

A numerical implementation of the variational iteration method for the Lienard equation

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Abstract. In this paper, by considering the variational iteration method, a kind of explicit exact and numerical solutions to the Lienard equation is obtained, and the numerical solutions has been compared with their known theoretical solution. The variational iteration method is based on Lagrange multipliers for identification of optimal value of parameters in a functional. Using this method, it is possible to find the exact solution or an approximate solution of the problem.

Keywords: variational iteration method, Lienar equation

1 Introduction

In this work, we consider the Lienard equation

$$x'' + f(x)x' + g(x) = h(t), \quad (1)$$

which is not only regarded as a generalization of the damped pendulum equation or a damped spring-mass system (where $f(x)x'$ is the damping force, $g(x)$ is the restoring force, and $h(t)$ is the external force), but also used as nonlinear models in many physically significant fields when taking different choices for $f(x)$, $g(x)$ and $h(t)$. For example, the choices $f(x) = \epsilon(x^2 - 1)$, $g(x) = x$ and $h(t) = 0$ lead equation of (1) to the Van der Pol equation served as a nonlinear model of electronic oscillation [3, 11]. Therefore, studying equation of (1) is of physical significance. In the general case, it is commonly believed that it is very difficult to find its exact solution by usual ways [4]. Kong studied the following special case of equation (1) in [2, 10]:

$$x''(t) + lx(t) + mx^3(t) + nx^5(t) = 0, \quad (2)$$

where l , m and n are real coefficients. Finding explicit exact and numerical solutions of nonlinear equations efficiently is of major importance and has widespread applications in numerical methods and applied mathematics. In this study, we will implement the variational iteration method (in short VIM) [5–8] to find exact solution and approximate solutions to the Lienard equation for a given nonlinearity.

2 Variational iteration method

The variational iteration method is proposed by the Chinese mathematician He [5–8] as a modification of a general Lagrange multiplier method [9]. It has been shown that this procedure is a powerful tool for solving various kinds of problems. To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

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$$Lu + Nu = g(x), \tag{3}$$

where L is linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term . According to the VIM [5–8], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)\{Lu_n + N\tilde{u}_n - g(\tau)\}d\tau, \tag{4}$$

where $\lambda(\tau)$ is a general Lagrangian multiplier [5–9] which can be identified optimally via the variational theory, the subscript n denotes the n th-order approximation and \tilde{u}_n is considered as a restricted variation [1], i.e. $\delta\tilde{u}_n = 0$. In the next section, the VIM has been used to study Eq. (2).

3 VIM for Lienard equation (2)

In this section, we consider VIM for Eq. (2). Using the VIM we have

$$x_{i+1}(t) = x_i(t) + \int_0^t \lambda(\tau)\{x_i''(\tau) + lx_i(\tau) + m\tilde{x}_i^3(\tau) + n\tilde{x}_i^5(\tau)\}d\tau, i \geq 0. \tag{5}$$

Where \tilde{x}_i is considered as restricted variational [1], i.e. $\delta\tilde{x}_i = 0$. To find optimal value of $\lambda(\tau)$, we have

$$\delta x_{i+1}(t) = \delta x_i(t) + \delta \int_0^t \lambda(\tau)\{x_i''(\tau) + lx_i(\tau) + m\tilde{x}_i^3(\tau) + n\tilde{x}_i^5(\tau)\}d\tau, \tag{6}$$

or

$$\delta x_{i+1}(t) = \delta x_i(t) + \delta \int_0^t \lambda(\tau)\{x_i''(\tau) + lx_i(\tau)\}d\tau = 0, \tag{7}$$

which result:

$$\delta x_{i+1}(t) = \delta x_i(t) + \delta \lambda(\tau)x_i'(\tau)|_{\tau=t} - \delta \lambda'(\tau)x_i(\tau)|_{\tau=t} + \int_0^t \{\lambda''(\tau) + l\lambda(\tau)\}x_i(\tau)d\tau = 0. \tag{8}$$

Therefore, the stationary conditions are obtained in the following form:

$$1 - \lambda'(\tau) = 0|_{\tau=t}, \tag{9}$$

$$\lambda(\tau) = 0|_{\tau=t}, \tag{10}$$

$$\lambda''(\tau) + l\lambda(\tau) = 0|_{\tau=t}. \tag{11}$$

Using above stationary conditions, we can find several values for $\lambda(\tau)$, such as:

$$\lambda(\tau) = \tau - t, \tag{12}$$

$$\lambda(\tau) = \frac{1}{\sqrt{-l}} \sinh(\sqrt{-l}(\tau - t)), \tag{13}$$

$$\lambda(\tau) = \sin(\tau) \cos(t) - \cos(\tau) \sin(t) = \sin(\tau - t). \tag{14}$$

Substituting each of above values (12, 13, 14) instead of $\lambda(\tau)$ into functional (5), we can obtain a new iteration formula, i.e: substituting (12) into (5), we can obtain the iteration formula

$$x_{i+1}(t) = x_i(t) + \int_0^t (\tau - t)\{x_i'' + lx_i + mx_i^3 + nx_i^5\}d\tau, \tag{15}$$

and substituting (13) into (5), we can obtain the iteration formula

$$x_{i+1}(t) = x_i(t) + \int_0^t \frac{1}{\sqrt{-l}} \sinh(\sqrt{-l}(\tau - t))\{x_i'' + lx_i + mx_i^3 + nx_i^5\}d\tau, \tag{16}$$

and also substituting (14) into (5), we can obtain the iteration formula

$$x_{i+1}(t) = x_i(t) + \int_0^t (\sin(\tau) \cos(t) - \cos(\tau) \sin(t)) \{x_i'' + lx_i + mx_i^3 + nx_i^5\} d\tau. \quad (17)$$

Using each of above iteration formulas, we can obtain a sequence which tends to the exact solution of the equation. we should point that difference in these formula is in their convergence speed. There exists also various values for initial approximation, for above iteration formulas (15, 16, 17), such as:

$$x_0(t) = x(0) + x'(0)t, \quad (18)$$

$$x_0(t) = x(0) \cos(\sqrt{-l}t) + x'(0) \sin(\sqrt{-l}t). \quad (19)$$

In this work, we consider the iteration formula (15) with initial approximation (18). similarly another forms can be used, but they don't give us the better results.

4 Implementation of the method

In this section, we will be concerned with the general initial values of the Lienard equation (2)

$$x(0) = \sqrt{\frac{-2l}{m}}, \quad x'(0) = -\frac{\sqrt{-l}}{\sqrt{\frac{-2l}{m}m}}, \quad (20)$$

where m and l are arbitrary constants.

We start with initial approximation $x_0(t) = x(0) + x'(0)t = \sqrt{\frac{-2l}{m}} - \frac{lt}{\sqrt{2m}}$, and using the iteration formula (15) and by MATLAB software, we can obtain

$$\begin{aligned} x_1(t) &= \frac{1}{1680} \sqrt{2} [1680 \sqrt{-lm} - 840l \sqrt{m^5}t + (-3360nl + 840m^2)l \sqrt{-lmt^2} \\ &+ (2800nl - 700m^2)l^2 \sqrt{mt^3} + (1400nl - 210m^2)l^2 \sqrt{-lmt^4} + (-420nl + 21m^2)l^3 \sqrt{mt^5} \\ &- 70nl^4 \sqrt{-lmt^6} + 5nl^5 \sqrt{mt^7}] / m^3 \\ x_2(t) &= \frac{\sqrt{2} \sqrt{-lm}}{m} - \frac{1}{2} \frac{\sqrt{2l}}{\sqrt{m}} t - \left(-\frac{1}{2} + 2 \frac{nl}{m^2}\right) \frac{\sqrt{2l} \sqrt{-lm}}{m} t^2 \\ &+ \left(-\frac{5}{12} + \frac{5}{3} \frac{nl}{m^2}\right) \frac{\sqrt{2l^2}}{\sqrt{m}} t^3 + \left(\frac{1}{12} - \frac{5}{6} \frac{nl}{m^2} + \frac{10}{3} \frac{n^2 l^2}{m^4}\right) \frac{\sqrt{2l^2} \sqrt{-lm}}{m} t^4 \\ &+ \left(-\frac{29}{120} + \frac{131}{60} \frac{nl}{m^2} - \frac{17}{3} \frac{n^2 l^2}{m^4}\right) \frac{\sqrt{2l^3}}{\sqrt{m}} t^5 \\ &+ \left(-\frac{19}{240} + \frac{337}{360} \frac{nl}{m^2} - \frac{132}{105} \frac{n^2 l^2}{m^4} - \frac{16}{3} \frac{n^3 l^3}{m^6}\right) \frac{\sqrt{2l^3} \sqrt{-lm}}{m} t^6 \\ &+ \left(-\frac{29}{672} + \frac{631}{1008} \frac{nl}{m^2} - \frac{197}{42} \frac{n^2 l^2}{m^4} + \frac{760}{63} \frac{n^3 l^3}{m^6}\right) \frac{\sqrt{2l^4}}{\sqrt{m}} t^7 \\ &+ \left(-\frac{607}{13440} + a \frac{nl}{m^2} + b \frac{n^2 l^2}{m^4} + \frac{572}{63} \frac{n^3 l^3}{m^6} + \frac{40}{7} \frac{n^4 l^4}{m^8}\right) \frac{\sqrt{2l^4} \sqrt{-lm}}{m} t^8 \\ &+ \left(\frac{457}{34560} - \frac{661}{1890} \frac{nl}{m^2} + \frac{14003}{9072} \frac{n^2 l^2}{m^4} + \frac{160}{81} \frac{n^3 l^3}{m^6} - \frac{140}{9} \frac{n^4 l^4}{m^8}\right) \frac{\sqrt{2l^5}}{\sqrt{m}} t^9 \\ &+ \left(-\frac{97}{43200} + \frac{10877}{100800} \frac{nl}{m^2} - \frac{4817}{3024} \frac{n^2 l^2}{m^4} + \frac{758}{81} \frac{n^3 l^3}{m^6} - \frac{488}{27} \frac{n^4 l^4}{m^8} - \frac{32}{9} \frac{n^5 l^5}{m^{10}}\right) \frac{l^5 \sqrt{2} \sqrt{-lm}}{m} t^{10} + \dots \\ &\vdots \end{aligned}$$

and so on. In the same manner the rest of components can be obtained using the iteration formula (15) and by MATLAB software. Therefore, the solution of $x(t)$ in a closed form is readily found to be [2].

$$x(t) = \sqrt{\frac{-2l(1 + \tanh(\sqrt{-l}t))}{m}}. \tag{21}$$

This result can be verified of the Taylor series of $x(t)$.

The numerical results of this example is given in Table 1. In the numerical calculation, $x_i(t)$ ($i = 1, 2$) is approximation of $x(t)$ which is calculated by the recurrence formula (15) for the values of $m = 4, n = -3, l = -1, t = 0.1(0.1)1$. By the variational iteration method, one can get a better result by calculating more terms of the sequence $\{x_i\}$.

Table 1. The numerical results for x_1 and x_2 in comparison with the exact solution of x

t	$ x(t) - x_1(t) $	$ x(t) - x_2(t) $
0.1	8.8312e-007	3.3588e-010
0.2	1.2932e-005	1.7493e-008
0.3	5.6140e-005	1.4734e-007
0.4	1.3721e-004	5.3415e-007
0.5	2.1031e-004	1.0564e-006
0.6	1.2368e-004	1.0878e-006
0.7	4.4551e-004	7.6947e-007
0.8	2.0649e-003	6.6036e-006
0.9	5.6264e-003	4.5760e-005
1.0	1.2430e-002	1.9575e-004

In the second example, we will consider the Lienard equation (2) with the initial conditions

$$x(0) = \sqrt{\frac{K}{2 + D}}, \tag{22}$$

where

$$K = 4\sqrt{\frac{3l^2}{3m^2 - 16nl}}, \quad D = -1 + \frac{\sqrt{3}m}{\sqrt{3m^2 - 16nl}},$$

where m and l are arbitrary constants.

We start with initial approximation $x_0(t) = x(0) + x'(0)t = \sqrt{\frac{K}{2+D}}$, and using the iteration formula (15) and by software MATLAB software, we can obtain:

$$\begin{aligned} x_1(t) &= \sqrt{\frac{K}{2 + D}} - \frac{1}{2} \sqrt{\frac{K}{2 + D}} \left[l + \frac{mK}{2 + D} + \frac{nK^2}{(2 + D)^2} \right] t^2, \\ x_2(t) &= \sqrt{\frac{K}{2 + D}} - \frac{1}{2} \sqrt{\frac{K}{2 + D}} \left[l + \frac{mK}{2 + D} + \frac{nK^2}{(2 + D)^2} \right] t^2 - \frac{1}{12} \sqrt{\frac{K}{2 + D}} \left[-\frac{1}{2} l^2 - 2 \frac{mlK}{2 + D} \right. \\ &\quad \left. - \frac{1}{2} \frac{nlK^2 + 3m^2K^2}{(2 + D)^2} - \frac{3}{2} \frac{mnK^3}{(2 + D)^3} - \frac{5}{2} \frac{nK^2A}{(2 + D)^4} \right] t^4 - \frac{1}{30} \sqrt{\left(\frac{K}{2 + D}\right)^3} \left[-\frac{1}{8} m(-2B^2 + B(-4l - 4 \frac{mK}{2 + D} \right. \\ &\quad \left. - 4 \frac{nK^2}{(2 + D)^2})) + \frac{5}{2} \frac{nK^2A}{(2 + D)^5} \right] t^6 + \frac{1}{56} \sqrt{\left(\frac{K}{2 + D}\right)^3} \left[\frac{1}{8} mB^3 + \frac{5}{4} \frac{nKA^3}{(2 + D)^7} \right] t^8 \\ &\quad - \frac{1}{288} \sqrt{\left(\frac{K}{2 + D}\right)^5} \frac{nA^4}{(2 + D)^8} t^{10} + \frac{1}{4224} \sqrt{\left(\frac{K}{2 + D}\right)^5} \frac{nA^5}{(2 + D)^{10}} t^{12}, \end{aligned}$$

where

$$A = 4l + 4lD + lD^2 + 2mK + mKD + nK^2, \quad B = l + \frac{mK}{2 + D} + \frac{nK^2}{(2 + D)^2},$$

and so on. In the same manner the rest of components can be obtained using the iteration formula (15) and by MATLAB software. Therefore, the solution of $x(t)$ in a closed form is readily found to be [2].

$$x(t) = \sqrt{\frac{K \operatorname{sech}^2(\sqrt{-l}t)}{2 + D \operatorname{sech}^2(\sqrt{-l}t)}},$$

where

$$K = 4 \sqrt{\frac{3l^2}{3m^2 - 16nl}}, \quad D = -1 + \frac{\sqrt{3}m}{\sqrt{3m^2 - 16nl}}.$$

This result can be verified of the Taylor series of $x(t)$.

The numerical results of this example is given in Table 2. In the numerical calculation, $x_i(t)$ ($i = 1, 2$) is approximation of $x(t)$ which is calculated by the recurrence formula (15) for the values of $m = 4$, $n = 3$, $l = -1$, $t = 0.1(0.1)1$. By the variational iteration method, one can get a better result by calculating more terms of the sequence $\{x_i\}$.

Table 2. The numerical results for x_1 and x_2 in comparison with the exact solution of x

t	$ x(t) - x_1(t) $	$ x(t) - x_2(t) $
0.1	2.0441e-005	4.4279e-008
0.2	3.2151e-004	2.7277e-006
0.3	1.5829e-003	2.9172e-005
0.4	4.8181e-003	1.5024e-004
0.5	1.1232e-002	5.1335e-004
0.6	2.2079e-002	1.3428e-003
0.7	3.8546e-002	2.9032e-003
0.8	6.1676e-002	5.4296e-003
0.9	9.2328e-002	9.0425e-003
1.0	1.3118e-001	1.3667e-002

5 Conclusions

In this paper, variational iteration method was employed successfully for solving the Lienard equation. The exact solutions are compared with the numerical solutions in Tab. 1 and Tab. 2. Tab. 1 and Tab. 2 show that we will achieve a very good approximation to the actual solution by using only few steps of the iteration formula (15). it is evident that the overall errors can be made smaller by calculating more terms of the sequence $\{x_i\}$. The results show that the variational iteration method is a powerful mathematical tool for finding the exact and approximate solutions of nonlinear equations.

In final, all results obtained by MATLAB software.

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