Principal eigenvalues of the $p$-Laplacian with the boundary condition involving indefinite weight

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Abstract. This paper deals with principal eigenvalues of the following class of boundary value problems

\[ \begin{aligned}
-\Delta_p u &= \lambda a(x) |u|^{p-2} u, \quad x \in \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} &= g(x,u), \quad x \in \partial \Omega,
\end{aligned} \]

where $\Omega$ is a bounded region in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $a(x)$ is an indefinite weight function and $g(x,u)$ is a Caratheodory function.

Keywords: principal eigenvalue, indefinite weight function, nonlinear boundary condition

1 Introduction

This paper deals with the existence of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the following class of nonlinear boundary condition problems

\[ \begin{aligned}
-\Delta_p u &= \lambda a(x) |u|^{p-2} u, \quad x \in \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} &= g(x,u), \quad x \in \partial \Omega,
\end{aligned} \]

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator with $p > 2$, $\Omega \subseteq \mathbb{R}^N$ is a connected bounded domain with a smooth boundary $\partial \Omega$, the outward unit normal to which is denoted by $n$. The function $a(x)$ is assumed to be continuous in $\partial \Omega$ and $\Omega$ which changes sign on $\Omega$. Here we say a function $a(x)$ changes sign if the measure of the sets $\{x \in \Omega; a(x) > 0\}$ and $\{x \in \Omega; a(x) < 0\}$ are both positive.

We consider a special type of function $g(x,u) : \Omega \times R \to R$, $g \in C^\beta(\partial \Omega \times R)$ which was excluded in [1]. More precisely we investigate the problem of the type

\[ \begin{aligned}
-\Delta_p u &= \lambda a(x) |u|^{p-2} u, \quad x \in \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + f(x) |u|^{p-2} &= 0, \quad x \in \partial \Omega,
\end{aligned} \]

where $f(x) : \partial \Omega \to R$ is a continuous function which satisfies: $f(x) \geq 0$ on $\partial \Omega$ or $f(x) < 0$ on $\partial \Omega$, and we find a necessary condition to have principal eigenvalues for the problem (1) in each case.

The operator $\Delta_p$ with $p \neq 2$ arises from a variety of physical phenomena. It is used in non-Newtonian fluids, in some reaction diffusion problems, as well as in flow through porous media. It also appears in nonlinear elasticity, glaceology, and petroleum extraction. Diaz\cite{4} collected detailed references on physical background and presented mathematical treatments of free boundary problem associated with $\Delta_p$.

The problem such as (1) in the case $p = 2$ and $f(x) = \alpha \in R$ have been studied in recent years because of associated nonlinear problem arising in the study of population genetics (see [5]). The study of the ordinary

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For every \( f(x) \), it is true that the result does not hold, i.e., there exist \( S \) of variational arguments and that an eigenfunction corresponding to \( \Delta \) was studied by many authors. Senn and Hess investigated the boundedness below of \( \mu \), while Guedda and Veron studied the one dimensional bifurcation phenomena of \( \Delta_p \).

In the case \( p = 2 \), eigenvalue problems of second order elliptic operator with Neumann boundary conditions were studied by many authors. Senn and Hess studied the existence and uniqueness of the eigenvalue Neumann problems with indefinite weight.

In this paper we consider the problem (1) that occurs in some mathematical models in applied sciences. It is also related to the class of boundary value problems arising in the theory of conformal transformation of Riemannian metrics. The case \( p = 2 \) was analyzed in [1] completely. We shall investigate how the principal eigenvalues of (1) depend on \( f(x) \), obtaining new results for the cases \( f(x) \equiv 0 \), \( f(x) > 0 \) on \( \partial \Omega \) and \( f(x) \) on \( \partial \Omega \). The case \( f(x) \equiv 0 \) on \( \partial \Omega \), seems to have been considered far less often than the case \( f(x) \geq 0 \) on \( \partial \Omega \), probably because it is more natural that the flux across the boundary should be outwards if there is a positive concentration at the boundary, and also because \( f(x) \geq 0 \) is an easier condition to use when applying the maximum principle to discuss positive solutions. Our approach in this paper follows the technique of Hess and Kato. However, the study is nonlinear with nonlinear boundary condition and the associated operator is not selfadjoint.

In the next section the existence and multiplicity of the principal eigenvalues of (1) depend on \( f(x) \) are proved.

### 2 Existence results

In this section we will setup an appropriate functional analysis framework for our problem. We work in the Sobolev space \( X = W^{1,p}(\Omega) \) with an ordinary norm

\[
||u||_X = ||\nabla u||_{L^p(\Omega)} = \left( \int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}}.
\]

We consider for fixed \( \lambda \), the eigenvalue problem

\[
\begin{align*}
-\Delta_p u - \lambda a(x)u|u|^{p-2} &= \mu u|u|^{p-2}, & x \in \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + f(x)|u|^{p-2} &= 0, & x \in \partial \Omega.
\end{align*}
\]

We denote the lowest eigenvalue of (2) by \( \mu(\lambda) \). Let

\[
S_\lambda = \{ J_\lambda(\phi) = \int_{\Omega} |\nabla \phi|^p dx - \lambda \int_{\Omega} a(x)|\phi|^p dx + \int_{\partial \Omega} f(x)|\phi|^p d\sigma; \phi \in X, ||u||_{L^p(\Omega)} = 1 \}.
\]

Suppose that \( f(x) \geq 0 \). It is easy to see that for fixed \( \lambda \), \( S_\lambda \) is bounded below and \( \mu(\lambda) = \inf S_\lambda \), by using variational arguments and that an eigenfunction corresponding to \( \mu(\lambda) \) does not change sign on \( \Omega \). Thus, clearly, \( \lambda \) is a principal eigenvalue of the problem (1) if and only if \( \mu(\lambda) = 0 \).

When \( f(x) < 0 \), there exists \( m < 0 \) such that \( m < f(x) < 0 \), by using the continuity of \( f(x) \) on \( \partial \Omega \). The boundedness below of \( S_\lambda \) is a consequence of the following lemma.

**Lemma 1.** For every \( \varepsilon > 0 \) there exists a constant \( C(\varepsilon) > 0 \) such that

\[
\int_{\partial \Omega} |\phi|^p d\sigma \leq \varepsilon \int_{\Omega} |\nabla \phi|^p dx + C(\varepsilon) \int_{\Omega} |\phi|^p dx
\]

for all \( \phi \in X \).

**Proof.** Suppose that the result does not hold, i.e., there exist \( \varepsilon_0 > 0 \) and a sequence \( \{ \phi_n \} \subset X \) such that \( ||\phi_n||_X = 1 \) and

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\[ \int_{\Omega} |\phi_n|^p \, d\sigma \geq \varepsilon_0 + n \int_{\Omega} |\phi_n|^p \, dx. \]  

Suppose first that \( \{||\phi_n||_{L^p(\Omega)}\} \) is unbounded. Then we assume without loss of generality that \( ||\phi_n||_{L^p(\Omega)} \to \infty \). Hence using Sobolev embedding theorem, we obtain \( ||\phi_n||_X \to \infty \), which is impossible. 

Suppose now that \( \{||\phi_n||_{L^p(\Omega)}\} \) is bounded. Then \( \{\phi_n\} \) is bounded in \( X \). So we may assume without loss of generality that \( \phi_n \to \phi_0 \) in \( X \) for some \( \phi_0 \in X \). Since \( X \) may be compactly embedded in \( L^p(\Omega) \) and \( L^p(\partial \Omega) \), it follows that \( \phi_n \to \phi_0 \) in \( L^p(\Omega) \) and \( L^p(\partial \Omega) \). Thus \( \{\phi_n\} \) is bounded in \( L^p(\partial \Omega) \). It follows from (3) that \( \phi_n \to 0 \) in \( L^p(\partial \Omega) \) and so in \( L^p(\partial \Omega) \), but this is impossible because of (3). \( \square \)

**Corollary 1.** Suppose that \( f(x) < 0 \) on \( \partial \Omega \), then \( S_{\lambda} \) is bounded below independently of \( \phi \in X \).

**Proof.** Choose \( \varepsilon > 0 \) such that \( \varepsilon < -\frac{1}{m} \). By using above lemma, there exists \( c(\varepsilon) > 0 \) such that

\[ \int_{\partial \Omega} |\phi|^p \, d\sigma \leq \varepsilon \int_{\Omega} |\nabla \phi|^p \, dx + C(\varepsilon) \int_{\Omega} |\phi|^p \, dx \]

for all \( \phi \in X \). Let \( \phi \in X \) with \( ||\phi_n||_{L^p(\Omega)} = 1 \) be arbitrary. Then

\[
\begin{align*}
J_\lambda(\phi) & \geq \int_{\Omega} |\nabla \phi|^p \, dx - \lambda \int_{\Omega} a(x)|\phi|^p \, dx + m \int_{\partial \Omega} |\phi|^p \, d\sigma \\
& \geq \int_{\Omega} |\nabla \phi|^p \, dx - \lambda \int_{\Omega} a(x)|\phi|^p \, dx + m\varepsilon \int_{\Omega} |\nabla \phi|^p \, dx + C(\varepsilon) \int_{\Omega} |\phi|^p \, dx \\
& \geq (1 + m\varepsilon) \int_{\Omega} |\nabla \phi|^p \, dx - \lambda \sup_{x \in \Omega} |a(x)| + mC(\varepsilon) \\
& \geq -\lambda \sup_{x \in \Omega} |a(x)| + mC(\varepsilon),
\end{align*}
\]

i.e., the set \( S_{\lambda} \) is bounded below. \( \square \)

Using above corollary we have \( \mu(\lambda) = \inf S_{\lambda} \) and that an eigenfunction corresponding to \( \mu(\lambda) \) does not changes sign on \( \Omega \). Thus it is again the case that \( \lambda \) is a principal eigenvalue of (1) if and only if \( \mu(\lambda) = 0 \).

For fixed \( \phi \in X \); \( \lambda \to J_\lambda(\phi) \) is affine and so a concave function. It follows that \( \lambda \to \inf J_\lambda(\phi) = \mu(\lambda) \) is a concave function. Also it is easy to see that \( \mu(\lambda) \to -\infty \) as \( \lambda \to \pm \infty \). Thus \( \mu(\lambda) \) is an increasing function until it attains its maximum, and is a decreasing function thereafter.

As can be seen from variational characterization of \( \mu(\lambda) \), \( \mu(0) > 0 \) for \( 0 < f(x) < \infty \), and so \( \mu(\lambda) \) has exactly two zeroes for \( 0 < f(x) < \infty \). Thus in this case (1) has exactly two principal eigenvalues; one positive and one negative.

In the case \( f(x) \leq 0 \) we have that \( \mu(\lambda) \leq 0 \), and the situation is less clear.

**Lemma 2.** Suppose that \( u_0 \) is a positive eigenfunction of (2) corresponding to the principal eigenvalue \( \mu(\lambda) \). Then

\[
\frac{d\mu}{d\lambda}(\lambda) = -\frac{\int_{\Omega} a(x)u_0^p \, dx}{\int_{\Omega} u_0^p \, dx},
\]

provided that

\[ u_0 \nabla u_0 \nabla u_0' = |\nabla u_0|^2 u_0', \quad (*) \]

where \( u_0' = \frac{du_0}{dx}(x, \lambda) \).

**Proof.** Regarding \( u_0 \) and \( \mu \) as functions of \( \lambda \), we have

\[
\begin{cases}
-\Delta_p u_0 - \lambda a(x)u_0^{p-1} = \mu u_0^{p-1}, & x \in \Omega, \\
|\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} + f(x)u_0^{p-1} = 0, & x \in \partial \Omega.
\end{cases}
\]

(4)

Then \( u_0' \) satisfies


\[(p - 1)(-\nabla(|\nabla u_0|^2 \nabla u'_0) - \lambda_0(x)u_0^{p-2} - \mu u'_0u_0^{p-2}) = a(x)u_0^{p-1} + \frac{d\mu}{d\lambda}u_0^{p-1}, \quad (5)\]

in \( \Omega \), and

\[(p - 2)|\nabla u_0|^4 \nabla u_0 \nabla u'_0 \frac{\partial u_0}{\partial n} + |\nabla u_0|^2 \frac{\partial u'_0}{\partial n} + (p - 1)f(x)u_0^{p-2}u'_0 = 0, \quad (6)\]
on \( \partial \Omega \).

Multiplying (5) by \( u_0 \) and integrating over \( \Omega \) gives

\[(p - 1)(p - 2) \int_{\partial \Omega} |\nabla u_0|^2 \nabla u_0 \nabla u'_0 \frac{\partial u_0}{\partial n} d\sigma + (p - 1) \int_{\partial \Omega} f(x)u_0^{p-1}u'_0 d\sigma + \int_\Omega |\nabla u_0|^2 \nabla u_0 \nabla u'_0 dx - \lambda \int_\Omega a(x)u_0^{p-1}u'_0 dx - \mu \int_\Omega u_0^{p-1}u'_0 dx = \mu \int_\Omega u_0^{p-1}u'_0 dx. \quad (7)\]

Here the boundary condition in (6) was used.

We now claim that the left hand side of (7) vanishes which completes the proof.

Multiplying (4) by \( u_0^p \) and integrating over \( \Omega \) gives

\[\int_\Omega |\nabla u_0|^2 \nabla u_0 \nabla u'_0 dx + \int_{\partial \Omega} f(x)u_0^{p-1}u'_0 d\sigma - \lambda \int_\Omega a(x)u_0^{p-1}u'_0 dx = \mu \int_\Omega u_0^{p-1}u'_0 dx. \]

So

\[\int_\Omega |\nabla u_0|^2 \nabla u_0 \nabla u'_0 dx - \lambda \int_\Omega a(x)u_0^{p-1}u'_0 dx - \mu \int_\Omega u_0^{p-1}u'_0 dx = - \int_{\partial \Omega} f(x)u_0^{p-1}u'_0 d\sigma. \quad (8)\]

Substituting (8) in (7), gives

\[(p - 1)(p - 2) \int_{\partial \Omega} |\nabla u_0|^2 \nabla u_0 \nabla u'_0 \frac{\partial u_0}{\partial n} d\sigma + (p - 1) \int_{\partial \Omega} f(x)u_0^{p-1}u'_0 d\sigma - \int_{\partial \Omega} f(x)u_0^{p-1}u'_0 d\sigma = \]

\[\int_\Omega a(x)u_0^{p-1}u'_0 dx + \frac{d\mu}{d\lambda} \int_\Omega u_0^{p-1}u'_0 dx. \quad (9)\]

By direct calculation we have

\[(p - 1)(p - 2) \int_{\partial \Omega} f(x)u_0^p(\frac{\nabla u_0 \nabla u'_0}{|\nabla u_0|^2}) d\sigma + (p - 2) \int_{\partial \Omega} f(x)u_0^{p-1}u'_0 d\sigma = \]

\[(p - 1)(p - 2) \int_{\partial \Omega} f(x)u_0^p(\frac{\nabla u_0 \nabla u'_0}{|\nabla u_0|^2}) - \frac{u'_0}{u_0} d\sigma = 0. \]

Hence the left hand side of (9) vanishes and so the result follows. \( \square \)

The above lemma shows that where \( \lambda \to \mu(\lambda) \) is an increasing (decreasing) function we have that \( \int_\Omega a(x)u_0^p dx < 0 (> 0) \), and at critical point we must have \( \int_\Omega a(x)u_0^p dx = 0 \), whenever \( u_0 \) satisfies the condition (\#).

The next lemma shows that \( \mu(\lambda) \) has a unique critical point.

**Lemma 3.** Suppose that \( u_0 \) is a positive eigenfunction of (2) corresponding to the principal eigenvalue \( \mu(\lambda_0) \), such that \( \int_\Omega a(x)u_0^p dx = 0 \). Then \( \mu(\lambda_0) > \mu(\lambda) \) for \( \lambda \neq \lambda_0 \).

**Proof.** Regarding \( \mu \) as function of \( \lambda \), we have

\[\begin{align*}
-\Delta u_0 - \lambda a(x)u_0^{p-1} &= \mu(\lambda_0)u_0^{p-1}, & x \in \Omega, \\
|\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} + f(x)u_0^{p-1} &= 0, & x \in \partial \Omega. 
\end{align*}\]

multiplying (10) by \( u_0 \) and integrating over \( \Omega \) gives

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\[
\int_{\Omega} |\nabla u_0|^p \, dx + \int_{\partial \Omega} f(x) u_0^p \, d\sigma = \mu(0) \int_{\Omega} u_0^p \, dx.
\]

Let \( v_0 = \frac{u_0}{||u_0||_X} \). Then
\[
\mu(\lambda_0) = \int_{\Omega} |\nabla v_0|^p \, dx + \int_{\partial \Omega} f(x) v_0^p \, d\sigma,
\]
and
\[
\mu(\lambda) \leq \int_{\Omega} |\nabla v_0|^p \, dx + \int_{\partial \Omega} f(x) v_0^p \, d\sigma - \lambda \int_{\Omega} a(x)v_0^p \, dx = \int_{\Omega} |\nabla v_0|^p \, dx + \int_{\partial \Omega} f(x) v_0^p \, d\sigma = \mu(\lambda_0).
\]

We now show that \( \mu(\lambda) < \mu(\lambda_0) \), whenever \( \lambda \neq \lambda_0 \).

Suppose otherwise. Then \( \mu = \mu(\lambda) = \mu(\lambda_0) \) satisfies
\[
-\Delta_p u_0 - \lambda_0 a(x)u_0^{p-1} = \mu u_0^{p-1} \quad \text{in } \Omega; \quad |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} + f(x)u_0^{p-1} = 0 \quad \text{on } \partial \Omega,
\]
and
\[
-\Delta_p u_0 - \lambda a(x)u_0^{p-1} = \mu u_0^{p-1} \quad \text{in } \Omega; \quad |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial n} + f(x)u_0^{p-1} = 0 \quad \text{on } \partial \Omega,
\]
while, \( \lambda \neq \lambda_0 \), and this is a contradiction. □

The above result shows that the unique global maximum of \( \mu(\lambda) \) occurs when \( \lambda = \lambda_0 \). Hence the graph of \( \lambda \rightarrow \mu(\lambda) \) may have 2, 1 and 0 intersections with the \( \mu \)-axis, and so (1) may have 2, 1 and 0 principal eigenvalues.

As we shall see in the next theorem when \( f(x) > 0 \) on \( \partial \Omega \), (1) has 2 principal eigenvalues, one positive and one negative.

When \( f(x) \equiv 0 \) on \( \partial \Omega \), i.e., we have nonlinear Neumann boundary condition, \( \mu(0) = 0 \) and the corresponding eigenfunction is a constant. Hence \( \frac{\partial u_0}{\partial \nu}(0) > 0 (= 0) < 0 \) as \( \int_{\Omega} a(x) \, dx < 0 (= 0) > 0 \). Thus when \( f(x) \equiv 0 \) on \( \partial \Omega \), \( \mu = 0 \) is a principal eigenvalue in all cases; if \( \int_{\Omega} a(x) \, dx < 0 \), there is an additional positive principal eigenvalue; and, if \( \int_{\Omega} a(x) \, dx > 0 \), there is an additional negative principal eigenvalue and, if \( \int_{\Omega} a(x) \, dx = 0 \), \( \mu = 0 \) is the only principal eigenvalue.

We now find a necessary condition on \( f(x) \), when \( f(x) < 0 \) on \( \partial \Omega \), that under it there exist principal eigenvalues to problem (1). We first assume that \( \int_{\Omega} a(x) \, dx < 0 \).

**Theorem 1.** There exists \( c(\lambda_0, u_0) < 0 \) such that the problem (1) has an eigenfunction \( u_0 \) corresponding to the principal eigenvalue \( \lambda_0 \) if \( f(x) \) satisfies
\[
c(\lambda_0, u_0) < \int_{\partial \Omega} f(x) \, d\sigma < 0.
\]

**Proof.** Suppose \( f(x) < 0 \) on \( \partial \Omega \) and \( u_0 \) is a positive eigenvalue of (1) corresponding to a positive principal eigenvalue \( \lambda_0 \). By using the maximum principle, it follows that \( u_0(x) > 0 \) for all \( x \in \Omega \cup \partial \Omega = \bar{\Omega} \). Since \( f(x) < 0 \), we have
\[
0 = \mu_1(\lambda_0) = \inf\left\{ \int_{\Omega} |\nabla \phi|^p \, dx - \lambda \int_{\Omega} a(x)|\phi|^p \, dx + \int_{\partial \Omega} f(x)|\phi|^p \, d\sigma \right\}
\leq \inf\left\{ \int_{\Omega} |\nabla \phi|^p \, dx - \lambda \int_{\Omega} a(x)|\phi|^p \, dx \right\} = \mu_2(\lambda_0),
\]
where \( \mu_1 \) is the eigenvalue of the problem (2) for \( f(x) \leq 0 \) and \( \mu_2 \) is the eigenvalue of the problem (2) for \( f(x) \equiv 0 \), i.e., the Neumann problem. Hence \( \lambda_0 < \mu_0 \) (the positive eigenvalue of the Neumann problem). Dividing (1) by \( u_0^{p-1} \) and integrating over \( \Omega \), we have

References

on \( \Omega \) positive principal eigenfunction \( u \)

Hence

By direct calculation

$$\int_{\Omega} \frac{-\Delta_{p} u_{0}}{u_{0}^{p-1}} \, dx = \lambda_{0} \int_{\Omega} a(x) \, dx.$$ 

Since \( \lambda_{0} \) is a principal eigenvalue of (1) with corresponding positive principal eigenfunction \( u_{0} \), we obtain

$$-\Delta_{p} u_{0}^{\gamma} = \lambda_{0} a(x) u_{0}^{p+\gamma-1}$$

on \( \Omega \) and so

$$-\int_{\partial \Omega} |\nabla u_{0}|^{p-2} \frac{\partial u_{0}}{\partial n} u_{0}^{\gamma} \, d\sigma + \gamma \int_{\Omega} |\nabla u_{0}|^{p} u_{0}^{\gamma-1} \, dx = \lambda_{0} \int_{\Omega} a(x) u_{0}^{p+\gamma-1} \, dx.$$ 

Hence

$$\lambda_{0} \int_{\Omega} a(x) u_{0}^{p+\gamma-1} \, dx = \int_{\partial \Omega} f(x) u_{0}^{p+\gamma-1} \, d\sigma + \gamma \int_{\Omega} |\nabla u_{0}|^{p} u_{0}^{\gamma-1} \, dx$$

and so the required result holds.

If \( f(x) \equiv 0 \) the surface integral term vanishes and the result follows easily. \( \square \)

The following theorem gives us another property of the eigenfunctions.

**Theorem 2.** Let \( f(x) \geq 0 \) on \( \partial \Omega \) and suppose that \( \lambda_{0} \neq 0 \) is a principal eigenvalue of (1) with corresponding positive principal eigenfunction \( u_{0} \). Then \( \lambda_{0} \int_{\Omega} a(x) u_{0}^{\nu} \, dx > 0 \) for all \( \nu \geq p \).

**Proof.** Suppose \( f(x) > 0 \). Multiplying (1) by \( u_{0}^{\nu} \) where \( \gamma \geq 1 \), we obtain

$$\int_{\Omega} \frac{-\Delta_{p} u_{0}}{u_{0}^{p-1}} \, dx = (1-p) \int_{\Omega} |\nabla u_{0}|^{p} u_{0}^{1-p} \, dx + \int_{\partial \Omega} f(x) \, d\sigma.$$ 

Hence

$$\int_{\partial \Omega} f(x) \, d\sigma = \int_{\Omega} |\nabla u_{0}|^{p} u_{0}^{1-p} \, dx + \lambda_{0} \int_{\Omega} a(x) \, dx.$$ 

Since \( \lambda_{0} < \mu_{0} \), \( f(x) \) can not be too negative, and the proof is complete. \( \square \)

**References**


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