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Nonlinear elliptic involving the p-Laplacian problems and stability of positive solutions

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Abstract. This paper deal with the stability of positive stationary solutions of boundary value problem of the form

$$\begin{cases} -\triangle_p u^m(x) = \lambda a(x) f(u(x)), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $\triangle_s z = div(|\ \nabla z\ |^{s-2}\ \nabla z)$ is the p-Laplacian operator; $s > 2$, $\lambda > 0$ is a parameter, $m > 1$, a(x) is a weight function that satisfies $a(x) \in C(\bar{\Omega})$ for all $x \in \Omega$. Here $f : [0, +\infty) \to R^-(R^+)$ is C^1 function. We establish if $\frac{f(u)}{u^{m(p-1)}}$ be strictly decreasing (increasing) then every positive stationary solution is stable.

Keywords: nonlinear elliptic, p-Laplacian problems, stability of positive solutions

1 Section heading

We will Consider the following p-Laplacian equation:

$$-\triangle_p u^m(x) = \lambda a(x) f(u(x)), \qquad x \in \Omega \tag{1}$$

$$u(x) = 0, \qquad x \in \partial\Omega \tag{2}$$

where $\triangle_s z = div(|\ \nabla z\ |^{s-2}\ \nabla z)$ denotes the p-Laplacian operator; $s > 2$ and λ is a positive parameter, $m > 1$ is a constant, $\Omega \subset R^N (N \geq 1)$ is a bounded region with smooth boundary $\partial\Omega$, weight a(x) satisfies $a(x) \in C(\bar{\Omega})$ and we assume that either $a(x) > 0$ or $a(x) < 0$ for all $x \in \Omega$.

Existence of solutions in this type of models (the case system, $p = 2$) have been studied in [3]. A large number of works have been made studying stability of solutions in the case when $m = 1$ (see [1, 2, 4, 5]). In [1], authors have shown that every non-negative stationary solution of boundary value problem

$$\begin{cases} -\triangle_p u(x) = \lambda g(x) f(u(x)), & x \in \Omega \\ Bu(x) = 0, & x \in \partial\Omega \end{cases}$$

is linearly unstable (stable). But in this article, we ponder the case when $m > 1$ and establish stability of positive stationary solutions of (1) and (2), under certain conditions.

It is well known that if u be any positive solution of (1) and (2), then the linearized equation about u consists of

$$-m(p-1)div(|\nabla u^m|^{p-2}\nabla(u^{m-1}\phi)) - \lambda a(x) f^{'}(u)\phi = \mu\phi, \qquad x \in \Omega \tag{3}$$

$$\phi(x) = 0, \qquad x \in \partial\Omega \tag{4}$$

where equation (3) obtained from the formal derivative of the operator $\triangle_p u^m$ with respect to u.

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Definition 1. *We say a positive solution u is stable if all eigenvalues of (3) is strictly positive, which can be inferred if the principal eigenvalue μ_1 (i.e., eigenvalue corresponding to positive eigenfunction) be positive. Otherwise we say u is unstable.*

2 Main results

In this section, we prove the following Theorem by determining the sign of the principal eigenvalue, μ_1, of the linearized equation of (1) about a positive solution $\bar{u}$. Stability of solutions follows from Theorem (4.2) of [6] that if $\mu_1 > 0$, then $\bar{u}$ is stable, otherwise $\bar{u}$ is unstable.

Theorem 1. *Let $f(u)$: $[0,\infty) \to R^-(R^+)$, $a(x) > 0(a(x) < 0)$, and $\frac{f(u)}{u^{m(p-1)}}$ be strictly decreasing (increasing), then every positive stationary solution of boundary value problem (1) and (2) is linearly stable.*

Proof. Let $\bar{u}$ be any positive stationary solution of (1) and (2), then the linearized equation about $\bar{u}$ consists of

$$-m(p-1)div(|\nabla\bar{u}^m|^{p-2}\nabla(\bar{u}^{m-1}\phi)) - \lambda a(x)f^{'}(\bar{u})\phi = \mu\phi, \qquad x \in \Omega \tag{5}$$

$$\phi(x) = 0, \qquad x \in \partial\Omega \tag{6}$$

presume that μ_1 is the principal eigenvalue and also $\psi(x)(\psi \geq 0)$ is a corresponding eigenfunction. Now multiplying (5) by $-\bar{u}$, (1) by $m(p-1)\psi$, add integrating by parts over Ω and add the two resulting expressions to get

$$-\mu_1\int_\Omega \psi(x)\bar{u}(x)dx = m(p-1)\int_\Omega[\bar{u}div(|\nabla\bar{u}^m|^{p-2}\nabla(\bar{u}^{m-1}\psi)) - \psi div(|\nabla\bar{u}^m|^{p-2}\nabla\bar{u}^m)]dx$$
$$+\lambda\int_\Omega \psi(x)a(x)[\bar{u}f'(\bar{u}) - m(p-1)f(\bar{u})]dx. \tag{7}$$

But by Green identity and Dirichlet boundary condition,

$$\int_\Omega \bar{u}div(|\nabla\bar{u}^m|^{p-2}\nabla(\bar{u}^{m-1}\psi))dx = \int_\Omega \bar{u}\nabla(|\nabla\bar{u}^m|^{p-2})\nabla(\bar{u}^{m-1}\psi)dx + \int_\Omega \bar{u}|\nabla\bar{u}^m|^{p-2}\triangle(\bar{u}^{m-1}\psi)dx$$
$$= -\int_\Omega |\nabla\bar{u}^m|^{p-2}\nabla\bar{u}\nabla(\bar{u}^{m-1}\psi)dx, \tag{8}$$

also similarly,

$$\int_\Omega \psi div(|\nabla\bar{u}^m|^{p-2}\nabla(\bar{u}^m))dx = -\int_\Omega |\nabla\bar{u}^m|^{p-2}\nabla\psi\nabla(\bar{u}^m)dx, \tag{9}$$

and hence

$$\int_\Omega[\bar{u}div(|\nabla\bar{u}^m|^{p-2}\nabla(\bar{u}^{m-1}\psi)) - \psi div(|\nabla\bar{u}^m|^{p-2}\nabla\bar{u}^m)]dx$$
$$= \int_\Omega |\nabla\bar{u}^m|^{p-2}(\nabla\psi\nabla\bar{u}^m - \nabla\bar{u}\nabla(\bar{u}^{m-1}\psi))dx$$
$$= (\frac{m-1}{m})\int_\Omega |\nabla\bar{u}^m|^{p-2}\nabla\psi\nabla\bar{u}^m dx$$
$$-(m-1)\int_\Omega |\nabla\bar{u}^m|^{p-2}\bar{u}^{m-2}(\nabla\bar{u})^2\psi dx, \tag{10}$$

on the other hand by Divergence theorem and since ψ is eigenfunction(thus $\psi = 0$ on $\partial\Omega$), we have

$$\int_\Omega |\nabla\bar{u}^m|^{p-2}\nabla\bar{u}^m\nabla\psi dx = \int_\Omega div(|\nabla\bar{u}^m|^{p-2}\nabla\bar{u}^m\psi)dx - \int_\Omega div(|\nabla\bar{u}^m|^{p-2}\nabla\bar{u}^m)\psi dx$$

$$= \int_{\partial\Omega} |\nabla\bar{u}^m|^{p-2}\nabla\bar{u}^m\psi.nds - \int_{\Omega} div(|\nabla\bar{u}^m|^{p-2}\nabla\bar{u}^m)\psi dx$$

$$= \lambda\int_{\Omega}\psi(x)a(x)f(\bar{u})dx, \tag{11}$$

therefore by replacing (11) in (10), (7) becomes

$$-\mu_1\int_{\Omega}\psi(x)\bar{u}(x)dx = (p-1)(m-1)\int_{\Omega}\psi[\lambda a(x)f(\bar{u}) - m|\nabla\bar{u}^m|^{p-2}\bar{u}^{m-2}(\nabla\bar{u})^2]dx$$

$$+\lambda\int_{\Omega}\psi(x)a(x)[\bar{u}f'(\bar{u}) - m(p-1)f(\bar{u})]dx. \tag{12}$$

Now observe that since $\frac{f(\bar{u})}{\bar{u}^{m(p-1)}}$ is strictly decreasing (increasing), $\bar{u} > 0$ and $a(x) > 0(< 0)$, we get

$$a(x)[\bar{u}f'(\bar{u}) - m(p-1)f(\bar{u})] < 0, \tag{13}$$

and because of $f(\bar{u}) < 0(> 0)$, $a(x) > 0(< 0)$ and $\bar{u} > 0$, we have

$$\lambda a(x)f(\bar{u}) - m|\nabla\bar{u}^m|^{p-2}\bar{u}^{m-2}(\nabla\bar{u})^2 < 0, \tag{14}$$

thus using (13) and (14) in (12) and knowing this fact that $\psi > 0$ in Ω, we obtain

$$-\mu_1\int_{\Omega}\psi(x)\bar{u}(x)dx < 0. \tag{15}$$

Hence $\mu_1 > 0$ and it show that every positive solution is stable(see[6]).

Next Theorem can be generalization of Theorem 1.

Theorem 2. *Suppose $f(x,u)$: $\Omega\times[0,\infty)\to R^-$ be a continuous function and $\frac{f(x,u)}{u^{m(p-1)}}$ be strictly decreasing for $u > 0$ at each fixed $x\in\Omega$, then every positive stationary solution of boundary value problem*

$$-\triangle_p u^m(x) = \lambda f(x,u), \qquad x\in\Omega \tag{16}$$

$$u(x) = 0, \qquad x\in\partial\Omega \tag{17}$$

is linearly stable.

Proof. The proof proceeds in the same way as for Theorem 1, i.e; let $\bar{u}$ be any positive stationary solution of (16) and (17). Instead of (12), we obtain

$$-\mu_1\int_{\Omega}\psi(x)\bar{u}(x)dx = (p-1)(m-1)\int_{\Omega}\psi[\lambda f(x,\bar{u}) - m|\nabla\bar{u}^m|^{p-2}\bar{u}^{m-2}(\nabla\bar{u})^2]dx$$

$$+\lambda\int_{\Omega}\psi(x)[\bar{u}f'(x,\bar{u}) - m(p-1)f(x,\bar{u})]dx.$$

and result follows via the appropriate analogue of Theorem 1.

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