New exact solutions for the generalized BBM and Burgers-BBM equations*

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Abstract. In this paper, based on the homogeneous balance principle and two improved projective Riccati Equations, exact solutions to the general form of BBM equation and the generalized BBM-Burgers equation with nonlinear terms of any order are obtained, which include several kinds of solitary wave solutions and periodic wave solutions. Some of them are found for the first time.

Keywords: BBM equation, generalized Burgers-BBM equation, solitary wave solutions, periodic wave solutions, improved projective Riccati Equations

1 Introduction

As we all know, nonlinear evolution equations (NEEs) are widely used to describe complex phenomena in various fields of science, such as fluid mechanics, plasma physics, solid state physics and optical fibres, etc. In order to better understand these nonlinear phenomena as well as further them in the practical life, it is important to seek their more exact solutions. Various effective methods have been developed recently, such as inverse scattering transformation, Hirota bilinear method, Backlund transformation, Darboux transformation, homogeneous balance method, Jacobian elliptic function expansion method and its generalization, F-expansion method and so on. Among them the Riccati equations method which was first presented by Conte et al in 1992 to seek more new solitary wave solutions to NEEs that can be expressed as polynomial in two elementary functions which satisfy a projective Riccati equation. In ref. Huang and Zhang developed a variable-coefficient projective Riccati equation method for certain nonlinear evolution equations. In this paper, we construct two extended Riccati equations which is more general and simple than the method in [3, 12], based on this method several new families of exact solutions to some NEEs are obtained.

Benjamin, Bona and Mahoney advocated that the PDE modeled the same physical phenomena equally well as the KdV equation, given the same assumptions and approximations that were originally used by Korteweg and de Vries. This PDE of Benjamin et al. is now often called the BBM equation, although it is also known as the regularized long wave (RLW) equation. Morrison et al. proposed the one-dimensional PDE, as an equally valid and accurate model for the same wave phenomena simulated by the KdV and RLW equations. This PDE is called the equal width (EW) equation because the solutions for solitary waves with a permanent form and speed, for a given value of the parameter \( \mu \), are waves with an equal width or wavelength for all wave amplitudes. The EW equation is transformed into the BBM equation by means of a simple change of the dependent variable \( u \rightarrow u + 1 \), and therefore, we often didn’t distinguish the both equations.

In this letter we consider the exact solutions for the BBM equation in its general form with nonlinear terms of any order \( p \) (GBBM),
and the generalized BBM-Burgers equation with nonlinear terms of any order and dissipative effects modeled by the term $\delta u_{xx}$ (GB-BBM).

$$u_t + \sigma u^p u_x - \mu u_{xxx} = 0 \quad (\sigma \mu \neq 0, p > 0) \tag{2}$$

where $\sigma, \delta, \mu, p$ are arbitrary constants, many research work has been done to the Eqs. (1) and (2)\cite{1, 4, 8, 19}. Here we construct solutions of Eqs. (1) and (2) by using two extended Riccati equations and obtained many new results.

The paper is arranged as follows. In section 2, we briefly describe the general Riccati method. In section 3, several types of exact solutions of the GBBM equation are obtained. In section 4, we research the exact solutions of GB-BBM equation. In section 5, some conclusions are given.

## 2 Summary of the general Riccati equations method

For a given nonlinear evolution equations (NEEs) in two independent variables $x$ and $t$

$$P(u, u_t, u_x, u_{xx}, \cdots) = 0 \tag{3}$$

by using the traveling wave transformation

$$u(x, t) = u(\xi), \xi = kx + wt + \xi_0 \tag{4}$$

where $k, w$ are constants to be determined later, $\xi_0$ is a arbitrary constant. Then Eq. (3) reduce to a nonlinear ordinary differential equations (ODEs):

$$O(u, u', u'', u''', \cdots) = 0 \tag{5}$$

where $"n$ denotes $\frac{d^n}{d\xi^n}$. In order to seek the traveling wave solutions of Eq. (5), we take the following transformations

$$u(\xi) = \sum_{i=0}^{n} A_i f^i(\xi) + \sum_{j=1}^{n} B_j f^{j-1}(\xi) g^j(\xi) \tag{6}$$

where $A_0, A_i, B_j, (i, j = 1, 2, \cdots n)$ are constants to be determined later. The parameter $n$ can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq. (3) or (5), $(n$ is usually a positive integer). If $n$ is a fraction or a negative integer, we make the following transformation:

(a) When $n = d/e$ is a fraction, we let $u(\xi) = v^{d/e}(\xi)$, then return to determine the balance constant $n$ again;

(b) When $n$ is a negative integer, we suppose $u(\xi) = v^n(\xi)$, then return to determine the balance constant $n$ again.

The new variable $f(\xi), g(\xi)$ satisfy the following extended Riccati equations:

### Case 1.

$$f'(\xi) = -q f(\xi) g(\xi), \quad g'(\xi) = q[1 - g^2(\xi) - r f(\xi)], \quad g^2(\xi) = 1 - 2 r f(\xi) + (r^2 + \varepsilon) f^2(\xi) \tag{7}$$

where $\varepsilon = \pm 1, r, q$ are arbitrary constants. It is easy to see that Eq. (7) admit the following solutions:

$$f_1(\xi) = \frac{a}{b \cosh(q\xi) + c \sinh(q\xi) + ar}, \quad g_1(\xi) = \frac{b \sinh(q\xi) + c \cosh(q\xi)}{b \cosh(q\xi) + c \sinh(q\xi) + ar} \tag{8}$$

when $\varepsilon = 1$: $a, b, c$ satisfies $c^2 = a^2 + b^2$. when $\varepsilon = -1$: $a, b, c$ satisfies $b^2 = a^2 + c^2$.

### Case 2.
\[ f'(\xi) = q f(\xi) g(\xi), \quad g'(\xi) = q[1 + g^2(\xi) - r f(\xi)], \quad g^2(\xi) = -1 + 2 r f(\xi) + (1 - r^2) f^2(\xi) \]  

Eq. (9) have the following solutions:

\[ f_2(\xi) = \frac{a}{b \cos(q\xi) + c \sin(q\xi) + ar}, \quad g_2(\xi) = \frac{b \sin(q\xi) - c \cos(q\xi)}{b \cos(q\xi) + c \sin(q\xi) + ar} \]

where \( a, b, c \) satisfies \( a^2 = b^2 + c^2 \).

Substituting (7) along with (6) and (9) along with (6) into Eq. (5) separately yields a set of algebraic equations for \( f_i(\xi) g_j(\xi)(i = 1, 2, \cdots; j = 0, 1) \). Setting the coefficients of \( f_i(\xi) g_j(\xi) \) to zero yields a set of nonlinear algebraic equations (NAEs) in \( A_0, A_i, B_j, (i, j = 1, 2, \cdots n) \) and \( k, w \). Solving the NAEs by Mathematica and Wu elimination\(^{21}\), we can obtain many exact solutions of Eq. (3) according to (4), (6), (8) and (10).

**Remark 1.** If we let \( q = -p, r = \frac{a}{q}, \varepsilon = \frac{h^2 - s^2}{q^2} \) in (7) and \( b = \frac{as}{q}, c = \frac{ah}{q} \) in (8), we can obtain the result of (1), (2), (7) and (8) in [20]. If we let \( q = p, r = \frac{a}{q}, q^2 = h^2 + s^2 \) in (9) and \( b = \frac{as}{q}, c = \frac{ah}{q} \) in (10), we can obtain the result of (1), (2), (9) and (10) in [20]. The Riccati equations (7) and (9) and solutions (8) and (10) are much simpler than the equations (1) and (2) and solutions (7), (8), (9) and (10) in [20]. We also find that the value of \( p \) can be extended to arbitrary constants while \( p = \pm 1 \) in [20].

**Remark 2.** In the ref. [12], the author considered the case for \( q = 1 \) and \( b = 5, c = 3, a = 4; b = 1, c = 0, a = 1; b = 0, c = 1, a = 1 \) in (8), so it is only a special case of Eq. (8).

In the following situations, we illustrate the method by considering the Eqs. (1) and (2).

### 3 Exact solutions of the generalized BBM equation

Let \( u(x, t) = u(\xi), \xi = k x + w t + \xi_0 \) and substitute it into Eq. (1) we have:

\[ w u'(\xi) + \sigma k u^p(\xi) u'(\xi) - \mu k^2 w u''(\xi) = 0 \]

Integrating the above equation once and setting the integrating constant to be equal to zero, we have

\[ u(\xi) + \frac{\sigma k}{w(p + 1)} u^{p+1}(\xi) - \mu k^2 u''(\xi) = 0 \]

According to the idea of (a) in Section 2, we get the value of \( n, n = 2/p \), therefore we make the following transformation

\[ u(\xi) = v^{2/p}(\xi) \]

Eq. (1) is reduced to the following form:

\[ v^2(\xi) + \frac{\sigma k}{w(p + 1)} v^4(\xi) - \frac{2 \mu k^2 (2 - p)}{p^2} v^2(\xi) - \frac{2 \mu k^2}{p} v(\xi) v''(\xi) = 0 \]

Let \( \alpha = \frac{\sigma k}{w(p + 1)}, \beta = \frac{2 \mu k^2 (2 - p)}{p^2}, \gamma = \frac{2 \mu k^2}{p} \), then we have

\[ v^2(\xi) + \alpha v^4(\xi) - \beta v^2(\xi) - \gamma v(\xi) v''(\xi) = 0 \]

Now we discuss the Eq. (15) using the method described in the previous section.

Based on the homogeneous balance principle we can suppose the solution of Eq. (15) is of the form

\[ v(\xi) = a_0 + a_1 f(\xi) + a_2 g(\xi) \]
Where $\xi = kx + wt + \xi_0$, $k, w, a_i (i = 0, 1, 2)$ are constants to be determined later.

**State 1**

Substituting (7) with (16) into (15) and collecting all terms with the same power in $f^i(\xi)g^j(\xi)$ ($i = 0, 1, 2, 3, 4; j = 0, 1$), setting the coefficients of these terms $f^i(\xi)g^j(\xi)$ to zero, we obtain a nonlinear algebraic equations (NAEs) with respect to the unknowns $a_0, a_1, a_2, k, w, q$. Solving these NAEs and after substitutions
\[
(\alpha = \frac{\sigma k}{w(p+1)}, \beta = \frac{2\mu k^2(2-p)}{p^2}, \gamma = \frac{2\mu k^2}{p}, \xi = kx + wt + \xi_0, u(\xi) = v^2(\xi))
\]
we could determine the following solutions:

**Case 1**

\[
r = 0, k = \pm \frac{p}{2q\sqrt{\mu}}, a_0 = a_2 = 0, a_1 = \pm \sqrt{\frac{\varepsilon qw(1+p)(2+p)}{\sigma p}}
\]

where $\varepsilon = \pm 1$, $\sigma \neq 0$, $p > 0$, $q \neq 0$, $\mu \neq 0$, $w$, are arbitrary constants. So do the following situations.

According to Eqs. (8), (13), (16) and (17) we obtain solitary wave solutions of Eq. (1):

**Family 1**

\[
v_1(\xi) = \frac{aA_1}{b \cosh[q(\pm \frac{p}{2q\sqrt{\mu}}x + wt + \xi_0)] + c \sinh[q(\pm \frac{p}{2q\sqrt{\mu}}x + wt + \xi_0)]},
\]
\[
u_1(\xi) = \frac{u_1^2(\xi)}{A_1} = \left\{\frac{aA_1}{b \cosh[q(\pm \frac{p}{2q\sqrt{\mu}}x + wt + \xi_0)] + c \sinh[q(\pm \frac{p}{2q\sqrt{\mu}}x + wt + \xi_0)]}\right\}^\frac{2}{p}
\]

where $A_1 = \pm \sqrt{\frac{\varepsilon qw(1+p)(2+p)}{\sigma p}}$.

**State 2**

In common, Substituting (9) with (16) into (15) we could determine the following solutions:

**Case 2**

\[
r = 0, k = \pm \frac{p}{2q\sqrt{-\mu}}, a_0 = a_2 = 0, a_1 = \pm \sqrt{\frac{\varepsilon qw(1+p)(2+p)}{\sigma p}} \cdot e = \pm 1
\]

According to Eqs. (10), (13), (16) and (18) we obtain periodic wave solutions of Eq. (1):

**Family 2**

\[
v_2(\xi) = \frac{aA_2}{b \cos[q(\pm \frac{p}{2q\sqrt{-\mu}}x + wt + \xi_0)] + c \sin[q(\pm \frac{p}{2q\sqrt{-\mu}}x + wt + \xi_0)]},
\]
\[
u_2(\xi) = \frac{u_2^2(\xi)}{A_2} = \left\{\frac{aA_2}{b \cos[q(\pm \frac{p}{2q\sqrt{-\mu}}x + wt + \xi_0)] + c \sin[q(\pm \frac{p}{2q\sqrt{-\mu}}x + wt + \xi_0)]}\right\}^\frac{2}{p}
\]

where $A_2 = \pm \sqrt{\frac{\varepsilon qw(1+p)(2+p)}{\sigma p}} \cdot e = \pm 1$.

If choosing $c = 0, b = a$ or $b = 0, c = a$ in $u_1$ we obtain the bell-shape solitary wave solutions, singular solitary wave solutions of Eq. (1) as the following
\[
u_{11}(\xi) = \left\{\frac{\varepsilon qw(1+p)(2+p)}{\sigma p} \cdot e \cdot \sec h^2[q(\pm \frac{p}{2q\sqrt{\mu}}x + wt + \xi_0)]\right\}^{\frac{1}{2}},
\]
\[
u_{12}(\xi) = \left\{\frac{\varepsilon qw(1+p)(2+p)}{\sigma p} \cdot e \cdot \csc h^2[q(\pm \frac{p}{2q\sqrt{\mu}}x + wt + \xi_0)]\right\}^{\frac{1}{2}}
\]

**Remark 3.** The solutions $u_{11}$ contain the result of (22) in [8]. If we select other value of $a, b, c$, we can obtain many new exact solutions of Eq. (1), to our knowledge, the two families of solutions are new.

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4 Exact solutions of the generalized Burgers-BBM equation

Commonly, exact solutions of the generalized Burgers-BBM equation are difficult to obtain because of the dissipation effects modeled by the term δuxx. Similarly, Eq. (2) is reduced to the following form after using the transformation Eq. (4) and Eq. (13) and integrating once with respect to ξ.

\[ v^2 + \alpha v^4 - mvv' - \beta v^2 - \gamma vv'' = 0 \]  \hspace{1cm} (19)

where \( \alpha = \frac{\sigma k}{w(p+1)} \), \( m = \frac{2\delta k^2}{p} \), \( \beta = \frac{2}\mu k^2(2-p) \), \( \gamma = \frac{2\mu k^2}{p} \), \( \sigma, \delta, \mu, p \) are arbitrary constants, \( k, w \) are constants to be determined later.

Using the same process we can obtain the six families of exact solutions of Eq.(19) and Eq. (2) as the following:

**Family 1**

\[ v_1(\xi) = B_1(1 - \frac{b \sinh(q\xi_1) + c \cosh(q\xi_1)}{b \cosh(q\xi_1) + c \sinh(q\xi_1)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0) \]

\[ u_1(\xi) = v_1^2(\xi) = \{B_1(1 - \frac{b \sinh(q\xi_1) + c \cosh(q\xi_1)}{b \cosh(q\xi_1) + c \sinh(q\xi_1)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0)\}^2 \]

\[ q\xi_1 = \pm \frac{pe}{\mu} \sqrt{\frac{-\mu}{2(2+p)}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0 \]

\[ B_1 = \pm \sqrt{\frac{w\delta(1+p)\sqrt{2+p}}{2\sigma(4+p)\sqrt{-2\mu}}} e = \pm 1, \varepsilon = \pm 1 \]

**Family 2**

\[ v_2(\xi_2) = B_1(1 + \frac{b \sinh(q\xi_2) + c \cosh(q\xi_2)}{b \cosh(q\xi_2) + c \sinh(q\xi_2)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0) \]

\[ u_2(\xi_2) = v_2^2(\xi_2) = \{B_1(1 + \frac{b \sinh(q\xi_2) + c \cosh(q\xi_2)}{b \cosh(q\xi_2) + c \sinh(q\xi_2)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0)\}^2 \]

\[ q\xi_2 = q\xi_1 = \pm \frac{pe}{\mu} \sqrt{\frac{-\mu}{2(2+p)}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0, e = \pm 1, \varepsilon = \pm 1 \]

**Family 3**

\[ v_3(\xi_3) = B_1(1 + \frac{b \sinh(q\xi_3) + c \cosh(q\xi_3)}{b \cosh(q\xi_3) + c \sinh(q\xi_3)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0) \]

\[ u_3(\xi_3) = v_3^2(\xi_3) = \{B_1(1 + \frac{b \sinh(q\xi_3) + c \cosh(q\xi_3)}{b \cosh(q\xi_3) + c \sinh(q\xi_3)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0)\}^2 \]

\[ q\xi_3 = q\xi_2 = q\xi_1 = \pm \frac{pe}{\mu} \sqrt{\frac{-\mu}{2(2+p)}} x + \frac{p\delta w}{\mu(4+p)} t + \xi_0 \]

**Family 4**

\[ v_4(\xi_4) = B_2(i + \frac{b \sin(q\xi_4) - c \cos(q\xi_4)}{b \cos(q\xi_4) + c \sin(q\xi_4)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{ip\delta w}{\mu(4+p)} t + i\xi_0) \]

\[ u_4(\xi_4) = v_4^2(\xi_4) = \{B_2(i + \frac{b \sin(q\xi_4) - c \cos(q\xi_4)}{b \cos(q\xi_4) + c \sin(q\xi_4)} \pm \sqrt{\frac{\delta}{\mu}} x + \frac{ip\delta w}{\mu(4+p)} t + i\xi_0)\}^2 \]

\[ q\xi_4 = iq\xi_1 = \pm \frac{ip}{\mu} \sqrt{\frac{-\mu}{2(2+p)}} x + \frac{ip\delta w}{\mu(4+p)} t + i\xi_0 \]

\[ B_2 = iB_1 = \pm \sqrt{\frac{w\delta(1+p)\sqrt{2+p}}{2\sigma(4+p)\sqrt{-2\mu}}} e = \pm 1, i = \sqrt{-1} \]

**Family 5**
\[ \begin{align*}
  v_5(\xi_5) &= B_2(i + a + b \sin(q\xi_5) - c \cos(q\xi_5)) \\
  u_5(\xi_5) &= v_5^2(\xi_5) = \{ B_2(i + a + b \sin(q\xi_5) - c \cos(q\xi_5)) \}^{2p} \\
  q\xi_5 &= q\xi_4 = \pm ipe \sqrt{\frac{-\mu}{2(2 + p)}} x \pm \frac{ip\delta w}{\mu(4 + p)} t + i\xi_0, e = \pm 1, i = \sqrt{-1}
\end{align*} \]

Family 6
\[ \begin{align*}
  v_6(\xi_6) &= B_2(i + b \sin(q\xi_6) - c \cos(q\xi_6)) \\
  u_6(\xi_6) &= v_6^2(\xi_6) = \{ B_2(i + b \sin(q\xi_6) - c \cos(q\xi_6)) \}^{2p} \\
  q\xi_6 &= q\xi_5 = q\xi_4 = \pm ipe \sqrt{\frac{-\mu}{2(2 + p)}} x \pm \frac{ip\delta w}{\mu(4 + p)} t + i\xi_0, e = \pm 1, i = \sqrt{-1}
\end{align*} \]

If choosing \( c = 0, b = a \) or \( b = 0, c = a \) in \( u_3 \) we obtain the solitary wave solutions of Eq. (2) as the following
\[ \begin{align*}
  u_{31}(\xi_{31}) &= \left( \frac{-\sigma(1 + p) \sqrt{2 + p}}{2\sigma(4 + p) \sqrt{2 - 2\mu}} (2 \pm 2 \tanh(q\xi_3) - \sec h^2(q\xi_3)) \right)^{\frac{1}{p}} \\
  u_{32}(\xi_{32}) &= \left( \frac{-\sigma(1 + p) \sqrt{2 + p}}{2\sigma(4 + p) \sqrt{2 - 2\mu}} (2 \pm 2 \coth(q\xi_3) + \csc h^2(q\xi_3)) \right)^{\frac{1}{p}}
\end{align*} \]

Remark 4. The solutions \( u_{31} \) contain the result of (35) in ref. [8], to our knowledge, the six families of solutions \( u_1(\xi) \) to \( u_6(\xi) \) are new.

5 Conclusion

In this work, many kinds of exact solutions to the GBBM equation and GBurgers-BBM equation are founded by using two extended Riccati equations, including solitary wave solutions and periodic wave solutions. Some solutions are not given in literature to our knowledge. Of course, this method can be also applied to other nonlinear wave equations[9, 10, 15–17].

References


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