ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 4 (2008) No. 3, pp. 237-240

The minimax admissible characteristic in gauss-markov mddel*

Wenliang Huang[1] , Yan Gao[2†]

[1] Department of Mathematics, Eeast China University of Science and Technology, Shanghai, 200237, China
[2] School of Management, University of Shanghai for Science and Technology, Shanghai, 200093, China

(*Received February 14 2008, Accepted May 1 2008*)

Abstract. The present paper discusses the minimax admissible characteristic in constant matrix linear model. For Gauss-Markov model, a necessary and sufficient condition for minimax and admissible estimation is proposed when variance is strictly positive.

Keywords: linear admissible estimate, minimax admissible estimate, quadratic loss function

1 Introduction

Let us consider Gauss-Markov model:

$$H : Y = X\beta + \varepsilon,\ E(\varepsilon) = 0,\ Var(\varepsilon) = \sigma^2 V. \tag{1.1}$$

where Y is a n-dimensional observable random vector, n is a $n \times p$ design matrix, β is one unknown p-dimension parameter vector; ε is a n-dimensional random vector, $V > 0$ is known and $\sigma^2 > 0$ is unknown. Rao[4] gave a necessary and sufficient condition on admissible estimation for (1.1) in the case $V > 0$. Wu extended Rao's results to $V \geq 0$[5]. On the other hand, Hoffman, Oslen, Seely, Birkes and Lamotte[2, 3] aimed at the general Gauss-Markov model, the result that AY in HL is $S\beta$ admissibility characterization estimate was given. In most applications, it is not enough to choose estimate in the side of admissibility characterization.

In this paper, we will study minimax and admissible estimation. $S\beta$ in

$$HL = \{AY : A \text{ is constant matrix}\}$$

is the minimax and admissible estimator; the loss function is different in dimension, for it has the minimax admissibility characterization; the quadratic loss function is

$$L(S\beta, \sigma^2, d) = \frac{(d - S\beta)'(d - S\beta)}{\sigma^2 + \beta' T\beta}, \quad R(S\beta, \sigma^2, d) = \frac{E\{(d - S\beta)'(d - S\beta)\}}{\sigma^2 + \beta' T\beta},$$

where $T = X'V^{-1}X (V > 0)$.

AY is said to be estimator value of $S\beta$, if $\sup_{(\beta,\sigma^2)} R(S\beta, \sigma^2, AY)$ admits the minimax value in HL. AY is said to be minimax and admissible estimator in HL, if AY is not only a minimax estimator, but also an admissible estimator. Throughout this paper, estimator is in the sense of the quadratic loss.

* This work was supported by National Science foundation of China (under grant:10671126) and Shanghai leading academic discipline project(under grant:T0502)
† Corresponding author. *E-mail address*: gaoyan1962@263.net, wlhuang@163.com

2 The minimax admissible characteristic

Definition 1. *For any matrix T with dimension $m \times n$, if there is a matrix A such that*
(1) $TAT = T$,
(2) $ATA = A$,
(3) $(TA)' = TA$,
(4) $(AT)' = AT$,
then A is called the Moore-Penrose inverse of T, denoted by T^*.

Lemma 1. *In the model H, if $S\beta$ is linearly estimate, then the following relation holds:*
(1) $\sup_{(\beta,\sigma^2)} R(S\beta,\sigma^2,AY) \geq tr\{AVA'\}$,
(2) $\sup_{(\beta,\sigma^2)} R(S\beta,\sigma^2,AY) \geq tr\{(AX-S)T^*(AX-S)'\}$.

Proof. Let

$$R(S\beta,\sigma^2,AY) = \frac{\sigma^2 tr\{AVA'\} + tr\{\beta'(AX-S)'(AX-S)\beta\}}{\sigma^2 + \beta' T\beta}. \quad (2.1)$$

Setting β in (2.1), we obtain (1) immediately. Passing the limit in (2.1) ($\sigma \to 0$), we get

$$\sup_{(\beta,\sigma^2)} R(S\beta,\sigma^2,AY) \geq \frac{tr\{\beta'(AX-S)'(AX-S)\beta\}}{\beta' T\beta}. \quad (2.2)$$

Introducing $\beta = T^*(AX-S)l$ to (2.2), where $l \in R^k$, it is obtained that

$$\begin{aligned}\frac{tr(\beta'(AX-S)'(AX-S)\beta)}{\beta' T\beta} &= \frac{(l(AX-S)'T^*(AX-S)l)^2}{l'(AX-S)'T^*(AX-S)l} \\ &= l'(AX-S)'T^*(AX-S)l. \quad (2.3)\end{aligned}$$

Let $q_1,\ldots,q_k$ be normal eigenvector of $(AX-S)'T^*(AX-S)$, and let $Q' = (q_1,\ldots,q_k)$. Denote $l = (q_1+\ldots+q_k)$, it is easy to verify that

$$\begin{aligned}l'(AX-S)'T^*(AX-S)l &= (q_1+\ldots+q_k)'(AX-S)(q_1+\ldots+q_k) \\ &= \sum_{i=1}^{k} q_i'(AX-S)'T^*(AX-S)q_i \\ &= tr\{Q(AX-S)'T^*(AX-S)Q'\} \\ &= tr\{(AX-S)'T^*(AX-S)\}.\end{aligned}$$

This implies (2). Thus, the proof of the theorem is completed.

Lemma 2. *Let L and S be two constant matrixes $k \times n$, $k \times p$. Suppose that $S\beta$ is linearly estimable under the model of H, then LY is the permissible estimator of $S\beta$ in HL if and only if*
(1) $L = LXT^-X'V^{-1}$
(2) $LXT^-X'L' \leq LXT^-S'$

The proof is trivial. It is omitted.

Theorem 1. *In the model H, if $S\beta$ is homogeneous linear estimate, then one has that*
(1) AY is the minimax estimator of $S\beta$ if and only if
A) $\frac{tr(ST^*S')}{4} \geq tr(AVA')$,
B) $\frac{tr(ST^*S')}{4} \geq tr\{(AX-S)T^*(AX-S)'\}$.
(2) AY is the minimax admissible estimator of $S\beta$ if and only if the conditions of A), B), and Lemma 2.2 holds.

Proof. "The necessity of (1)". Because AY is the Minimax estimator of $S\beta$, it is plain that

$$\sup_{(\beta,\sigma^2)} R(S\beta, \sigma^2, \frac{1}{2}ST^*X'V^{-1}Y) \geq \sup_{\beta,\sigma^2} R(S\beta, \sigma^2, AY). \qquad (2.4)$$

Thus, one has that

$$\begin{aligned} R(S\beta, \sigma^2, \frac{1}{2}ST^*XV^{-1}Y) &= \frac{\sigma^2 tr((\frac{1}{2}ST^*X'V^{-1})V(\frac{1}{2}ST^*X'V_{-1}))}{\sigma^2+\beta'T\beta} \\ &+ \frac{tr(\frac{1}{2}ST^*X'V^{-1}X-S)\beta\cdot\beta'(\frac{1}{2}ST^*X'V^{-1}X-S)}{\sigma^2+\beta'T\beta} \\ &= \frac{\sigma^2 tr(\frac{1}{4}ST^*X'V^{-1}XT^*S')+tr((-\frac{1}{2}S)\beta\beta'(-\frac{1}{2}S)')}{\sigma^2+\beta'T\beta} \\ &= \frac{\frac{1}{4}\sigma^2 tr(ST^*S')+\frac{1}{4}tr(S\beta\beta'S')}{\sigma^2+\beta'T\beta}. \qquad (2.5) \end{aligned}$$

Choosing an orthogonal matrix $Q=(q_1',\ldots,q_k')$. Then, there exists

$$\begin{aligned} trS\beta\beta'S' &= tr(QST^*X'V^{-1}X\beta\beta'X'V^{-1}T^*S'Q') \\ &= \sum_{i=1}^{k} q_1'ST^*X'V^{-1}X\beta\beta'X'V^{-1}T^*S'q_i \\ &= \sum_{i=1}^{k}(q_i'ST^*X'V^{-1}X\beta)^2. \qquad (2.6) \end{aligned}$$

Using Schwartz inequality in (2.6), one has that

$$\begin{aligned} (q_i'ST^*x'V^{-1}X\beta)^2 &= (q_i'ST^*XV^{-\frac{1}{2}}V^{-\frac{1}{2}}X\beta)^2 \\ &\leq q_i'ST^*X'V^{-1}XT^*S'q_i\beta'X'V^{-1}X\beta \\ &= q_i'ST^*S'q_i\beta'T\beta. \qquad (2.7) \end{aligned}$$

It implies that

$$\begin{aligned} tr(S\beta\beta'S') &\leq \sum_{i=2}^{k} q_i'ST^*S'q_i\beta'T\beta \\ &= (\sum_{i=1}^{k} q_i'ST^*S'q_i)\beta'T\beta \\ &= tr(QST^*S'Q')\beta'T\beta \\ &= tr(ST^*S')\beta'T\beta \end{aligned}$$

Thus, one has

$$\sup_{(\beta,\sigma^2)} R(S\beta, \sigma^2, \frac{1}{2}ST^*x'V^{-1}Y) \leq \frac{tr(ST^*S')}{4}. \qquad (2.8)$$

From (2.4) (2.8) and Lemma 2.1 we complete the proof of the proposition that A) and B) are true.

"The sufficiency of (1)". Supposing A meets the conditions A) and B), similar to the way proved the necessity, we can obtain

$$\begin{aligned}\sup_{(\beta,\sigma^2)} R(S\beta,\sigma^2,AY) &= \sup_{\beta,\sigma^2} \frac{\sigma^2 tr(AVA') + tr\{(AX-S)\beta\beta'(AX-S)'\}}{\sigma^2+\beta'T\beta} \\ &\le \sup_{(\beta,\sigma^2)} \frac{\sigma^2 tr(AVA') + tr\{\beta'T\beta(AX-S)T^*(AX-S)'\}}{\sigma^2+\beta'T\beta} \\ &\le \frac{tr(ST^*S')}{4}. \qquad (2.9)\end{aligned}$$

Supposing BY is one of estimators of the $S\beta$, by the lemma 2.1, we get:

$$\begin{aligned}\sup_{(\beta,\sigma^2)} R(S\beta,\sigma^2,BY) &\ge \frac{1}{2}[tr(BVB') + tr(BX-S)T^*(BX-S)'] \\ &= \frac{1}{2}\left[trB(V-XT^*X')B' + 2tr(BX-\frac{1}{2}S)T^*(BX-\frac{1}{2}S)' + \frac{1}{2}ST^*S'\right] \\ &\ge \frac{1}{4}tr(ST^*S'). \qquad (2.10)\end{aligned}$$

Because of $V - XT^*X' \ge 0$, the equation(2.10) is true. By virtue of (2.9) and (2.10), the sufficiency of (1) is proved.

Likewise, we can prove (2).

3 The relationship between admissibility in hl and minimax admissibility

Definition 2. *In the normal-quadratic-loss function $L(S\beta,\sigma^2,d)$, if an equation has the following forms*

$$R(S\beta,\sigma^2,LY) = R(S\beta,\sigma^2,AY).$$

where $\beta \in R^p$, $\sigma^2 > 0$, then we call LY and AY is the R-equivalent and call LY the R-unique Minimax estimator of $S\beta$. If AY is the Minimax estimator of $S\beta$ as well, we can say that AY and LY are the R-equivalent.

Theorem 2. *In the quadratic-loss function $R(S\beta,\sigma,d)$, if LY is a R-unique Minimax estimator of $S\beta$, then LY is admissible.*

Proof. If LY is not admissible, the there must be another estimator of AY which is defined by

$$R(S\beta,\sigma^2,AY) \le R(S\beta,\sigma^2,LY).$$

At any time, in a particular value of β and σ^2. as following , $\beta \in R$ and $\sigma^2 > 0$, which in equation is true.It is easy to see that AY is also an Minimax estimator, therefore, one can find that

$$R(S\beta,\sigma^2,AY) \ne R(S\beta,\sigma^2,LY).$$

And the conclusion reverse the assuming that LY is the R-unique Minimax estimator of $S\beta$, so that LY is admissible is proved.

References

[1] G. Casella, W. Strawderman. Estimating a bounded normal mean. *Ann, Statistics*, 1981, **9**: 870–878.
[2] K. Hoffmann. Admissibility of linear estimates with respect to restricted parameter set. math. *Operations for statistic Ser. Statistic*, 1977, **3**: 425–438.
[3] A. Olsen, T. Seely, D. Birkes. Invariant quadratic estimation for two variance components. *Ann, Statistic*, 1976, **4**: 878–890.
[4] C. Rao. Estimation of parameters in a linear model. *Ann, Statistic*, 1997, **4**: 1037–1073.
[5] Q. Wu. Admissibility of linear estimates of coefficients under matrix loss function. 1983, 155–158.
[6] J. Zhan, J. Chen. Admissibility of linear estimator of regression coefficients under quadratic loss. *ActaMath. Apply. Sinica*, 1992, **8**: 237–244.