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Existence and nonexistence of positive solutions for quasilinear elliptic systems

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Abstract. We prove existence and nonexistence and uniqueness of positive solutions

$$\begin{cases} -\Delta_p u = \lambda f(v) + \theta & in\ \Omega \\ -\Delta_q v = \mu g(u) + \beta & in\ \Omega \\ u = v = 0 & on\ \partial\Omega \end{cases}$$

where $f, g : [0, \infty) \to [0, \infty)$, Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $\Delta_q v = div(|\nabla v|^{q-2}\nabla v)$, $p, q > 1$, and $\lambda, \mu, \theta, \beta$ are positive parameters.

Keywords: positive solutions, quasilinear elliptic systems, schauder fixed point

1 Introduction

Consider the quasilinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda f(v) + \theta & in\ \Omega \\ -\Delta_q v = \mu g(u) + \beta & in\ \Omega \\ u = v = 0 & on\ \partial\Omega \end{cases} \qquad (I)$$

where $f, g : [0, \infty) \to [0, \infty)$, Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $\Delta_q v = div(|\nabla v|^{q-2}\nabla v)$, $p, q > 1$, and $\lambda, \mu, \theta, \beta$ are positive parameters.

Dalmasso[2] studied existence and uniqueness of positive solutions to (I) when $p = q = 2, \theta = \beta = 0$ and $f(cg(x))$ is sublinear at 0 and ∞ for every $c > 0$. Related results in the case when $f(0) < 0$ or $g(0) < 0$ are obtained in [5]. D. D. Hai[4] studied existence and uniqueness of positive solution to (I) when $\theta = \beta = 0$.

In this paper we are interested in the existence and uniqueness of positive solutions for the quasilinear system (I) when $\lambda^{\frac{1}{p-1}}(f(c\mu^{\frac{1}{q-1}}(g(x)+\frac{\beta}{\mu})^{\frac{1}{q-1}})+\frac{\theta}{\lambda})^{\frac{1}{p-1}}$ is sublinear at 0 and ∞ for every $c > 0$. we also show under additional assumptions that (I) has no positive solutions for all $\lambda, \mu, \theta, \beta > 0$. Our approach depends on fixed points arguments and maximum principles.

2 Main results

(H_1)**:** $f, g : [0, \infty) \to [0, \infty)$ are continuous, nondecreasing, and $g(x) > 0$ for $x > 0$.
(H_2)**:** For each $c > 0$,

$$\limsup_{x\to 0^+} \frac{\lambda^{\frac{1}{p-1}}(f(c\mu^{\frac{1}{q-1}}(g(x)+\frac{\beta}{\mu})^{\frac{1}{q-1}}+\frac{\theta}{\lambda})^{\frac{1}{p-1}}}{x} = \infty$$

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(H_3)**:** For each $c > 0$,

$$\liminf_{x\to\infty} \frac{\lambda^{\frac{1}{p-1}}(f(c\mu^{\frac{1}{q-1}}(g(x)+\frac{\beta}{\mu})^{\frac{1}{q-1}}+\frac{\theta}{\lambda})^{\frac{1}{p-1}}}{x} = 0.$$

(H_4)**:** There exist $C_1, C_2 > 0$ such that $f(v) \geq C_1 v^{p-1},\ \ g(u) \geq C_2 u^{q-1}$ for all $u, v > 0$.

Then we have

Theorem 1. *Let $(H_1) - (H_2)$ hold. then (I) has a positive solution (u, v) for all $\lambda, \mu > 0$.*

Theorem 2. *Let f, g satisfy (H_1). Suppose that there exist positive numbers r, s with $rs < (p-1)(q-1)$ such that*

$$\frac{f(x)+\theta}{x^r} \quad \textit{and} \quad \frac{g(x)+\beta}{x^s}$$

are nonincreasing for $x \geq 0$. then (I) has at most one positive solution.

Theorem 3. *Suppose that (H_4) hold, then there exist positive number $\lambda^*, \mu^* > 0$ such that (I) has no positive solution for $\lambda > \lambda^*, \mu > \mu^*$.*

Let ϕ, ψ, satisfy

$$-\Delta_p \phi = 1 \ \ in \ \ \Omega, \ \ \phi = 0 \ \ on \ \ \partial\Omega \tag{1}$$

$$-\Delta_q \phi = 1 \ \ in \ \ \Omega, \ \ \psi = 0 \ \ on \ \ \partial\Omega \tag{2}$$

Let D be a sub-domain of Ω with $\bar{D} \subset \Omega$. Let

$$h(x) = \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases}$$

and let $\tilde{\phi}, \tilde{\psi}$ be the solution of

$$\begin{cases} -\Delta_p \tilde{\phi} = h & in\ \Omega \\ \tilde{\phi} = 0 & on\ \Omega \end{cases}$$

and

$$\begin{cases} -\Delta_p \tilde{\psi} = h & in\ \Omega \\ \tilde{\psi} = 0 & on\ \Omega \end{cases}$$

respectively. By the strong maximum principle (see [8]), there exist positive numbers M, m such that $\tilde{\phi} \geq M\phi$ in Ω and $\phi, \psi, \tilde{\phi}, \tilde{\psi} \geq m$ in $\bar{D}$.

Without loss of generality, we assume that $\lambda = \mu = 1$ in the proof of theorems 1 and 2.

Proof of theorem 1. By (H_2), there exist $\varepsilon \in (0, 1)$ such that

$$M(f(m(g(\varepsilon m)+\beta)^{\frac{1}{q-1}}+\theta)^{\frac{1}{p-1}} \geq \varepsilon$$

for each $\omega \in C(\Omega)$, let $u = T\omega$ be the solution of

$$\begin{cases} -\Delta_p u = f(v) + \theta & in\ \Omega \\ -\Delta_q v = g(\max(\omega, \varepsilon\phi)) + \beta & in\ \Omega \\ u = v = 0 & on\ \Omega \end{cases}$$

Then $T : C(\bar{\Omega}) \to C(\bar{\Omega})$ is completely continuous (see e.g. [3, 4]). By (H_3), there exist a number $R > |\phi|_\infty$ such that

$$(f(|\psi|_\infty(g(R)+\beta)^{\frac{1}{q-1}})+\theta)^{\frac{1}{p-1}}|\phi|_\infty \leq R.$$

We claim that $T:\bar{B}(0,R)\to\bar{B}(0,R)$, where $\bar{B}(0,R)$ denotes the closed ball centered at 0 with radius R in $C(\bar{\Omega})$. Indeed, let $\omega\in C(\bar{\Omega})$ with $|\omega|_\infty\leq R$. Then we have

$$-\Delta_q v = g(max(\omega,\varepsilon\phi))+\beta \geq g(R)+\beta \qquad in\ \ \Omega$$

which implies by the maximum principles that

$$v \leq (g(R)+\beta)^{\frac{1}{q-1}}\psi.$$

Thus

$$-\Delta_p u = f(v)+\theta \leq f((g(R)+\beta)^{\frac{1}{q-1}}\psi)+\theta \leq f(|\psi|_\infty(g(R)+\beta)^{\frac{1}{q-1}})+\theta$$

and therefore

$$u \leq (f(|\psi|_\infty(g(R)+\beta)^{\frac{1}{q-1}})+\theta)^{\frac{1}{p-1}}\phi$$

consequently

$$|u|_\infty \leq (f(|\psi|_\infty(g(R)+\beta)^{\frac{1}{q-1}})+\theta)^{\frac{1}{p-1}}|\phi|_\infty \leq R$$

proving the claim.

$$-\Delta_q v = g(max(u,\varepsilon\phi))+\beta \geq \begin{cases} g(\epsilon m)+\beta & x\in D \\ 0 & x\in \Omega\backslash D \end{cases}$$

By the Schauder fixed theorem, T has a fixed point u with $|u|_\infty\leq R$. Next we verify that $u\geq\varepsilon\phi$. Since

$$-\Delta_q v = g(max(u,\varepsilon\phi))+\beta \geq \begin{cases} g(\varepsilon m) & x\in D \\ 0 & x\in \Omega\backslash D \end{cases}$$

It follows from the maximum principle that

$$v \geq (g(\varepsilon m)+\beta)^{\frac{1}{q-1}}\tilde{\psi}.$$

Using this in the equation for u gives

$$-\Delta_p u = f(v)+\theta \geq \begin{cases} f(m(g(\varepsilon m)+\beta)^{\frac{1}{q-1}})+\beta & x\in D \\ 0 & x\in \Omega\backslash D \end{cases}$$

and therefore

$$u \geq (f(m(g(\varepsilon m)+\beta)^{\frac{1}{q-1}}+\theta)^{\frac{1}{p-1}}\tilde{\phi} \geq M(f(m(g(\varepsilon m)+\beta)^{\frac{1}{q-1}}+\theta)^{\frac{1}{p-1}}\phi \geq \varepsilon\phi.$$

Since $g(x)>0$ for $x>0$, we have $v>0$ in Ω by the strong maximum principle. This completes the proof of theorem 1.

Proof of theorem 2. Let (u,v) and (u_1,v_1) be positive solutions of (I). As in [1, 8], we define $\delta = sup\{\varepsilon > 0 : v\geq\varepsilon v_1\ \ in\ \ \Omega\}$. Then $v\geq\delta v_1$. If $\delta<1$, then we have

$$-\Delta_p u = f(v)+\theta \geq f(\delta v_1)+\theta \geq \delta^r(f(v_1)+\theta).$$

Since

$$-\Delta_p(\delta^{\frac{r}{p-1}}u_1) = \delta^r(f(v_1)+\theta),$$

it follows that

$$u \geq \delta^{\frac{r}{p-1}} u_1.$$

Using this in the equation for v gives

$$-\Delta_q v = g(u) + \beta \geq g(\delta^{\frac{r}{p-1}} u_1) + \beta \geq \delta^{\frac{rs}{p-1}}(g(u_1) + \beta),$$

which implies

$$v \geq \delta^{\frac{rs}{(p-1)(q-1)}} v_1,$$

a contradiction with the definition of δ. Thus $\delta = 1$, i,e, $v \geq v_1$, and so $v = v_1$, $u = u_1$, proving theorem 2.
Proof of theorem 3. Suppose that (I) has a positive solution (u, v) for every $\lambda, \mu > 0$. By (H_3) then we have

$$-\Delta_p u = \lambda f(v) + \theta \geq \lambda C_1 v^{p-1}, \quad -\Delta_q v = \mu g(u) + \beta \geq \mu C_2 u^{q-1}.$$

Hence

$$\frac{-\Delta_p u}{v^{p-1}} \geq \lambda C_1, \quad \frac{-\Delta_q v}{u^{q-1}} \geq \mu C_2.$$

Or

$$-\, div(\frac{|\nabla u|^{p-2}\nabla u}{v^{p-1}}) \geq \lambda C_1, \tag{3}$$

$$-\, div(\frac{|\nabla v|^{q-2}\nabla v}{u^{q-1}}) \geq \mu C_2. \tag{4}$$

Let $B = B(x_1, \frac{r}{2})$ and $\phi \in C_0^\infty(B)$ with $\phi \geq 0$ on B and $\phi \neq 0$. Multiplying (3), (4) by ϕ, and integrating, we obtain

$$\int_B \frac{|\nabla u|^{p-2}\nabla u.\nabla\phi}{v^{p-1}} dx \geq \lambda C_1 \int_B \phi dx,$$

$$\int_B \frac{|\nabla v|^{q-2}\nabla v.\nabla\phi}{u^{q-1}} dx \geq \mu C_2 \int_B \phi dx.$$

By Holder's inequality,

$$\int_B \frac{|\nabla u|^{p-2}\nabla u.\nabla\phi}{v^{p-1}} dx \leq \int_B \frac{|\nabla u|^{p-1}|\nabla\phi|}{v^{p-1}} dx \leq \frac{p-1}{p}\int_B \frac{|\nabla u|^p}{v^p} dx + \frac{1}{p}\int_B |\nabla\phi|^p dx, \tag{5}$$

$$\int_B \frac{|\nabla v|^{q-2}\nabla v.\nabla\phi}{u^{q-1}} dx \leq \int_B \frac{|\nabla v|^{q-1}|\nabla\phi|}{u^{q-1}} dx \leq \frac{q-1}{q}\int_B \frac{|\nabla v|^q}{u^q} dx + \frac{1}{q}\int_B |\nabla\phi|^q dx. \tag{6}$$

Combining (3),(5) and (4), (6) we get

$$\frac{p-1}{p}\int_B \frac{|\nabla u|^p}{v^p} dx \geq \lambda C_1 \int_B \phi dx - \frac{1}{p}\int_B |\nabla\phi|^p dx, \tag{7}$$

$$\frac{q-1}{q}\int_B \frac{|\nabla v|^q}{u^q} dx \mu C_2 \int_B \phi dx - \frac{1}{q}\int_B |\nabla\phi|^q dx. \tag{8}$$

Next we show that

$$\int_B \frac{|\nabla u|^p}{v^p} dx \leq K_1, \tag{9}$$

$$\int_B \frac{|\nabla v|^q}{u^q} dx \leq K_2, \tag{10}$$

where K_1, K_2 are constant independent of u, v, λ, μ.

To this ends, let $\psi \in C_0^\infty(B_1)$ with $\psi \geq 0$ on $B_1 = B(x,r)$. Multiplying (3) by ψ^p and (4) by ψ^q and integrating we obtain

$$p\int_{B_1} \frac{\psi^{p-1}|\nabla u|^{p-2}\nabla u \nabla \psi}{v^{p-1}}dx - (p-1)\int_{B_1} \frac{|\nabla u|^p \psi^p}{v^p}dx \geq \lambda C_1 \int_{B_1} \psi^p dx \geq 0,$$
$$q\int_{B_1} \frac{\psi^{q-1}|\nabla v|^{q-2}\nabla v \nabla \psi}{u^{q-1}} - (q-1)\int_{B_1} \frac{|\nabla v|^q \psi^q}{u^q} \geq \mu C_2 \int_{B_1} \psi^q dx \geq 0.$$

From this and holder's inequality, we deduce

$$(p-1)\int_{B_1} \frac{|\nabla u|^p \psi^p}{v^p}dx \leq p\int_{B_1} \frac{\psi^{p-1}|\nabla u|^{p-2}\nabla u \nabla \psi}{v^{p-1}}dx \leq \frac{p-1}{p}\int_{B_1} \frac{|\nabla u|^p \psi^p}{v^p}dx + p^{p-1}\int_{B_1} |\nabla \psi|^p dx,$$
$$(q-1)\int_{B_1} \frac{|\nabla v|^q \psi^p}{u^q}dx \leq q\int_{B_1} \frac{\psi^{q-1}|\nabla v|^{q-2}\nabla v \nabla \psi}{u^{q-1}}dx \leq \frac{q-1}{q}\int_{B_1} \frac{|\nabla v|^q \psi^q}{u^q}dx + q^{q-1}\int_{B_1} |\nabla \psi|^q dx,$$

which implies

$$\frac{(p-1)^2}{p}\int_{B_1} \frac{|\nabla u|^p \psi^p}{v^p}dx \leq p^{p-1}\int_{B_1} |\nabla \psi|^p dx$$
$$\frac{(q-1)^2}{q}\int_{B_1} \frac{|\nabla v|^q \psi^q}{u^q}dx \leq q^{q-1}\int_{B_1} |\nabla \psi|^q dx$$

and (9),(10) follows if we choose ψ so that $\psi = 1$ on B. In view of (7), (8) we reach a contradiction if λ, μ is sufficiently large. This completes the proof of theorem 3.

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