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On the stability of nonnegative solutions to classes of p-Laplacian systems

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Abstract. The purpose of this paper is to investigate stability properties for non-trivial non-negative stationary solutions to the classes of systems of the form

$$\begin{cases} -\triangle_{p_i} u_i(x) = f_i(u_1, u_2, ..., u_n) & x \in \Omega, \\ Bu_i(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\triangle_s$ denotes the s-Laplacian operator defined by

$$\triangle_s z := div(|\nabla z|^{s-2}\nabla z);\ s > 1,$$

where $\Omega \subset R^n (n \geq 1)$ is a bounded domain having a smooth boundary $Bz(x) = \alpha h(x)z + (1-\alpha)\frac{\partial z}{\partial n}$ where $\alpha \in [0, 1]$ is a constant, $h : \partial\Omega \to R^+$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, and $f_i \in C^1[0, +\infty)$ for $i = 1, ..., n$. Then we infer stability(instability) in the case when system is cooperative and strictly coupled ($\frac{\partial f_i}{\partial u_j} \geq 0, i \neq j, \sum_{j=1, j\neq i}^n (\frac{\partial f_i}{\partial u_j})^2 > 0$) and competitive and strictly coupled ($\frac{\partial f_i}{\partial u_j} \leq 0, i \neq j, \sum_{j=1, j\neq i}^n (\frac{\partial f_i}{\partial u_j})^2 > 0$).

Keywords: stability of nonnegative solutions, p-Laplacian systems

1 Introduction

In this article, we address the stability of non-trivial non-negative stationary solutions of the boundary value problem

$$-\triangle_{p_i} u_i(x) = f_i(u_1, u_2, ..., u_n) \qquad x \in \Omega, \tag{1}$$

$$Bu_i(x) = 0 \qquad x \in \partial\Omega, \tag{2}$$

for $i = 1, ..., n$, where $\triangle_s$ denotes the s-Laplacian operator defined by $\triangle_s z := div(|\nabla z|^{s-2}\nabla z);\ s > 1$. For $i = 1, ..., n$, $f_i : [0, +\infty)^n \to R$ are C^1 functions and Ω is a bounded region in $R^n; n \geq 1$ having smooth boundary

$$Bz(x) = \alpha h(x)z(x) + (1-\alpha)\frac{\partial z(x)}{\partial n}, \tag{3}$$

here B is a boundary operator and $(\frac{\partial}{\partial n})$ denotes the outward conormal derivative, $\alpha \in [0, 1]$ is a constant, $h : \partial\Omega \to R^+$ is a smooth function with $h \equiv 1$ when $\alpha = 1$, i.e; the boundary condition may be of Dirichlet, Neumann or mixed type (robin boundary condition).

This problem concerning a single equation (n=1) was studied by several authors (see [7, 10]). In [7], authors gave a simple proof of the results. This results can be summed up in the theorem below.

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Theorem 1. *Let $n = 1$ and $f : [0, +\infty) \to R$ is a continuous function with the weight $m(x)$, then*
(1). $\frac{f(u)}{u^{(p-1)}}$ be strictly increasing (decreasing) and $m(x) > 0$ for all $x \in \Omega$, then every non-trivial non-negative stationary solution is linearly unstable(stable), while
(2). $\frac{f(u)}{u^{(p-1)}}$ be strictly decreasing (increasing) and $m(x) < 0$ for all $x \in \Omega$, then every non-trivial non-negative stationary solution is linearly unstable(stable).

Concerning systems with Dirichlet boundary and $p_i = 2$, both cooperative and competitive, Castro, Chhetri and Shivaji established sufficient conditions on the nonlinearity for the solutions to be stable and unstable (see [1]). Also in [9] stability of non-negative stationary solutions of symmetric cooperative semi-linear systems with $p_i = 2$ and some convex (resp. concave) nonlinearity condition was studied. In the case $p_i = 2, n = 1$, problem has been extensively studied (refer to [2,4,6,8,11,12]). In this article, we will show a generalization of theorem 1 and theorems in [1] to a system of equations given in (1) and (2).

As a reminder, if $u = (u_1, u_2, ..., u_n)$ be every non-trivial non-negative stationary solution of system (1) and (2), then the linearized equation about u consists of

$$-(p_i - 1)div(|\nabla u_i|^{p_i-2}\nabla w_i) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}(u)w_j(x) = \eta w_i(x) \qquad x \in \Omega, \tag{4}$$

$$Bw_i(x) = 0 \qquad x \in \partial\Omega, \tag{5}$$

for $i = 1, ..., n$, where above Equation obtained from the formal derivative of the operator $\triangle_s$.

Definition 1. *Let η_1 denote the first eigenvalue of (1) and (2). $u = (u_1, u_2, ..., u_n)$ will be called a linearly stable solution if $\eta_1 > 0$. Otherwise u is called linearly unstable.*

Now we state our results as follows:

Theorem 2. *(i) Suppose for $i = 1, ..., n$, $\sum_{j=1}^{n} u_j(x)\frac{\partial f_j}{\partial u_i}(u) < (p_i - 1)f_i(u)$, for all $u := (u_1, u_2, ..., u_n)$; $u_k \geq 0$, $k = 1, ..., n$. Then any non-trivial non-negative stationary solution of (1) and (2) is linearly stable.*
(ii) suppose for $i = 1, ..., n$, $\sum_{j=1}^{n} u_j(x)\frac{\partial f_j}{\partial u_i}(u) > (p_i - 1)f_i(u)$, for all $u := (u_1, u_2, ..., u_n)$; $u_k \geq 0$, $k = 1, ..., n$. Then any non-trivial non-negative stationary solution of (1) and (2) is linearly unstable.

Theorem 3. *(i) Let system be cooperative, and for $i = 1, ..., n$, $f_i(u)/u_i^{p_i-1}$ with respect to u_i be strictly increasing, then every non-trivial non-negative stationary solution of (1) and (2) is linearly unstable.*
(i) Let system be competitive, and for $i = 1, ..., n$, $f_i(u)/u_i^{p_i-1}$ with respect to u_i be strictly decreasing, then every non-trivial non-negative stationary solution of (1) and (2) is linearly stable.

The purpose of this paper is to prove Theorem 2 and 3 by analyzing the linearized equation.

2 Proofs of theorems

Proof of Theorem 2. Let $u := (u_1, u_2, ..., u_n)$ be every non-trivial non-negative stationary solution of (1) and (2), and $(\psi_1, ..., \psi_n)$ a corresponding eigenfunction that satisfying $\psi_i \geq 0$ for $i = 1, ..., n$ (in [5] sufficient conditions are given for the nonnegativity of the first eigenfunction). Calculate $(1)(p_i - 1)\psi_i - (4)u_i$ for each $i = 1, ..., n$, and integrate parts over Ω yields and add results,

$$(-\eta_1)\int_\Omega u_i(x)\psi_i(x)dx = (p_i - 1)\int_\Omega \{u_i(x)div(|\nabla u_i|^{p_i-2}\nabla\psi_i) - \psi_i(x)div(|\nabla u_i|^{p_i-2}\nabla u_i)\}dx$$

$$+\int_\Omega \sum_{j=1}^{n} u_i(x)\frac{\partial f_i}{\partial u_j}(u)\psi_j(x)dx - (p_i - 1)\int_\Omega f_i(u)\psi_i(x)dx. \tag{6}$$

Now we use the Green identity, which applies that second term be zero. Because

$$\int_\Omega \{u_i div(|\nabla u_i|^{p_i-2}\nabla\psi_i) - \psi_i div(|\nabla u_i|^{p_i-2}\nabla u_i)\}dx = \int_{\partial\Omega} |\nabla u_i|^{p_i-2}\{u_i\frac{\partial\psi_i}{\partial n} - \psi(s)\frac{\partial u_i}{\partial n}\}ds,$$

on the other hand when $\alpha = 1$, using the boundary condition, we obtain $u_i = 0$ and $\psi_i = 0$ for each $s \in \partial\Omega$ and when $\alpha \neq 1$, then

$$\int_{\partial\Omega} |\nabla u_i|^{p_i-2}\{u_i\frac{\partial\psi_i}{\partial n} - \psi(s)\frac{\partial u_i}{\partial n}\}ds = \int_{\partial\Omega} |\nabla u_i|^{p_i-2}\{\frac{\alpha h(s)\psi(s)}{(1-\alpha)}\}(u_i - u_i)ds = 0,$$

therefore

$$(-\eta_1)\int_\Omega u_i(x)\psi_i(x)dx = \int_\Omega \sum\nolimits_{j=1}^{n} u_i(x)\frac{\partial f_i}{\partial u_j}(u)\psi_j(x)dx - (p_i - 1)\int_\Omega f_i(u)\psi_i(x)dx, \tag{7}$$

now add (7) over $i = 1, ..., n$, then rearrange the terms to get

$$(-\eta_1)\int_\Omega u_i(x)\psi_i(x)dx = \int_\Omega \sum\nolimits_{i=1}^{n} \psi_i(x)\{(\sum\nolimits_{j=1}^{n} u_j(x)\frac{\partial f_j}{\partial u_i}(u)) - (p_i - 1)f_i(u)\}dx.$$

For $i = 1, ..., n$, $u_i > 0$ and $\psi_i > 0$ in Ω, so we conclude η_1 is positive in case (i) and negative in (ii). This completes the proof of theorem 2.

Proof of Theorem 3. As before let us $u := (u_1, u_2, ..., u_n)$ be any non-trivial non-negative stationary solution of (1) and (2), and use of the same way as for theorem 2 to get (7), and add (7) over $i = 1, ..., n$, but one can rearrange on the other way, that is

$$\int_\Omega \sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1, j\neq i}^{n} u_i(x)\frac{\partial f_i}{\partial u_j}(u)\psi_j(x)dx + \int_\Omega \sum\nolimits_{i=1}^{n} \psi_i(x)[u_i(x)\frac{\partial f_i}{\partial u_i}(u) - (p_i - 1)f_i(u)]dx$$

$$= (-\eta_1)\int_\Omega u_i(x)\psi_i(x)dx.$$

On the other hand for $i = 1, 2, ..., n$, $u_i(x)\frac{\partial f_i}{\partial u_i}(u) - (p_i - 1)f_i(u)$ are positive in the case (i) and are negative in the case (ii), by using the following formula

$$\frac{d}{du_i}(\frac{f_i(u_i)}{u_i^{p_i-1}})(u) = \frac{u_i(x)\frac{\partial f_i}{\partial u_i}(u) - (p_i - 1)f_i(u)}{u_i^{p_i}} \qquad (\forall u \in R^+).$$

But because of for all $i = 1, 2, ..., n$, $u_i > 0$ and $\psi_i > 0$ in Ω, thus we have $\eta_1 < 0$ in the case (i) and so u is linearly unstable and $\eta_1 > 0$ in the case (ii) and so u is linearly stable.

Remark 1. This results even could be easily extended to the general elliptic operator of the form $Lu := div(A\nabla u)$, that is $(Lu)_i = div(A_i\nabla u_i)$ as in [11], where all $A_i : \Omega \to R^{n\times n}$ are symmetric and uniformly positive definite.

3 Application

Consider

$$\begin{cases} -\triangle_p u(x) = \lambda u^\alpha v^\gamma & x \in \Omega, \\ -\triangle_q v(x) = \lambda u^\delta v^\beta & x \in \Omega, \\ Bu = 0 = Bv & x \in \partial\Omega, \end{cases} \tag{I}$$

where λ is positive parameter and Ω is as defined before, $\alpha, \beta \geq 0$, $\gamma, \delta > 0$. Readers are referred to [3] where the authors concerned with the existence and nonexistence of positive solution of system (I). It is easy to see that if $\alpha + 1 < p$ and $\beta + 1 < q$, then by theorem 2, every non-trivial non-negative stationary solution of (I) is linearly unstable. The same result is easily obtained by theorem 3.

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