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Exponential stability of approximate solutions for neutral stochastic differential delay equations with Markovian switching and possion jumps*

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Abstract. In this paper, the relation of exponential stability in mean square between approximate solutions and explicit solutions has been investigated. Not only the stability and convergence of approximate solutions have been proved, but also the equidistant condition of the stability between them for the neutral stochastic differential delay equations with Markovian swithching and Poisson jumps has been obtained. Moreover, some well-known results on the stability in the literature have been improved without Lyapunov Functions.

Keywords: exponential stability, Euler-Maruyama method, Poisson jumps, approximations

1 Introduction

In the recent years,stochastic differential delay equations receive more and more attention, because it can render better explanations to the phenomena. Many results on the stability have been obtained [6, 9, 12, 13, 15]. On the other hand, the numerical methods on stochastic differential equations and stochastic functional differential equations have also been well established,such as Hu[5], Higham et al.[1, 3, 4], Klodeden and Platen[7], Mao[11, 14] and the references therein. However, there exist a number of difficulties encountered in the study of stability for approximate solutions. Recently, Mao has proved the equidistant condition on the stability between approximate solutions and explicit solutions, and the investigator will get better results by using the method.

To the best of the authors knowledge to date,the equidistant condition is obtained only for stochastic differential delay equations which don't include Markov chain and Possion jumps. Neutral differential equations are often used to describe the dynamical systems which not only depend on present and past state but also involve derivatives with delay, so it is very popular. Now we will have a try to study neutral stochastic differential delay equations with Markov chain and Possion jumps to fill this gap.

In this paper,we will further research this topic. Our focus is on exponential stability in mean square of approximate solutions for neutral stochastic differential delay equations with Markovian switching and Poission jumps. We first establish the approximate solutions and convergence of approximate solutions. Later, we research the condition which can cause the stability of approximate solutions and the stability of explicit solutions to be equidistant, some well-known results are generalized.

The structure of this paper is as follows: In section 2, we first present some notions and hypothesis; later, the structures of the discrete Euler-Maruyama approximate solutions and continuous Euler-Maruyama approximate solutions have been established. In section 3, the main results and some useful lemmas under

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global Lipschitz condition have been obtained. In the last section, The proof of the main results has been completed.

1.1 Preliminaries and approximations

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P\}$ be a complete probability space with a filtration satisfying the usual conditions, i.e., the filtration is continuous on the right and $\mathcal{F}_0$ contains all P-zero sets. Let $B(t) = (B_1(t), B_2(t), \cdots, w_m(t))_T$ be an m-dimensional Brownian motion defined on the probability space. Let $C([-\tau, 0]; R^n)$ denote the family of functions φ from $[-\tau, 0]$ to R^n that are right-continuous and have limits on the left. $C([-\tau, 0]; R^n)$ is equipped with the norm $\|\varphi\| = \sup_{-\tau\leq s\leq 0} |\varphi(s)|$, where $|\cdot|$ is the Euclidean norm in R^n, i.e., $|x| = \sqrt{x^T x}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : x = 1\}$. Denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ the family of all bounded, $\mathcal{F}_0$ measurable, $C([-\tau, 0]; R^n)$ -valued random variables. Let $p > 0, t \geq 0$, $L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ denote by the family of all $\mathcal{F}_t$ measurable, $C([-\tau, 0]; R^n)$ -valued random variables $\varphi = \{\varphi(\theta) : -\tau \leq \theta \leq 0\}$, and $\sup_{-\tau\leq\theta\leq 0} E|\varphi(\theta)| < \infty$.

Let $\{r(t), t \in R_+ = [0, +\infty)\}$ be a right-continuous Markov chain on the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P\}$ taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P(r(t+\Delta) = j | r(t) = i) = \begin{cases} r_{ij}\Delta + o(\Delta)\ , & if \;\; i \neq j \\ 1 + r_{ii}\Delta + o(\Delta)\ , & if \;\; i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, if $i \neq j$. While

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is known that almost every sample path of $r(t)$ is right continuous step function with a finite number of simple jumps in any finite sub-interval of R_+.

Let $\{v(dt, du), t \in R_+, \sigma \in R\}$ be a centered Poisson random measure with parameter $\pi(du)dt$.

Consider the following neutral stochastic differential delay equations with Poisson jumps and Markovian switching:

$$\begin{aligned} d[y(t) - G(y(t-\tau))] =& f(y(t), y(t-\tau), r(t))dt + g(y(t), y(t-\tau), r(t))dB(t) \\ &+ \int_{-\infty}^{+\infty} h(y(t), y(t-\tau), u)v(dt, du), \end{aligned} \tag{1}$$

with the initial condition $y_0 = \xi$, where $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$ is independent of B and v, and $|G(x)-G(y)| \leq K|x - y|$ for all $x, y \in R^n$,where $K \in (0, 1)$ is constant.

Assume that

$$\begin{aligned} G :&\ L^p_{\mathcal{F}_0}([-\tau, 0]; R^n) \to R^n \\ f :&\ L^p_{\mathcal{F}_t}([-\tau, 0]; R^n) \times L^p_{\mathcal{F}_t}([-\tau, 0]; R^n) \times S \to R^n \\ g :&\ L^p_{\mathcal{F}_t}([-\tau, 0]; R^n) \times L^p_{\mathcal{F}_t}([-\tau, 0]; R^n) \times S \to R^{n\times m} \\ h :&\ L^p_{\mathcal{F}_t}([-\tau, 0]; R^n) \times L^p_{\mathcal{F}_t}([-\tau, 0]; R^n) \times R \to R^n. \end{aligned}$$

moreover, $f(0, 0, i) = 0, g(0, 0, i) = 0, h(0, 0, u) = 0$.

We shall impose the following hypotheses:

(H_1) (The Global Lipschitz condition) For any $d > 0$, there exits a $C_d > 0$ such that

$$|f(x,y,i)-f(\overline{x},\overline{y},i)|^2 \vee |g(x,y,i)-g(\overline{x},\overline{y},i)|^2 \vee \int_{-\infty}^{+\infty} |h(x,y,u)-g(\overline{x},\overline{y},u)|^2\pi(d\sigma)$$
$$\leq K_1(|x-\overline{x}|^2+|y-\overline{y}|^2),$$

for every $x, y, \overline{x}, \overline{y} \in R^n$ with $|x| \vee |y| \vee |\overline{x}| \vee |\overline{y}| \leq d$, which implies that

$$|f(x,y,i)|^2 \leq |f(x,y,i)-f(0,0,i)|^2 \leq K_1(|x|^2+|y|^2)$$
$$|g(x,y,i)|^2 \leq |g(x,y,i)-g(0,0,i)|^2 \leq K_1(|x|^2+|y|^2)$$
$$\int_{-\infty}^{+\infty} |h(x,y,u)|\pi(du) \leq \int_{-\infty}^{+\infty} |h(x,y,u)-h(0,0,u)|^2\pi(du) \leq K_1(|x|^2+|y|^2).$$

(H_2) (The Hölder continuity of the initial data) There exist constants $\overline{K}$ and $\gamma \in (0,1)$ such that for all $-\tau \leq s < t \leq 0$,

$$E|\xi(t)-\xi(s)|^2 \leq \overline{K}(t-s)^\gamma.$$

Under (H_1), the NSDDEJ (2.1) has a unique continuous solution on $t \geq -\tau$, with any initial data $y_0 = \xi \in L^2_{\mathfrak{F}_0}([-\tau,0];R^n)$ given at time $t=0$. We shall denote this solution by $y(t;0,\xi)$. In this paper we consider exponential stability in mean square of the origin, which is defined as follows.

Definition 1. *The NSDDEJ (2.1) is said to be exponentially stable in mean square if there is a pair of positive constants λ and M such that for any initial data $\xi \in L^2_{\mathfrak{F}_0}([-\tau,0];R^n)$*

$$E|y(t;0,\xi)|^2 \leq ME\|\xi\|^2 e^{-\lambda t} \quad \forall t \geq 0, \tag{2}$$

where λ is regarded as the rate constant and M is regarded as the growth constant.

Due to the property of the NSDDEJ (2.1) and (H_1), the existence and uniqueness of the solution to the equation (1) for initial data $y_s = \xi \in L^2_{\mathfrak{F}_s}([-\tau,0];R^n)$ given at time $t=s$ is guaranteed,which has the following property

$$y(t;0,\xi) = y(t;s,y_s) \quad \forall 0 \leq s < t < \infty.$$

We can now define the Euler-Maruyama approximate solution to the NSDDEJ (2.1) with discrete Markov Chain $\{r^\Delta_k = r(k\Delta), k=0,1,2,\cdots\}$. Given a stepsize $\Delta = \tau/N > 0$ for some positive integer N such that $0 < \Delta < 1$ and $\tau = N\Delta$. Compute the discrete approximations $X(t) = y(k\Delta)$ by setting $X_0(t) = \xi(t)$ on $-\tau \leq t \leq 0, r^\Delta_0 = i_0$ and forming $X_0 = \xi(t), -\tau \leq t \leq 0$,

$$\begin{aligned} X((k+1)\Delta) = {} & X(k\Delta) - G(X(k\Delta - N\Delta)) + G(X((k+1)\Delta - N\Delta)) \\ & + f(X(k\Delta), X(k\Delta - N\Delta), r^\Delta_k)\Delta + g(X(k\Delta), X(k\Delta - N\Delta), r^\Delta_k)\Delta B_k \\ & + \int_{-\infty}^{+\infty} h(X(k\Delta), X(k\Delta - N\Delta), u)v(\Delta t, du), \forall k \geq 0, \end{aligned} \tag{3}$$

where $\Delta B_k = B((k+1)\Delta) - B(k\Delta)$.

We define this discrete EM solution as $X(k\Delta;0,\xi)$ with the initial data ξ given at time $t=0$. Following definition 1, we may now define exponential stability in mean square for discrete EM approximate solution.

Definition 2. *The discrete EM approximate solution is said to be exponentially stable in mean square on the NSDDEJ (2.1) if there is a pair of positive constants γ and $\overline{H}$ such that for any initial data $\xi \in L^2_{\mathfrak{F}_0}([-\tau,0];R^n)$*

$$E|X(k\Delta;0,\xi)|^2 \leq \overline{H}E\|\xi\|^2 e^{-\gamma k\Delta} \quad \forall k \geq 0. \tag{4}$$

Let

$$Z_1(t) = \sum_{k=0}^{\infty} I_{[k\Delta,(k+1)\Delta)}(t)X(k\Delta),$$
$$Z_2(t) = \sum_{k=0}^{\infty} I_{[k\Delta,(k+1)\Delta)}(t)X(k\Delta - N\Delta),$$
$$\overline{r}(t) = \sum_{k=0}^{\infty} I_{[k\Delta,(k+1)\Delta)}(t)r_k^{\Delta},$$

then we defined the conditions Euler-Maruyama approximate solution as: $X(t) = \xi(t), -\tau \le t \le 0,$

$$\begin{aligned} X(t) =&\xi(0) - G(\xi) + G(Z_2(t)) + \int_0^t f(Z_1(s), Z_2(s), \overline{r}(s))ds \\ &+ \int_0^t g(Z_1(s), Z_2(s), \overline{r}(s))dB(s) + \int_0^t \int_{-\infty}^{+\infty} h(Z_1(s), Z_2(s), u)v(ds, du). \end{aligned} \tag{5}$$

where I_E denoting the indicator function for the set E. Now we define this continuous EM solution as $x(t; 0, \xi)$ with the initial data ξ given at time $t = 0$, and define the exponential stability in mean square for continuous EM approximate solution.

Definition 3. *The continuous EM approximate solution is said to be exponentially stable in mean square on the NSDDEJ (2.1) if there is a pair of positive constants γ and H such that for any initial data $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$*

$$E|X(t; 0, \xi)|^2 \le HE\|\xi\|^2 e^{-\gamma t} \quad \forall t \ge 0. \tag{6}$$

Lemma 1. *If (H_1) hold, then the discrete EM approximate solution on the NSDDEJ (2.1) is said to be exponentially stable in mean square if and only if the continuous EM approximate solution on the NSDDEJ (2.1) is said to be exponentially stable in mean square. Moreover, the rate constants are the same, and the growth constants $\overline{H}$ and H can be made arbitrarily close with sufficiently small Δ.*

Proof. It is easy to prove necessary part of the lemma from (6) to (4). Following proof is the sufficient part from (4) to (6). Fix ξ and write $X(t; o, \xi) = X(t)$, for any $t \ge 0$ choose $k \ge 0$ for $t \in [k\Delta, (k+1)\Delta)$. Note that

$$\begin{aligned} X(t) =&X(k\Delta) + G(x(k\Delta - N\Delta)) + G(Z_2(t)) \\ &+ f(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})(t - k\Delta) \\ &+ g(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})(B(t) - B(k\Delta)) \\ &+ \int_{-\infty}^{+\infty} h(X(k\Delta), X(k\Delta - N\Delta), u)v(t - k\Delta, du) \\ =&X(k\Delta) + f(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})(t - k\Delta) \\ &+ g(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})(B(t) - B(k\Delta)) \\ &+ \int_{-\infty}^{+\infty} h(X(k\Delta), X(k\Delta - N\Delta), u)v(t - k\Delta, du). \end{aligned}$$

By (H_1), we have

$$\begin{aligned}E|X(t)|^2 \le&(1+\varepsilon)E|X(k\Delta)|^2 + (1+\varepsilon^{-1})E|f(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})(t-k\Delta)\\&+ g(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})(B(t)-B(k\Delta))\\&+\int_{-\infty}^{+\infty} h(X(k\Delta), X(k\Delta - N\Delta), u)\upsilon(t-k\Delta, du)|^2\\\le&(1+\varepsilon)E|X(k\Delta)|^2 + (1+\varepsilon^{-1})[3\Delta^2 E|f(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})|^2\\&+3\Delta E|g(X(k\Delta), X(k\Delta - N\Delta), r_k^{\Delta})|^2\\&+3\Delta E\int_{-\infty}^{+\infty} |h(X(k\Delta), X(k\Delta - N\Delta), u)|^2\pi(du)]\\\le&(1+\varepsilon)E|X(k\Delta)|^2 + (1+\varepsilon^{-1})\Delta 3(\Delta+2)K_1[E|X(k\Delta)|^2 + E|X(k\Delta - N\Delta)|^2],\end{aligned}$$

where $\varepsilon > 0$ is arbitrary, using (4) we have

$$E|X(t)|^2 \le \overline{H}E\|\xi\|^2 e^{-\gamma k\Delta}[(1+\varepsilon) + \Delta 3(\Delta+2)K_1(1+\varepsilon^{-1})(1+e^{\gamma\tau})].$$

Let

$$H = \overline{H}[(1+\varepsilon) + \Delta 3(\Delta+2)K_1(1+\varepsilon^{-1})(1+e^{\gamma\tau})]e^{\gamma\Delta}. \tag{7}$$

Hence

$$E|X(t)|^2 \le HE\|\xi\|^2 e^{-\gamma(k+1)\Delta} \le HE\|\xi\|^2 e^{-\gamma t}.$$

From (7), we know that H can be made arbitrarily close to $\overline{H}$ by choose ε and sufficiently small Δ, the proof is completed.

Due to the property of (5) and (H_1), the solution defined by following from $X(t;0,\xi)$ for the initial data $\xi \in L^2_{\mathfrak{F}_s}([-\tau, 0]; R^n)$ given at time $s \ge 0$

$$\begin{aligned}X(t;s,\xi) =&\xi - G(Z_2(s)) + G(Z_2(t)) + \int_s^t f(Z_1(\theta), Z_2(\theta), \overline{r}(\theta))d\theta\\&+\int_s^t g(Z_1(\theta), Z_2(\theta), \overline{r}(\theta))dB(\theta) + \int_s^t\int_{-\infty}^{+\infty} h(Z_1(\theta), Z_2(\theta), u)\upsilon(d\theta, du),\end{aligned}$$

which has the property

$$X(t;0,\xi) = X(t;s,X_s) \quad \forall 0 \le s < t < \infty.$$

2 Main result

From previous section,the EM approximate solution $X_\Delta(t;0,\xi)$ depends on the stepsize Δ, but in this section we write $X_\Delta(t;0,\xi) = X(t;0,\xi)$.

Theorem 1. *If (H_1) hold, and the NSDDEJ (2.1) is exponentially stable in mean square, namely*

$$E|y(t;0,\xi)|^2 \le ME\|\xi\|^2 e^{-\lambda t} \quad \forall t \ge 0,$$

for all $\xi \in L^2_{\mathfrak{F}_s}([-\tau, 0]; R^n)$. Then there exists a Δ^* such that for every $\Delta < \Delta^*$, the EM approximate solution is exponentially stable in mean square on the NSDDEJ (2.1) with γ and H, which are independent of Δ, namely

$$E|X(t;0,\xi)|^2 \le HE\|\xi\|^2 e^{-\gamma t} \quad \forall t \ge 0.$$

Moreover,$\gamma = \frac{1}{2}\lambda$ and $H = 2MC_1 e^{\frac{1}{2}T}$, where $C_1 = (\sqrt{K} + \frac{K}{1-\sqrt{K}} + \frac{4+4(\tau+2)K_1\tau}{1-K})exp\{\frac{4\tau(\tau+2)K_1}{1-K}\}$ and $T = \tau(9 + In[\frac{4logM}{\lambda\tau}])$ are constants which are independent of Δ.

To obtain the proof of this theorem, we must present some lemmas.

Lemma 2. *If* (H_1) *hold, then*

$$\sup_{-\tau\le t\le\tau} E|X(t;0,\xi)|^2 \le C_1 E\|\xi\|^2,$$

where C_1 *is constant which is independent of* Δ.

Proof. We write $X(t) = X(t;0,\xi)$. Using the It$\hat{o}$isometry, H$\ddot{o}$ inequality and (H_1), we have

$$\begin{aligned} E|X(t)|^2 &= E|G(Z_2(t)) - G(\xi) + J(t)|^2 \\ &\le \frac{1}{K}E|G(Z_2(t)) - G(\xi)|^2 + \frac{1}{1-K}E|J(t)|^2 \\ &\le KE|Z_2(t)-\xi|^2 + \frac{1}{1-K}E|J(t)|^2 \\ &\le \sqrt{K}E|Z_2(t)|^2 + \frac{K}{1-\sqrt{K}}E\|\xi\|^2 + \frac{1}{1-K}E|J(t)|^2, \end{aligned}$$

where

$$\begin{aligned} J(t) =& \xi + \int_0^t f(Z_1(s), Z_2(s), \overline{r}(s))ds + \int_0^t g(Z_1(s), Z_2(s), \overline{r}(s))dB(s) \\ &+ \int_0^t \int_{-\infty}^{+\infty} h(Z_1(s), Z_2(s), u)\upsilon(ds, du). \end{aligned}$$

Compute

$$\begin{aligned} E|J(t)|^2 \le& 4E\|\xi\|^2 + 4E|\int_0^t f(Z_1(s), Z_2(s), \overline{r}(s))ds|^2 + 4E|\int_0^t g(Z_1(s), Z_2(s), \overline{r}(s))dB(s)|^2 \\ &+ 4E|\int_0^t \int_{-\infty}^{+\infty} h(Z_1(s), Z_2(s), u)\upsilon(ds, du)|^2 \\ \le& 4E\|\xi\|^2 + 4\tau E\int_0^t |f(Z_1(s), Z_2(s), \overline{r}(s))|^2 ds + 4E\int_0^t |g(Z_1(s), Z_2(s), \overline{r}(s))|^2 dB(s) \\ &+ 4E\int_0^t \int_{-\infty}^{+\infty} |h(Z_1(s), Z_2(s), u)|^2 \pi(du)ds \\ \le& 4E\|\xi\|^2 + 4(\tau+2)K_1\int_0^t (E|Z_1(s)|^2 + E|Z_2(s)|^2)ds. \end{aligned}$$

Putting this inequality into above

$$\begin{aligned} E|X(t)|^2 \le& \sqrt{K}E|Z_2(t)|^2 + \frac{K}{1-\sqrt{K}}E\|\xi\|^2 + \frac{4}{1-K}E\|\xi\|^2 \\ &+ \frac{4(\tau+2)K_1}{1-K}\int_0^t (E|Z_1(s)|^2 + E|Z_2(s)|^2)ds. \end{aligned}$$

Now for any $t_1 \in [0,\tau]$, by taking supremum for $t \in [0, t_1)$ on both side of inequality, we have

$$\begin{aligned} \sup_{0\le t\le t_1} E|X(t)|^2 \le& \sqrt{K}\sup_{0\le t\le t_1} E|Z_2(t)|^2 + (\frac{K}{1-\sqrt{K}} + \frac{4}{1-K})E\|\xi\|^2 \\ &+ \frac{4(\tau+2)K_1}{1-K}\sup_{0\le t\le t_1}\int_0^t (E|Z_1(s)|^2 + E|Z_2(s)|^2)ds \\ \le& (\sqrt{K} + \frac{K}{1-\sqrt{K}} + \frac{4}{1-K})E\|\xi\|^2 + \frac{4\tau(\tau+2)K_1}{1-K}E\|\xi\|^2 \\ &+ \frac{4(\tau+2)K_1}{1-K}\int_0^{t_1} \sup_{0\le r\le s} E|X(r)|^2 ds. \end{aligned}$$

From above inequality, we obtain

$$\sup_{0\le t\le t_1} E|X(t)|^2 \le (\sqrt{K} + \frac{K}{1-\sqrt{K}} + \frac{4+4(\tau+2)K_1\tau}{1-K})E\|\xi\|^2 + \frac{4(\tau+2)K_1}{1-K}\int_0^t \sup_{0\le r\le s} E|X(r)|^2 ds.$$

Now we get the result by Gronwall inequality

$$\sup_{0\le t\le t_1} E|X(t)|^2 \le (\sqrt{K} + \frac{K}{1-\sqrt{K}} + \frac{4+4(\tau+2)K_1\tau}{1-K}) \times exp\{\frac{4\tau(\tau+2)K_1}{1-K}\}E\|\xi\|^2.$$

Let

$$C_1 = (\sqrt{K} + \frac{K}{1-\sqrt{K}} + \frac{4+4(\tau+2)K_1\tau}{1-K})exp\{\frac{4\tau(\tau+2)K_1}{1-K}\},$$

we have

$$\sup_{-\tau\le t\le\tau} E|X(t)|^2 \le C_1 E\|\xi\|^2.$$

Lemma 3. *If* (H_1) *hold, then*

$$\sup_{0\le t\le\tau+T} E|X(t;0,\xi)|^2 \le C_2 E\|\xi\|^2,$$

where C_2 *is constant which is independent of* Δ.

Proof. We write $X(t) = X(t;0,\xi)$, for any $t \in [\tau, \tau+T]$, applying Lemma 1, we have

$$\begin{aligned} E|X(t)|^2 &= E|G(Z_2(t)) - G(Z_2(\tau)) + J(t)|^2 \\ &\le \frac{1}{K}E|G(Z_2(t)) - G(Z_2(\tau))|^2 + \frac{1}{1-K}E|J(t)|^2 \\ &\le KE|Z_2(t) - Z_2(\tau)|^2 + \frac{1}{1-K}E|J(t)|^2 \\ &\le \sqrt{K}E|Z_2(t)|^2 + \frac{K}{1-\sqrt{K}}E|Z_2(\tau)|^2 + \frac{1}{1-K}E|J(t)|^2, \end{aligned}$$

where

$$J(t) = X_\tau + \int_\tau^t f(Z_1(s), Z_2(s), \overline{r}(s))ds + \int_\tau^t g(Z_1(s), Z_2(s), \overline{r}(s))dB(s) + \int_\tau^t \int_{-\infty}^{+\infty} h(Z_1(s), Z_2(s), u)\upsilon(ds, du).$$

Compute

$$
\begin{aligned}
E|J(t)|^2 \leq & 4E|X_\tau|^2 + 4E|\int_\tau^t f(Z_1(s), Z_2(s), \overline{r}(s))ds|^2 + 4E|\int_\tau^t g(Z_1(s), Z_2(s), \overline{r}(s))dB(s)|^2 \\
& + 4E|\int_\tau^t \int_{-\infty}^{+\infty} h(Z_1(s), Z_2(s), u)v(ds, du)|^2 \\
\leq & 4E|X_\tau|^2 + 4TE\int_\tau^t |f(Z_1(s), Z_2(s), \overline{r}(s))|^2 ds + 4E\int_\tau^t |g(Z_1(s), Z_2(s), \overline{r}(s))|^2 dB(s) \\
& + 4E\int_\tau^t \int_{-\infty}^{+\infty} |h(Z_1(s), Z_2(s), u)|^2 \pi(du)ds \\
\leq & 4E|X_\tau|^2 + 4(T+2)K_1\int_\tau^t (E|Z_1(s)|^2 + E|Z_2(s)|^2)ds \\
\leq & 4C_1E\|\xi\|^2 + 4(T+2)K_1\int_\tau^t (E|Z_1(s)|^2 + E|Z_2(s)|^2)ds.
\end{aligned}
$$

Putting this inequality into above

$$
\begin{aligned}
E|X(t)|^2 \leq \sqrt{K}E|Z_2(t)|^2 + \frac{C_1K}{1-\sqrt{K}}E\|\xi\|^2 + \frac{4C_1}{1-K}E\|\xi\|^2 \\
+ \frac{4(T+2)K_1}{1-K}\int_\tau^t (E|Z_1(s)|^2 + E|Z_2(s)|^2)ds.
\end{aligned}
$$

By taking the suprememem for $t \in [\tau, \tau+T]$ on the both side of the inequality, we have

$$
\begin{aligned}
\sup_{0\leq r\leq t} E|X(r)|^2 \leq & \sqrt{K}\sup_{0\leq r\leq t} E|Z_2(r)|^2 + (\frac{C_1K}{1-\sqrt{K}} + \frac{4C_1}{1-K})E\|\xi\|^2 \\
& + \frac{4T(T+2)K_1}{1-K}\int_\tau^t \sup_{0\leq r\leq s}(E|Z_1(r)|^2 + E|Z_2(r)|^2)ds \\
\leq & \sqrt{K}\sup_{0\leq r\leq t} E|X(t)|^2 + (\sqrt{K} + \frac{C_1K}{1-\sqrt{K}} + \frac{4C_1}{1-K})E\|\xi\|^2 \\
& + \frac{4T(T+2)K_1}{1-K}E\|\xi\|^2 + \frac{8(T+2)K_1}{1-K}\int_\tau^t \sup_{0\leq r\leq s} E|X(r)|^2 ds.
\end{aligned}
$$

Now we have the result by Gronwall inequality

$$
\begin{aligned}
\sup_{0\leq r\leq t} E|X(r)|^2 \leq & (\frac{\sqrt{K}}{1-\sqrt{K}} + \frac{C_1K}{(1-\sqrt{K})^2} + \frac{4C_1 + 4T(T+2)K_1}{(1-K)(1-\sqrt{K})}) \\
& \times exp\{\frac{8T(T+2)K_1}{(1-K)(1-\sqrt{K})}\}E\|\xi\|^2.
\end{aligned}
$$

Let

$$
C_2 = (\frac{\sqrt{K}}{1-\sqrt{K}} + \frac{C_1K}{(1-\sqrt{K})^2} + \frac{4C_1 + 4T(T+2)K_1}{(1-K)(1-\sqrt{K})})exp\{\frac{8T(T+2)K_1}{(1-K)(1-\sqrt{K})}\}.
$$

Hence

$$
\sup_{0\leq t\leq \tau+T} E|X(t)|^2 \leq C_2E\|\xi\|^2.
$$

Lemma 4. *If (H_1) hold, then for any $T > 0$*

$$
E|X(t;0,\xi) - Z_1(t;0,\xi)|^2 \leq C_3E\|\xi\|^2\Delta \quad \forall t \in [0, \tau+T]
$$

where C_3 is constant which is independent of Δ.

Proof. We write $X(t;0,\xi) = X(t)$ and $Z_1(t;0,\xi) = Z_1(t)$. For any $t \in [0, \tau + T]$, exist $k \geq 0$ such that $t \in [k\varDelta, (k+1)\varDelta)$, compute

$$\begin{aligned} X(t) - Z_1(t) =& X(t) - X(k\varDelta) \\ =& G(Z_2(t)) - G(Z_2(k\varDelta)) + \int_{k\varDelta}^{t} f(Z_1(s), Z_2(s), \overline{r}(s))ds \\ &+ \int_{k\varDelta}^{t} g(Z_1(s), Z_2(s), \overline{r}(s))dB(s) + \int_{k\varDelta}^{t}\int_{-\infty}^{+\infty} h(Z_1(s), Z_2(s), u)\upsilon(ds, du) \\ =& \int_{k\varDelta}^{t} f(Z_1(s), Z_2(s), \overline{r}(s))ds + \int_{k\varDelta}^{t} g(Z_1(s), Z_2(s), \overline{r}(s))dB(s) \\ &+ \int_{k\varDelta}^{t}\int_{-\infty}^{+\infty} h(Z_1(s), Z_2(s), u)\upsilon(ds, du). \end{aligned}$$

Hence

$$\begin{aligned} E|X(t) - Z_1(t)|^2 \leq& 3\varDelta \int_{k\varDelta}^{t} |f(Z_1(s), Z_2(s), \overline{r}(s))|^2 ds + 3\int_{k\varDelta}^{t} |g(Z_1(s), Z_2(s), \overline{r}(s))|^2 ds \\ &+ 3\int_{k\varDelta}^{t}\int_{-\infty}^{+\infty} |h(Z_1(s), Z_2(s), u)|^2 \pi(du)ds \\ \leq& (3\varDelta + 6)K_1 \int_{k\varDelta}^{t} (E|Z_1(s)|^2 + E|Z_2(t)|^2)ds \\ \leq& 6(\eta + 2)K_1 C_2 E\|\xi\|^2 \varDelta, \end{aligned}$$

where $\eta = \tau \wedge 1$.

Let $C_3 = 6(\eta + 2)K_1 C_2$, we have

$$E|X(t) - Z_1(t)|^2 \leq C_3 E\|\xi\|^2 \varDelta.$$

Lemma 5. *If $(H_1), (H_2)$ hold, then for any $T > 0$*

$$E|X(t-\tau;0,\xi) - Z_2(t;0,\xi)|^2 \leq C_4 \varDelta^{\gamma}, \quad \forall t \in [0, \tau + T],$$

where C_4 is a constant independent of $\varDelta$.

Proof. We write $X(t-\tau;0,\xi) = X(t-\tau), Z_2(t;0,\xi) = Z_2(t)$, for any $t \in [0, \tau + T]$, there exists $k \geq 0$ such that $t \in [k\varDelta, (k+1)\varDelta)$, then

$$X(t-\tau) - Z_2(t) = X(t - N\varDelta) - X(k\varDelta - N\varDelta).$$

To show the desired result,we consider the following three case:

Case 1: If $t - N\varDelta \geq k\varDelta - N\varDelta \geq 0$, by the definition of $Z_2(t)$ and lemma 3, we obtain

$$
\begin{aligned}
E|X(t-N\Delta)-X(k\Delta-N\Delta)|^2 =&E|G(Z_2(t-N\Delta))-G(Z_2(k\Delta-N\Delta))\\
&+\int_{k\Delta-N\Delta}^{t-N\Delta} f(Z_1(s),Z_2(s),\overline{r}(s))ds\\
&+\int_{k\Delta-N\Delta}^{t-N\Delta} g(Z_1(s),Z_2(s),\overline{r}(s))dB(s)\\
&+\int_{k\Delta-N\Delta}^{t-N\Delta}\int_{-\infty}^{+\infty} h(Z_1(s),Z_2(s),u)\upsilon(ds,du)|^2\\
=&E|\int_{k\Delta-N\Delta}^{t-N\Delta} f(Z_1(s),Z_2(s),\overline{r}(s))ds\\
&+\int_{k\Delta-N\Delta}^{t-N\Delta} g(Z_1(s),Z_2(s),\overline{r}(s))dB(s)\\
&+\int_{k\Delta-N\Delta}^{t-N\Delta}\int_{-\infty}^{+\infty} h(Z_1(s),Z_2(s),u)\upsilon(ds,du)|^2\\
\leq&3(\Delta+2)K_1\int_{k\Delta-N\Delta}^{t-N\Delta}(E|Z_1(s)|^2+E|Z_2(s)|^2)ds\\
\leq&6C_2K_1E\|\xi\|^2(\eta+2)\Delta.
\end{aligned}
$$

Case 2: If $0\geq t-N\Delta\geq k\Delta-N\Delta$. Then,

$$
E|X(t-N\Delta)-X(k\Delta-N\Delta)|^2=E|\xi(t-N\Delta)-\xi(k\Delta-N\Delta)|^2\leq\overline{K}\Delta^\gamma,
$$

since the hypothesis of (H_2).
Case 3: If $t-N\Delta\geq 0\geq k\Delta-N\Delta$. Then,

$$
\begin{aligned}
E|X(t-\tau)-Z_2(t)|^2 =&E|X(t-N\Delta)-\xi(k\Delta-N\Delta)|^2\\
=&E|X(t-N\Delta)-\xi(0)+\xi(0)-\xi(k\Delta-N\Delta)|^2\\
=&2E|X(t-N\Delta)-\xi(0)|^2+2E|\xi(0)-\xi(k\Delta-N\Delta)|^2.
\end{aligned}
$$

In view of (H_2) and case 1, we obtain

$$
\begin{aligned}
E|X(t-\tau)-Z_2(t)|^2 \leq&12C_2K_1(\eta+2)E\|\xi\|^2\Delta+2\overline{K}\Delta^\gamma\\
\leq&[12C_2K_1(\eta+2)E\|\xi\|^2+2\overline{K}]\Delta^\gamma
\end{aligned}
$$

so the desired result is obtained by letting

$$
C_4=12C_2K_1(\eta+2)E\|\xi\|^2+2\overline{K}.
$$

Lemma 6. *Write $y(t)=y(t;\tau,X_\tau)$ which is the solution to the NSDDEJ (2.1) with initial data X_τ given at $t=\tau$, where $X_\tau=\{X(u):0\leq u\leq\tau\}$. If (H_1) hold, then*

$$
\sup_{\tau\leq t\leq\tau+T} E|X(t)-y(t)|^2\leq C\Delta \quad \forall T>0
$$

where C is constant which is independent of Δ.

Proof. Note that

$$\begin{aligned}X(t)-y(t)=&G(Z_2(t))-G(y(t-\tau))+\int_\tau^t f(Z_1(s),Z_2(s),\overline{r}(s))ds\\&-\int_\tau^t f(y(s),y(s-\tau),r(s))ds+\int_\tau^t g(Z_1(s),Z_2(s),\overline{r}(s))dB(s)\\&-\int_\tau^t g(y(s),y(s-\tau),r(s))dB(s)+\int_\tau^t\int_{-\infty}^{+\infty} h(Z_1(s),Z_2(s),u)\upsilon(ds,du)\\&-\int_\tau^t\int_{-\infty}^{+\infty} h(y(s),y(s-\tau),u)\upsilon(ds,du).\end{aligned}$$

Hence

$$\begin{aligned}E|X(t)-y(t)|^2&=E|G(Z_2(t))-G(y(t-\tau))+J(t)|^2\\&\le\frac{1}{K}E|G(Z_2(t))-G(y(t-\tau))|^2+\frac{1}{1-K}E|J(t)|^2\\&\le KE|Z_2(t)-y(t-\tau)|^2+\frac{1}{1-K}E|J(t)|^2\\&\le\sqrt{K}E|X(t-\tau)-y(t-\tau)|^2+\frac{K}{1-\sqrt{K}}E|Z_2(t)-X(t-\tau)|^2\\&+\frac{1}{1-K}E|J(t)|^2,\end{aligned}$$

where

$$\begin{aligned}J(t)=&\int_\tau^t f(Z_1(s),Z_2(s),\overline{r}(s))-f(y(s),y(s-\tau),r(s))ds\\&+\int_\tau^t g(Z_1(s),Z_2(s),\overline{r}(s))-g(y(s),y(s-\tau),r(s))dB(s)\\&+\int_\tau^t\int_{-\infty}^{+\infty} h(Z_1(s),Z_2(s),u)-h(y(s),y(s-\tau),u)\upsilon(ds,du),\end{aligned}$$

compute

$$\begin{aligned}E|J(t)|^2\le&3TE\int_\tau^t|f(Z_1(s),Z_2(s),\overline{r}(s))-f(y(s),y(s-\tau),r(s))|^2ds\\&+3E\int_\tau^t|g(Z_1(s),Z_2(s),\overline{r}(s))-g(y(s),y(s-\tau),r(s))|^2ds\\&+3E\int_\tau^t\int_{-\infty}^{+\infty}|h(Z_1(s),Z_2(s),u)-h(y(s),y(s-\tau),u)|^2\pi(du)ds\\\le&6TE\int_\tau^t|f(Z_1(s),Z_2(s),\overline{r}(s))-f(Z_1(s),Z_2(s),r(s))|^2ds\\&+6E\int_\tau^t|g(Z_1(s),Z_2(s),\overline{r}(s))-g(Z_1(s),Z_2(s),r(s))|^2ds\\&+3(2T+3)K_1\int_\tau^t(E|Z_1(s)-y(s)|^2+E|Z_2(s)-y(s-\tau)|^2).\end{aligned}$$

Using the method of [16],we obtain

$$E|J(t)|^2\le6(T+1)C_5\varDelta+3(2T+3)K_1\int_\tau^t(E|Z_1(s)-y(s)|^2+E|Z_2(s)-y(s-\tau)|^2),$$

where $C_5=16C_2K_1T\gamma\varDelta$ and $\gamma=\max_{1\le i\le N}(-\gamma_{ii})$.

By taking the supremum, applying Lemma 4 and Lemma 5, we have

$$\sup_{\tau\le r\le t} E|X(r)-y(r)|^2 \le \sqrt{K}\sup_{\tau\le r\le t} E|X(r)-y(r)|^2 + \frac{K}{1-\sqrt{K}}C_4\Delta + \frac{6(T+1)C_5}{1-K}\Delta + \frac{6(2T+3)K_1T(C_3E\|\xi\|^2+C_4)}{1-K}\Delta + \frac{12(2T+3)K_1}{1-K}\int_\tau^t \sup_{\tau\le r\le s} E|X(r)-y(r)|^2 ds,$$

$$\sup_{\tau\le r\le t} E|X(r)-y(r)|^2 \le [\frac{C_4K}{(1-\sqrt{K})^2}E\|\xi\|^2 + \frac{6(T+1)C_5}{(1-K)(1-\sqrt{K})} + \frac{6(2T+3)K_1T(C_3E\|\xi\|^2+C_4)}{(1-K)(1-\sqrt{K})}]\Delta + \frac{12(2T+3)K_1}{(1-K)(1-\sqrt{K})}\int_\tau^t \sup_{\tau\le r\le s} E|X(r)-y(r)|^2 ds.$$

We observe the result by Gronwall inequality

$$\sup_{\tau\le r\le t} E|X(r)-y(r)|^2 \le [\frac{C_4K}{(1-\sqrt{K})^2}E\|\xi\|^2 + \frac{6(T+1)C_5}{(1-K)(1-\sqrt{K})} + \frac{6(2T+3)K_1T(C_3E\|\xi\|^2+C_4)}{(1-K)(1-\sqrt{K})}] \times exp\{\frac{12(2T+3)K_1T}{(1-K)(1-\sqrt{K})}\}\Delta.$$

Let

$$C = [\frac{C_4KE\|\xi\|^2}{(1-\sqrt{K})^2} + \frac{6(T+1)C_5}{(1-K)(1-\sqrt{K})} + \frac{6(2T+3)K_1T(C_3E\|\xi\|^2+C_4)}{(1-K)(1-\sqrt{K})}]exp\{\frac{12(2T+3)K_1T}{(1-K)(1-\sqrt{K})}\},$$

thus

$$\sup_{\tau\le t\le \tau+T} E|X(t)-y(t)|^2 \le C\Delta.$$

Theorem 2. *Let (H_1) hold, if for some $\Delta>0$, the EM approximate solution is exponentially stable on the NSDDEJ (2.1), and following inequality hold:*

$$\frac{\beta_3\Delta}{E\|\xi\|^2} + 2\sqrt{\frac{\beta_3H\Delta}{E\|\xi\|^2}}e^{-\frac{1}{2}\gamma(T-2\tau)} + He^{-\gamma(T-2\tau)} \le e^{-\frac{1}{2}\gamma T}, \tag{8}$$

where $T=\tau(9+In[\frac{4logH}{\gamma\tau}])$, $\beta_3 = 2C(T-\tau)$ and $In[\frac{4logH}{\gamma\tau}]$ is the integer part of $\frac{4logH}{\gamma\tau}$. Then the NSDDEJ (2.1) is exponentially stable, namely

$$E|y(t;0,\xi)|^2 \le ME\|\xi\|^2e^{-\lambda t}, \quad \forall t\ge 0.$$

Moreover, $\lambda=\frac{1}{2}\gamma, M=C_1e^{\frac{1}{2}\lambda T}[\frac{\beta_4\Delta}{E\|\xi\|^2}+2\sqrt{\frac{\beta_4H\Delta}{E\|\xi\|^2}}+H]$ and $\beta_4=C(T-\tau)$.

To obtain the proof of this theorem,we must give the following lemma.

Lemma 7. *If (H_1) hold, then*

$$\sup_{0\le t\le\tau} E|y(t;0,\xi)|^2 \le C_1E\|\xi\|^2,$$

where C_1 was defined in lemma 2. Moreover

$$\sup_{\tau\le t\le \tau+T} E|X(t)-y(t)|^2 \le C\Delta \quad \forall T>0,$$

where $X(t)=X(t;\tau,y_\tau)$ and C has been defined.

Proof. The proof of this theorem is very similar for Lemma 2 and Lemma 6, so we omit it.

Applying Theorem 1 and Theorem 2, we obtain the equivalent condition:

Theorem 3. *If (H_1) hold, then the NSDDEJ (2.1) is exponentially stable in mean square if and only if for some Δ, the EM approximate solution is exponentially stable in mean square on the NSDDEJ (2.1) with following inequality hold:*

$$\frac{\beta_3 \Delta}{E\|\xi\|^2} + 2\sqrt{\frac{\beta_3 H \Delta}{E\|\xi\|^2}} e^{-\frac{1}{2}\gamma(T-2\tau)} + He^{-\gamma(T-2\tau)} \leq e^{-\frac{1}{2}\gamma T},$$

where $T = \tau(9 + In[\frac{4logH}{\gamma\tau}])$, $\beta_3 = 2C(T-\tau)$.

3 Proof of theorem

Proof of theorem 1: We write $X(t) = X(t;0,\xi), y(t) = y(t;\tau, X_\tau)$. The property of exponential stability on the NSDDEJ (2.1) is that:

$$E|y(t)|^2 \leq ME\|X_\tau\|^2 e^{-\lambda(t-\tau)} \quad \forall t \geq \tau. \tag{9}$$

By the definition of T, we have

$$Me^{-\lambda(T-2\tau)} \leq e^{-\frac{3}{4}\lambda T}. \tag{10}$$

Applying Lemma 6 and (9), we obtain

$$\begin{aligned}\sup_{T-\tau\leq t\leq 2T-\tau} E|X(t)|^2 \leq& (1+\alpha)\beta_1\Delta + (1+\alpha^{-1})ME\|X_\tau\|^2 e^{-\lambda(T-2\tau)} \\ \leq& [(1+\alpha)\frac{\beta_1}{E\|\xi\|^2}\Delta + (1+\alpha^{-1})Me^{-\lambda(T-2\tau)}] \sup_{-\tau\leq t\leq\tau} E|X(t)|^2,\end{aligned}$$

where $\beta_1 = 2C(T-\tau)$.

Let $\alpha = \sqrt{\frac{Me^{-2\lambda(T-\tau)}E\|\xi\|^2}{\beta_1\Delta}}$, we have

$$\sup_{T-\tau\leq t\leq 2T-\tau} E|X(t)|^2 \leq K_2(\Delta) \sup_{-\tau\leq t\leq\tau} E|X(t)|^2, \tag{11}$$

where $K_2(\Delta) = \frac{\beta_1\Delta}{E\|\xi\|^2} + 2\sqrt{\frac{\beta_1\Delta}{E\|\xi\|^2}M\Delta} e^{-\frac{1}{2}(T-2\tau)} + Me^{-\lambda(T-2\tau)}$.

By (10), we obtain that

$$K_2(0) = K_2(\Delta)Me^{-\lambda(T-2\tau)} \leq e^{-\frac{3}{4}\lambda T}.$$

Because of the monotonically increasing for $K_2(\Delta)$, then exist Δ^*, for all $\Delta < \Delta^*$, we observe $K_2(\Delta) \leq e^{-\frac{1}{2}\lambda T}$.

Using (11),we obtain that

$$\sup_{T-\tau\leq t\leq 2T-\tau} E|X(t)|^2 \leq e^{-\frac{1}{2}\lambda T} \sup_{-\tau\leq t\leq\tau} E|X(t)|^2. \tag{12}$$

For any integer $i \geq 0$, write $X(t) = X(t; iT, X_{iT}), \forall t \geq iT$. Repeat the method of (12),we have

$$\sup_{(i+1)T-\tau\leq t\leq(i+2)T-\tau} E|X(t)|^2 \leq e^{-\frac{1}{2}\lambda T} \sup_{iT-\tau\leq t\leq iT+\tau} E|X(t)|^2. \tag{13}$$

Applying (13),we obtain that

$$\sup_{(i+1)T-\tau\le t\le (i+2)T-\tau} E|X(t)|^2 \le e^{-\frac{1}{2}\lambda T} \sup_{iT-\tau\le t\le (i+1)T\tau} E|X(t)|^2$$
$$\le e^{-\frac{1}{2}(i+1)\lambda T} \sup_{-\tau\le t\le T-\tau} E|X(t)|^2. \quad (14)$$

By Lemma 6 and (9),we observe that

$$\sup_{\tau\le t\le T-\tau} E|X(t)|^2 \le (1+\alpha)\beta_2 \Delta + (1+\alpha^{-1}) M E\|X_\tau\|^2$$
$$\le [(1+\alpha)\frac{\beta_2}{E\|\xi\|^2}\Delta + (1+\alpha^{-1})M] \sup_{-\tau\le t\le \tau} E|X(t)|^2,$$

where $\beta_2 = C(T-\tau)$. Let $\alpha = \sqrt{\frac{ME\|\xi\|^2}{\beta_2 \Delta}}$, we have

$$\sup_{\tau\le t\le T-\tau} E|X(t)|^2 \le [\frac{\beta_2}{E\|\xi\|^2}\Delta + 2\sqrt{\frac{\beta_2 \Delta M}{E\|\xi\|^2}} + M] \sup_{-\tau\le t\le \tau} E|X(t)|^2$$

and let Δ^* be smaller, so that $\frac{\beta_2}{E\|\xi\|^2}\Delta + 2\sqrt{\frac{\beta_2\Delta M}{E\|\xi\|^2}} + M \le 2M$, for any $\Delta < \Delta^*$, we observe that

$$\sup_{\tau\le t\le T-\tau} E|X(t)|^2 \le 2M \sup_{-\tau\le t\le \tau} E|X(t)|^2. \quad (15)$$

Using (15) and (14),we have

$$\sup_{(i+1)T-\tau\le t\le (i+2)T-\tau} E|X(t)|^2 \le 2Me^{-\frac{1}{2}(i+1)\lambda T} \sup_{-\tau\le t\le \tau} E|X(t)|^2.$$

Applying Lemma 2, we obtain that

$$\sup_{(i+1)T-\tau\le t\le (i+2)T-\tau} E|X(t)|^2 \le 2Me^{-\frac{1}{2}(i+1)\lambda T} C_1 E\|\xi\|^2,$$

while

$$\sup_{0\le t\le T-\tau} E|X(t)|^2 \le 2MC_1 E\|\xi\|^2.$$

Hence

$$E|X(t)|^2 \le 2MC_1 e^{\frac{1}{2}\lambda T} E\|\xi\|^2 e^{-\frac{1}{2}\lambda t}$$
$$\le HE\|\xi\|^2 e^{-\gamma t} \quad t\ge 0.$$

Proof of theorem 2: Because the proof is very similar to the proof of theorem 1, we choose the different part. Write $y(t;o,\xi) = y(t), X(t;\tau,y_\tau) = X(t)$, the property is that

$$E|X(t)|^2 \le HE\|y_\tau\|^2 e^{-\lambda(t-\tau)} \quad \forall t\ge \tau. \quad (16)$$

Applying Lemma 7 and (15),we have

$$\sup_{T-\tau\le t\le 2T-\tau} E|y(t)|^2 \le [\frac{\beta_3}{E\|\xi\|^2}\Delta + 2\sqrt{\frac{\beta_3\Delta H}{E\|\xi\|^2}} e^{-\frac{1}{2}\gamma(T-2\tau)} + He^{-\gamma(T-2\tau)}] \sup_{-\tau\le t\le \tau} E|y(t)|^2. \quad (17)$$

Using (8),we obtain that

$$\sup_{T-\tau\le t\le 2T-\tau} E|y(t)|^2 \le e^{-\frac{1}{2}\gamma T} \sup_{-\tau\le t\le \tau} E|y(t)|^2. \quad (18)$$

Repeat the method of (17), we observe that

$$\sup_{(i+1)T-\tau\leq t\leq(i+2)T-\tau} E|y(t)|^2 \leq e^{-\frac{1}{2}(i+1)\gamma T} \sup_{-\tau\leq t\leq T-\tau} E|y(t)|^2 \quad i\geq 0, \tag{19}$$

while

$$\sup_{\tau\leq t\leq T-\tau} E|y(t)|^2 \leq [\frac{\beta_4}{E\|\xi\|^2}\Delta + 2\sqrt{\frac{\beta_4 \Delta M}{E\|\xi\|^2}} + H] \sup_{-\tau\leq t\leq\tau} E\|y(t)\|^2, \tag{20}$$

where $\beta_4 = C(T-\tau)$.

Hence

$$\sup_{\tau\leq t\leq T-\tau} E|y(t)|^2 \leq C_1[\frac{\beta_4}{E\|\xi\|^2}\Delta + 2\sqrt{\frac{\beta_4 \Delta M}{E\|\xi\|^2}} + H]E\|\xi\|^2. \tag{21}$$

Using (20) and (21), we have

$$\begin{aligned}\sup_{\tau\leq t\leq T-\tau} E|y(t)|^2 \leq & C_1 e^{\frac{1}{2}\gamma T}[\frac{\beta_4}{E\|\xi\|^2}\Delta + 2\sqrt{\frac{\beta_4 \Delta M}{E\|\xi\|^2}} + H]E\|\xi\|^2 e^{-\frac{1}{2}\gamma t} \\ \leq & ME\|\xi\|^2 e^{-\lambda t} \quad t\geq 0.\end{aligned}$$

Proof of theorem 3: The one part of proof is the proof of theorem 2, so we only give the other. Applying theorem 1, we obtain that exist $\Delta^* > 0$, for any $\Delta < \Delta^*$, the EM approximate solution is exponential stability on the NSDDEJ (2.1). Moreover, γ, H are independent of Δ. By the definition of T, we have

$$He^{-\gamma(T-2\tau)} \leq e^{-\frac{3}{4}\gamma T} < e^{-\frac{1}{2}\gamma T}.$$

Hence, the inequality in the theorem 1 hold with sufficiently small Δ.

All the proof have been completed.

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