^{-\frac{1}{2}} = \frac{\sqrt{2}}{|4\lambda - \rho^2 - \rho \sqrt{-3\rho^2 + 12\lambda}|^{1/2}} \neq 0, \infty,
\]

where

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Let (a) the other hand, we know that the integral due to the fact that \[ \int \triangle \] are convergent. Also, it is clear that the problem (1)-(2) has no trivial solution \[ u \equiv 0 \] with \[ u(0) = \rho \] at any \[ \lambda \in S_\rho^c = (0, \infty) - S_\rho \] where
\[
S_\rho^c = \begin{cases} 
(0, \rho^2), & \text{if } \rho < 0, \\
(0, \frac{\rho^2}{4}), & \text{if } \rho > 0.
\end{cases}
\]

Also, it is clear that the problem (1)-(2) has no trivial solution \[ u \equiv \rho \] on \([0, 1]\) at any \[ \lambda \in S_\rho^c \], since \[ u \equiv \rho \] dose not satisfy (1) when \[ \lambda \in S_\rho^c \]. Hence the proof of part (iv) is complete.

For any \( \rho \in \mathbb{R} \), it is clear that the constant solution \( u \equiv \rho \) satisfies the problem (1)-(2) when \( \lambda = \rho^2 \) also there exists no nontrivial solution \( u \) with \( u(0) = \rho \) at \( \lambda = \rho^2 \), because \( \rho^2 \) dose not belong to \( S_\rho \). Here the proof of part (iii) of the Theorem 1 is complete.

Lemma 5. Let \( u \) be a nontrivial solution of (1)-(2) with \( u(0) = \rho \) and \( k = 0, 1, 2, \ldots \) interior critical points at \( \lambda \in S_\rho \) then:

(a) \( \rho > 0 \) and \( \lambda \in \left( \frac{\rho^2}{4}, \frac{\rho^2}{3} \right) \) if and only if \( u \) is sign-changing and belong to the sets \( A_k \) or \( B_k \).
(b) $\rho > 0$ and $\lambda = \frac{\rho^2}{4}$ if and only if $u$ is nonnegative and belong to the sets $A_k$ or $B_k$.

(c) $\rho > 0$ and $\lambda \in (\frac{\rho^2}{4}, \rho^2)$ if and only if $u$ is positive and belong to the sets $A_k$ or $B_k$.

(d) $\rho > 0$ and $\lambda \in (\rho^2, \infty)$ if and only if $u$ is positive and belong to the sets $C_k$ or $D_k$.

(e) $\rho = 0$ and $\lambda \in (0, \infty)$ if and only if $u$ is nonnegative and belong to the sets $C_k$ or $D_k$.

(f) $\rho < 0$ and $\lambda \in (\rho^2, \infty)$ if and only if $u$ is sign-changing and belong to the sets $C_k$ or $D_k$.

Proof. (a) If $\rho > 0$ and $\lambda \in (\frac{\rho^2}{4}, \frac{\rho^2}{4})$, by Lemma 1(e), $\rho_0 < 0 < \rho$. Thus by Lemma 3(a), $u(x) \in [\rho_0, \rho]$. Hence $u$ must be sign-changing. Also since $\rho > \rho_0$, $u$ must be decreasing on the interval $(0, x_0)$. Hence according to Lemma 3(c), (5) and (6) hold. On the other hand, $u$ may be decreasing or increasing on the interval $(kx_0, 1)$. Thus by Lemma 3(b), one can conclude that $u(x) \in A_k$ or $B_k$. Inverse, if $u(x) \in A_k$ or $B_k$ and be sign-changing solution, then must be $\rho > 0$ and $\rho_0 < 0$, hence $\lambda$ must belong to $[\frac{\rho^2}{4}, \frac{\rho^2}{4}]$ (by Lemma 1(e)), but we have showed that $\lambda \neq \frac{\rho^2}{4}$ (in the proof of part (a) of Lemma 4 and in the case $\rho > 0$), therefore $\lambda \in (\frac{\rho^2}{4}, \frac{\rho^2}{4})$.

(b) If $\rho > 0$ and $\lambda = \frac{\rho^2}{4}$, by Lemma 1(e), $\rho_0 = 0$. Thus by Lemma 3(a), $u(x) \in [0, \rho]$. Hence $u$ must be nonnegative. Also $u$, by Lemma 3(c), satisfies (5) and (6). Thus by Lemma 3(b), $u(x) \in A_k$ or $B_k$. Inverse, if $u(x) \in A_k$ or $B_k$ and be nonnegative solution, then must be $\rho > 0$ and $\rho_0 = 0$, hence $\lambda$ must equal to $\frac{\rho^2}{4}$ (by Lemma1(e)).

(c) If $\rho > 0$ and $\lambda \in (\frac{\rho^2}{4}, \rho^2)$, by Lemma 1(e), $\rho > \rho_0 > 0$. Thus by Lemma 3(a), $u(x) \in [\rho_0, \rho]$. Hence $u$ must be positive. Also $u$ satisfies (5) and (6). Thus by Lemma 3(b), $u(x) \in A_k$ or $B_k$. Inverse, if $u(x) \in A_k$ or $B_k$ and be positive solution, then must be $\rho > 0$ and $\rho_0 > 0$, hence $\lambda$ must belong to $(\frac{\rho^2}{4}, \rho^2)$ (by Lemma1(e)).

(d) If $\rho > 0$ and $\lambda \in (\rho^2, \infty)$, by Lemma 1(e), $\rho_0 > \rho > 0$. Thus by Lemma 3(a), $u(x) \in [\rho, \rho_0]$. Hence $u$ must be positive. Also since $\rho_0 > \rho$, $u$ must be initially increasing. Thus according to Lemma 3(c), (7) and (8) hold. Thus by Lemma 3(b), $u(x) \in C_k$ or $D_k$. Inverse, if $u(x) \in C_k$ or $D_k$ and be positive solution, then must be $\rho_0 > \rho > 0$, hence $\lambda$ must belong to $(0, \infty)$ (by Lemma 1(e)).

(e) If $\rho = 0$ and $\lambda \in (0, \infty)$, by Lemma 3(a), $u(x) \in [0, \rho_0]$ where $\rho_0 = \sqrt{3\lambda}$ and it yields that $u$ must be nonnegative. Also $u$ is initially increasing and hence by Lemma 3(d), (7) and (8) hold. Thus by Lemma 3(b), $u(x) \in C_k$ or $D_k$, Inverse, if $u(x) \in C_k$ or $D_k$ and be nonnegative solution, then must be $\rho_0 > 0$ and $\rho = 0$, hence $\lambda$ must belong to $(0, \infty)$ (by Lemma 1(e)).

(f) Since for any $\rho < 0$ and $\lambda \in (\rho^2, \infty)$, $\rho_0 > 0$ (by Lemma 1(e)) and hence, by Lemma 3(a), $u(x) \in [\rho, \rho_0]$. Thus $u$ must be sign-changing. Also $u$ must be increasing on the interval $(0, x_0)$. Hence according to Lemma 3(c), (7) and (8) hold. On the other hand, $u$ may be decreasing or increasing on the interval $(kx_0, 1)$. Thus $u(x) \in C_k$ or $D_k$. Inverse, if $u(x) \in C_k$ or $D_k$ and be sign-changing solution, then must be $\rho_0 > 0$ and $\rho < 0$, hence $\lambda$ must belong to $[\rho^2, \infty)$ (by Lemma 1(e)). But we have showed that $\lambda \neq \rho^2$, therefore $\lambda \in (\rho^2, \infty)$.

Now, by back-tracking argument, one can prove from the Lemmas 3, 4 and 3 that the problem (1)-(2) has exactly one solution $u$ at $\lambda$ as described in the parts (i) and (ii) of the Theorem 1. Hence the proof of parts (i) and (ii) of the Theorem 1 is complete. $\triangle$

References


