

A study on one dimensional nonlinear thermoelasticity by Adomian decomposition method

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Abstract. In this work, Adomian decomposition method is applied to solve the Cauchy problem arising in one dimensional nonlinear thermoelasticity. The numerical solutions obtained by this method are then compared with the exact solution to show the efficiency of this method. The method has been shown to effectively, easily and accurately solve a large class of nonlinear problems with approximations converging rapidly to accurate solutions.

Keywords: Adomian decomposition method, cauchy problem, nonlinear one dimensional thermoelasticity

1 Introduction

Most scientific problems and phenomena in different fields of science and engineering occur nonlinearly, especially in fluid mechanics, solid state physics, plasma physics, chemical physics, plasma waves and thermoelasticity. Except in a limited number of these problems, we encounter difficulties in finding their exact analytical solutions. Thermoelasticity problems have gained a considerable attention for their importance and applications. Much effort was paid on existence of the solution of thermoelasticity problems and various numerical techniques were used^[1, 6, 8-10, 14, 16, 18].

Recently, numerical methods, which do not require discretization of space-time variables or linearization of the nonlinear equations, are introduced for finding analytical solutions of nonlinear problems, among which Adomian decomposition method (ADM)^[2, 3, 5, 7, 12, 15, 17, 20, 21] is the most effective, convenient and accurate one for both weakly and strongly nonlinear problems. ADM is to split the given equation into linear and nonlinear parts, invert the highest-order derivative operator contained in the linear operator in both sides, calculate Adomian's polynomials, and finally find the successive terms of the series solution by recurrent relation using Adomian's polynomials^[4, 19].

The aim of this work is to employ ADM to solve a real-life problem that exhibits coupling between the mechanical and thermal fields. Let consider the following nonlinear system arising in thermoelasticity [10, 16, 18]:

$$u_n - a(u_x, \theta)u_{xx} + b(u_x, \theta)\theta_x = f(x, t) \quad (1)$$

$$c(u_x, \theta)\theta_t + b(u_x, \theta)u_{xt} - d(\theta)\theta_{xx} = g(x, t) \quad (2)$$

subject to the initial conditions of:

$$u(x, 0) = u^0(x), u_t(x, 0) = u^1(x), \theta(x, 0) = \theta^0(x) \quad (3)$$

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where $u(x, t)$ is the body displacement from equilibrium and $\theta(x, t)$ is the difference of the body's temperature from a reference $T_0 = 0$, subscripts denote partial derivatives and a, b, c and d are given smooth functions. For more details about the physical meaning of the model, see [10, 13].

In the next section, a brief outline of ADM is explained. Then, an artificial model is used to illustrate the effectiveness of the method. Finally, the results obtained by ADM and the exact solution are compared graphically.

2 Fundamentals of Adomian decomposition method

Let us discuss a brief outline of the Adomian Decomposition method. For this, we consider a general nonlinear equation in the form:

$$Lu + Ru + Nu = g \quad (4)$$

where L is the highest order derivative which is assumed to be easily invertible, R the linear differential operator of less order than L , Nu presents the nonlinear terms and g is the source term. Applying the inverse operator L^{-1} to the both sides of Eq. (4), and using the given conditions we obtain:

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \quad (5)$$

where the function $f(x)$ represents the terms arising from integration the source term $g(x)$, using given conditions. For nonlinear differential equations, the nonlinear operator $Nu = F(u)$ is represented by an infinite series of the so-called Adomian polynomials

$$F(x) = \sum_{m=0}^{\infty} A_m \quad (6)$$

The polynomials A_m are generated for all kind of nonlinearity so that A_0 depends only on u_0 , A_1 depends on u_0 and u_1 , and so on. The Adomian polynomials introduced above show that the sum of subscripts of the components of u for each term of A_m is equal to n ^[19].

The Adomian method defines the solution $u(x)$ by the series

$$u = \sum_{m=0}^{\infty} u_m \quad (7)$$

In the case of $F(u)$, the infinite series is a Taylor expansion about u_0 , as follows:

$$F(u) = F(u_0) + F'(u_0)(u - u_0) + F''(u_0)\frac{(u - u_0)^2}{2!} + F'''(u_0)\frac{(u - u_0)^3}{3!} + \dots \quad (8)$$

By rewriting Eq. (7) as $u - u_0 = u_1 + u_2 + u_3 + \dots$, substituting it into Eq. (8) and then equating two expressions for found in Eq. (8) and (Eq. 6), defines formulas for the Adomian polynomials in the form of

$$F(u) + A_1 + A_2 + \dots = F(u_0) + F'(u_0)(u_1 + u_2 + \dots) + F''(u_0)\frac{(u_1 + u_2 + \dots)^2}{2!} + \dots \quad (9)$$

By equating terms in Eq. (9), the first few Adomian's polynomials A_0, A_1, A_2, A_3 and A_4 are given:

$$A_0 = F(u_0) \quad (10)$$

$$A_1 = u_1 F'(u_0) \quad (11)$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) \quad (12)$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) \quad (13)$$

$$A_4 = u_4 F'(u_0) + \left(\frac{1}{2!} u_1^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!} u_1^2 u_2 F'''(u_0) + \frac{1}{4!} u_1^4 F^{(iv)}(u_0) \quad (14)$$

⋮

Now that the A_m are known, Eq. (6) can be substituted in Eq. (5) to specify the terms in the expansion for the solution of Eq. (7).

3 Numerical example

In this section, we define a, b, c, d, u^0, u^1 and θ^0 by [18]:

$$a(u_x, \theta) = 2 - u_x\theta, b(u_x, \theta) = 2 + u_x\theta, c(u_x, \theta) = 1, d(u_x, \theta) = \theta \quad (15)$$

$$u^0(x) = \frac{1}{1+x^2}, u^1(x) = 0, \theta^0(x) = \frac{1}{1+x^2} \quad (16)$$

And replace the right-hand side of Eqs. (1) and (2) by

$$f(x, t) = \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3}a(w, v) - \frac{2x(1+t)}{(1+x^2)^2}b(w, v) \quad (17)$$

$$g(x, t) = \frac{1}{1+x^2}c(w, v) - \frac{4xt}{(1+x^2)^2}b(w, v) - \frac{2(1+t)(3x^2-1)}{(1+x^2)^3}d(w, v) \quad (18)$$

where a, b, c and d are defined by Eq. (15) and

$$w = w(x, t) = -\frac{2x(1+t^2)}{(1+x^2)^2}, v = v(x, t) = \frac{1+t}{1+x^2} \quad (19)$$

The exact solution of Eqs. (1) and (2) is given by [18]:

$$u(x, t) = \frac{(1+t^2)}{1+x^2}, \theta(x, t) = \frac{1+t}{1+x^2} \quad (20)$$

4 Implementation of ADM to thermoelasticity problem

In order to apply ADM to nonlinear thermoelasticity problem, we rewrite Eqs. (1) and (2) in the following operator form:

$$L_n u = 2L_{xx}u - 2L_x\theta - u_x u_{xx}\theta - u_x\theta\theta_x + f(x, t) \quad (21)$$

$$L_t \theta = -2L_{xt}u - u_x u_{xt}\theta + \theta\theta_x + g(x, t) \quad (22)$$

where the notations

$$L_{tt} = \frac{\partial^2}{\partial t^2}, L_{xx} = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, L_{xt} = \frac{\partial^2}{\partial x \partial t} \quad (23)$$

are the linear operators. By using the inverse operators, we can write Eqs. (21) and (22) in the following form:

$$u(x, t) = u(x, 0) + tu_t(x, 0) + L_n^{-1}[2L_{xx}u - 2L_x\theta - u_x u_{xx}\theta - u_x\theta\theta_x + f(x, t)] \quad (24)$$

$$\theta(x, t) = \theta(x, 0) + L_t^{-1}[-2L_{xt}u - u_x u_{xt}\theta + \theta\theta_x + g(x, t)] \quad (25)$$

where the inverse operators are defined by

$$L_{tt}^{-1} = \int_0^t \int_0^t (\square) dt dt, L_t^{-1} = \int_0^t (\square) dt \quad (26)$$

Using Eq. (16), we obtain:

$$u(x, t) = u(x, 0) + L_{tt}^{-1}[2L_{xx}u - 2L_x\theta - N_1(u, \theta) - N_2(u, \theta) + f(x, t)] \quad (27)$$

$$\theta(x, t) = \theta(x, 0) + L_t^{-1}[-2L_{xt}u - N_3(u, \theta) + N_4(u, \theta) + g(x, t)] \quad (28)$$

The solutions $u(x, t)$ and $\theta(x, t)$ can be decomposed by an infinite series, as follows:

$$u(x, t) = \sum_{i=0}^{\infty} u_i(x, t), \theta(x, t) = \sum_{i=0}^{\infty} \theta_i(x, t) \quad (29)$$

The nonlinear operators $N_1(u, \theta)$, $N_2(u, \theta)$, $N_3(u, \theta)$ and $N_4(u, \theta)$ are defined by the following infinite series^[11]:

$$N_i(u, \theta) = \sum_{n=0}^{\infty} A_{in}, i = 1, 2, 3, 4 \quad (30)$$

Where A_{in} is called Adomian polynomials^[4, 11, 19] and defined by:

$$A_{i,n} = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N_i \left[\sum_{k=0}^n \lambda^k u_k, \sum_{k=0}^n \lambda^k \theta_k \right] \right]_{\lambda=0} \quad (31)$$

Hence we obtain the components series solution by the following recursive relations:

$$u_0(x, t) = u(x, 0), \theta_0(x, t) = \theta(x, 0) \quad (32)$$

$$u_{n+1} = L_{tt}^{-1}[2L_{xx}u_n - 2L_x\theta_n - A_{1,n} - A_{2,n} + f(x, t)] \quad (33)$$

$$\theta_{n+1} = L_t^{-1}[-2L_{xt}u_n - A_{3,n} - A_{4,n} + g(x, t)] \quad (34)$$

where $n \geq 0$. Adomian's polynomials formula, Eq. (31), is easy to set computer code to get as many polynomials as we need in the calculation. We can give the first few Adomian's polynomials of the A_{in} as:

$$A_{1,0} = u_{0x}u_{0xx}\theta_0 \quad (35)$$

$$A_{1,1} = u_{0x}u_{0xx}\theta_1 + u_{0x}u_{1xx}\theta_0 + u_{1x}u_{0xx}\theta_0 \quad (36)$$

$$A_{1,2} = u_{1x}u_{1xx}\theta_1 + u_{0x}u_{1xx}\theta_2 + u_{0x}u_{1xx}\theta_1 + u_{0x}u_{2xx}\theta_0 + u_{1x}u_{0xx}\theta_1 + u_{2x}u_{0xx}\theta_0 \quad (37)$$

$$A_{2,0} = u_{0x}\theta_0\theta_{0x} \quad (38)$$

$$A_{2,1} = u_{0x}\theta_0\theta_{1x} + u_{0x}\theta_1\theta_{0x} + u_{1x}\theta_0\theta_{0x} \quad (39)$$

$$A_{2,2} = u_{0x}\theta_0\theta_{2x} + u_{0x}\theta_1\theta_{1x} + u_{0x}\theta_2\theta_{0x} + u_{1x}\theta_0\theta_{1x} + u_{1x}\theta_1\theta_{0x} + u_{2x}\theta_0\theta_{0x} \quad (40)$$

$$A_{3,0} = u_{0x}u_{0xt}\theta_0 \quad (41)$$

$$A_{3,1} = u_{0x}u_{0xt}\theta_1 + u_{0x}u_{1xt}\theta_0 + u_{1x}u_{0xt}\theta_0 \quad (42)$$

$$A_{3,2} = u_{0x}u_{0xt}\theta_2 + u_{0x}u_{1xt}\theta_1 + u_{0x}u_{2xt}\theta_0 + u_{1x}u_{0xt}\theta_1 + u_{1x}u_{1xt}\theta_0 + u_{2x}u_{0xt}\theta_0 \quad (43)$$

$$A_{4,0} = \theta_0\theta_{0xx} \quad (44)$$

$$A_{4,1} = \theta_0\theta_{1xx} + \theta_1\theta_{0xx} \quad (45)$$

$$A_{4,2} = \theta_0\theta_{2xx} + \theta_2\theta_{0xx} + \theta_1\theta_{1xx} \quad (46)$$

and so on, the rest of the polynomials can be constructed in a similar manner. Using the recursive relations, Eqs. (33) and (34), and Adomian's polynomials formula, Eq. (31), with the initial conditions, Eq. (3), gives:

$$u_0(x, t) = \frac{1}{1+x^2} \quad (47)$$

$$\begin{aligned} u_1(x, t) = & \frac{1}{105(1+x^2)^6} (10t^7(x-3x^3) + 14t^6(x+x^2-3x^3+x^4) + 42t^5(x+x^2-3x^3+x^4) \\ & + 35t^4(1+x+2x^2-6x^3-2x^4-8x^6-3x^8) + 70t^3(2x^2-7x^3+2x^4-6x^5-4x^7-x^9) \\ & + 105t^2(1+5x^2+10x^4+10x^6+5x^8+x^{10})) \end{aligned} \quad (48)$$

$$\theta_0(x, t) = \frac{1}{1 + x^2} \quad (49)$$

$$\theta_1(x, t) = \frac{1}{15(1+x^2)^5} (24t^5x^2 + 30t^4x^2 + 10t^3(1 + 2x^2 - 3x^4) + 30t^2(1 - 2x - 6x^3 - 3x^4 - 6x^5 - 2x^7) + 15t(1 + 4x^2 + 6x^4 + 4x^6 + x^8)) \quad (50)$$

Proceeding in the same way, we can obtain $u_2(x, t)$, $\theta_2(x, t)$, and higher order approximations. Here, the numerical results are evaluated using $n = 2$ terms approximation of the recursive relations.

Fig. 1 shows the comparison of the $u(x, t)$ obtained by ADM with that of the exact solution at various values of time. It is apparent that the results are in good agreement and the solution by ADM converges to that of the exact solution. The absolute errors of ADM at various values of time are also depicted in Fig. 2. A comparison of the $\theta(x, t)$ obtained by ADM with that of the exact solution at various values of time is shown in Fig. 3, which admits the efficiency and reliability of the method. The convergence of the solution by ADM is obviously seen in Fig. 4.

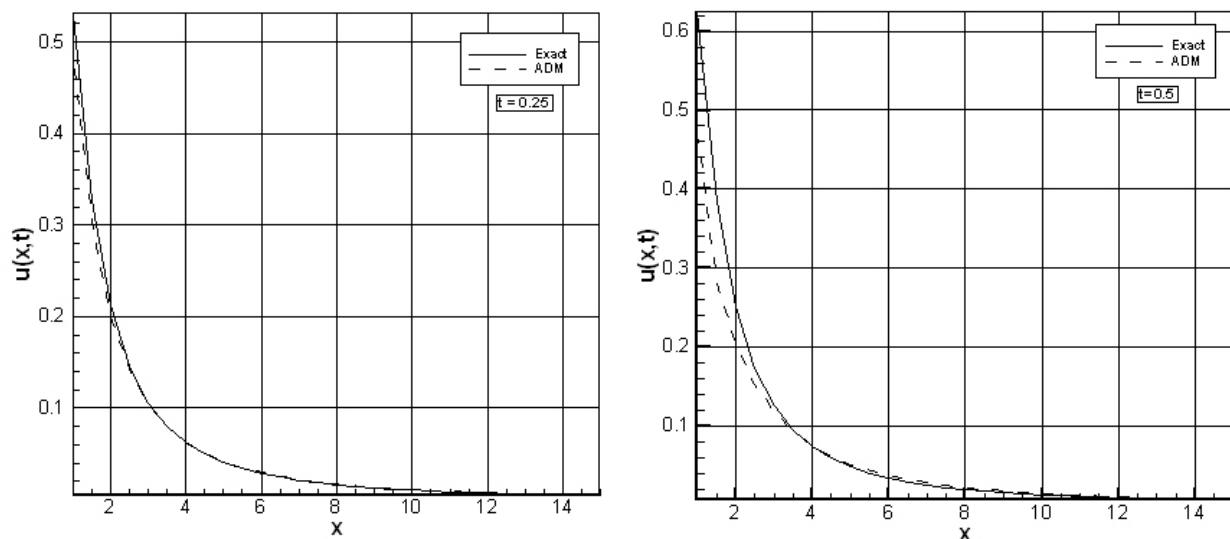


Fig. 1. The comparison of the $u(x, t)$ obtained by adm with that of the exact solution at various values of time

5 Conclusion

In this paper ADM has been successfully used to solve the Cauchy problem arising in one dimensional nonlinear thermoelasticity. The method needs less work in comparison with the traditional methods and decreases considerable volume of calculations. The decomposition procedure of Adomian will be obtained easily without linearization of the problem. In this approach the solution is found in the form of a convergent series with easily computed components. The results obtained by decomposition method are compared with those of the exact solution, which shows very good agreement, even using only few terms of the recursive relations. In general, this method provides highly accurate numerical solutions and can be applied to wide class of nonlinear problems.

In conclusion, ADM is an accurate, efficient and reliable method for both linear and nonlinear problems. It also has many merits in comparison with other methods, such as:

1. Adomian decomposition method does not require small parameters which are needed by perturbation method.
2. Adomian decomposition method avoids linearization and physically unrealistic assumptions.

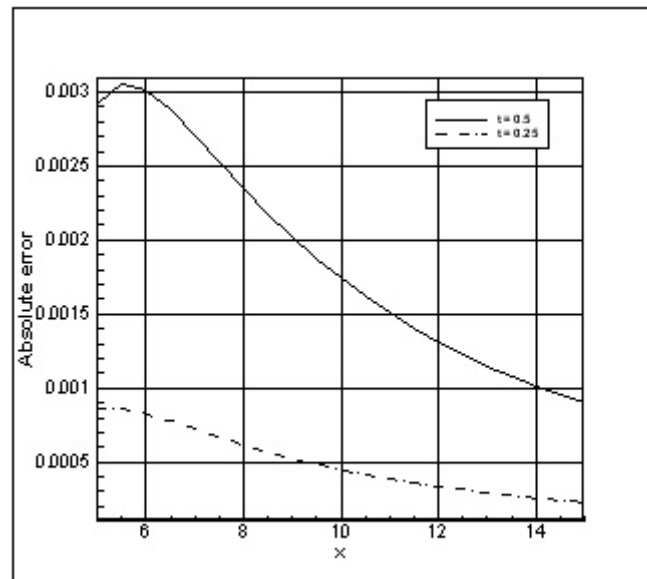


Fig. 2. The absolute errors of adm at various values of time

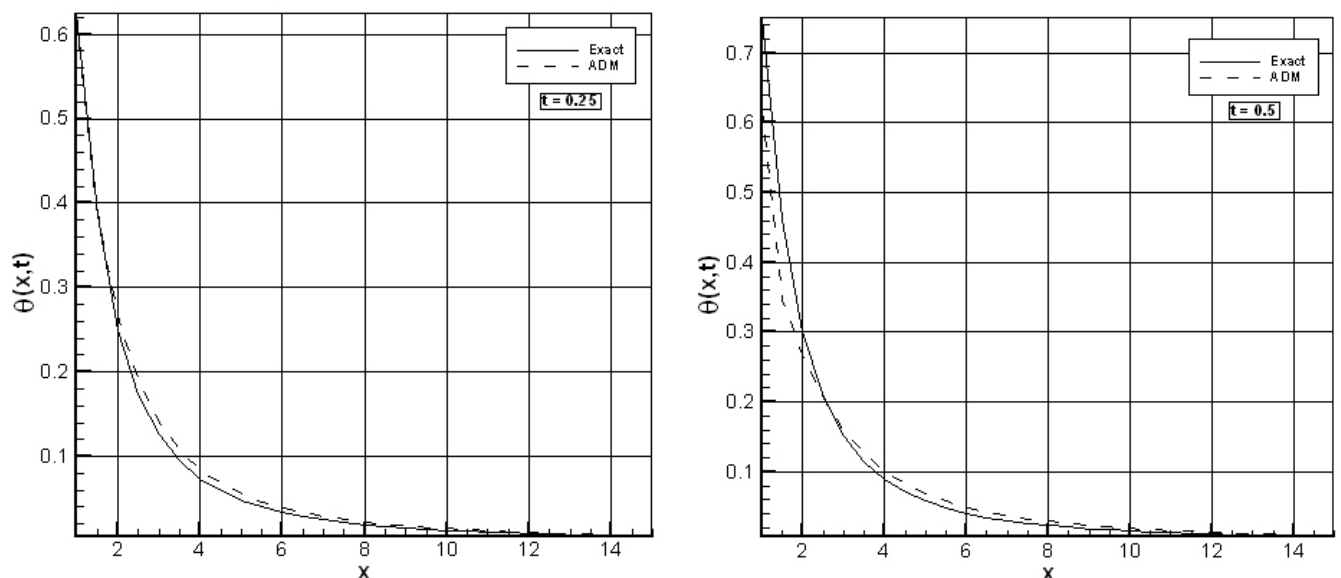


Fig. 3. The comparison of the $\theta(x, t)$ obtained by adm with that of the exact solution at various values of time

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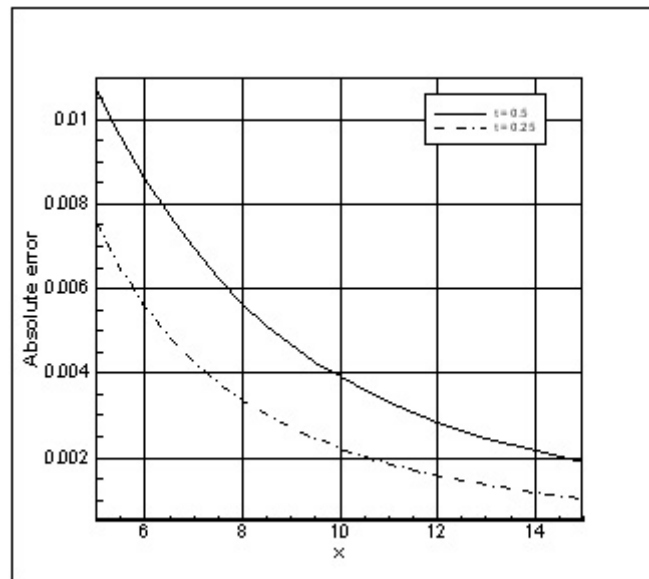


Fig. 4. The absolute errors of adm at various values of time for $\theta(x, t)$

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