On positive weak solutions for a class of semipositone equations

G. A. Afrouzi*, M. Baghery, S. H. Rasouli

Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar, Iran

(Received September 25 2006, Accepted February 2 2007)

Abstract. In this paper, we study existence of positive weak solution for the semipositone problem
\[-\Delta_p u = \lambda u^\alpha - c, \quad x \in \Omega,\]
\[u = 0, \quad x \in \partial \Omega,\]
with zero Dirichlet boundary conditions in bounded domain \(\Omega \subset \mathbb{R}^N\) where \(\Delta_p\) denotes the p-Laplacian operator defined by \(\Delta_p z = \text{div}(|\nabla z|^{p-2} \nabla z); p > 1, \alpha < p - 1\) and \(c > 0\) is a parameter.

We establish positive constants \(c_0(\Omega)\) and \(\lambda^*(\Omega, c)\) such that the above equation has a positive solution when \(c \leq c_0\) and \(\lambda \leq \lambda^*\).

Keywords: semipositone problem, p-Laplacian, positive solution, Method of sub- and supersolution

1 Introduction

In this paper, we consider the existence of positive weak solution of the equation
\[
\left\{ \begin{array}{ll}
-\Delta_p u = \lambda u^\alpha - c, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{array} \right.
\]
where \(\Delta_p z = \text{div}(|\nabla z|^{p-2} \nabla z); p > 1, \alpha < p - 1\) and \(c > 0\) is a parameter. We also consider the equation in the bounded domain \(\Omega \subset \mathbb{R}^N (N \geq 2)\) with \(\partial \Omega\) of class \(C^2\) and connected.

Problems involving the “p-Laplacian” arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [7]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [3, 5]). We refer to [4, 6] for additional results in p-laplacian problems.

2 Existence results

Let \(W^{1,p}_0 = W^{1,p}_0(\Omega)\) denote the usual Sobolev space. We first give the definition of weak subsolution and supersolution of (1). Our approach is based on the method of sub-super solutions, see [1, 2].

Definition 1. We say that \(\psi \in W^{1,p}_0(\Omega)\) \((\phi \in W^{1,p}_0(\Omega))\) is a subsolution (a supersolution) to (1) if for any \(v \in W\) where \(W = \{v \in C^\infty_0(\Omega) \mid v \geq 0, v \in \Omega\}\), we have
\[
\int_\Omega |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla v \, dx - \int_\Omega [\psi^{p-1} - c] v \, dx \leq 0 (\geq 0).
\]
Now if there exists sub and super solutions \(\psi\) and \(\phi\) respectively such that \(0 \leq \psi \leq \phi\) for \(x \in \Omega\), then (1) has a positive solution \(u \in W^{1,p}_0(\Omega)\) such that \(\psi \leq u \leq \phi\). This follows from a result in [2].

* E-mail address: afrouzi@umz.ac.ir.

Published by World Academic Press, World Academic Union
Let $\lambda_1$ be the first eigenvalue of the $\Delta_p$ with Dirichlet boundary conditions and $\phi_1 \in C^1(\Omega)$ be a corresponding eigenfunction such that $\phi_1 > 0$ in $\Omega$ and $\|\phi_1\|_\infty = 1$. Let $m > 0, \delta > 0$ and $n > 0$ be such that

$$|\nabla \phi_1|^p - \lambda_1 \phi_1^p \geq m, \quad x \in \bar{\Omega}_\delta,$$

$$\phi_1 \geq n, \quad x \in \Omega \setminus \bar{\Omega}_\delta,$$

where $\bar{\Omega}_\delta = \{x \in \Omega; d(x, \partial \Omega) \leq \delta\}$.

Now we shall establish:

**Theorem 1.** There exist positive constants $c_0 = c_0(\Omega)$ and $\lambda^* = \lambda^*(\Omega, c)$ such that (1) has a positive solution for $c \leq c_0$ and $\lambda \geq \lambda^*$.

**Proof.** Let $\lambda_1, \phi_1, m, \delta, n$ and $\Omega_\delta$ be as describe in section 1. We establish the above theorem by the method of sub- and supersolution.

We shall varify that $\psi = ((p-1)/p)\phi_1^{p/(p-1)}$ is a subsolution of (1). A calculation shows that $\nabla \psi = \phi_1^{1/(p-1)} \nabla \phi_1$ and

$$\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w \, dx = \int_{\Omega} |\nabla \phi_1|^{p-2} \phi_1 \nabla \phi_1 \cdot \nabla w \, dx = \int_{\Omega} |\nabla \phi_1|^{p-2} \phi_1 \nabla \phi_1 w \, dx - \int_{\Omega} |\nabla \phi_1|^p w \, dx = \int_{\Omega} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w \, dx.$$

Now, in $\Omega_\delta$ we have $[\lambda_1 \phi_1^p - |\nabla \phi_1|^p] \leq -m$. Hence if $c \leq c_0 = m$ then $[\lambda_1 \phi_1^p - |\nabla \phi_1|^p] \leq [\lambda \phi^\alpha - c]$ in $\bar{\Omega}_\delta$.

Next, in $\Omega - \Omega_\delta$ we have $[\lambda_1 \phi_1^p - |\nabla \phi_1|^p] \leq \lambda_1$ while

$$\lambda \phi^\alpha - c \geq \lambda n^\alpha - c$$

Hence if $\lambda \geq \lambda^* = (\lambda_1 + c)/n^\alpha$ then $[\lambda_1 \phi_1^p - |\nabla \phi_1|^p] \leq \lambda \phi^\alpha - c$. Hence if $c \leq c_0$ and $\lambda \leq \lambda^*$ then (3) is satisfy and $\psi$ is a subsolution.

Next, we construct a supersolution $z$ of (1). Let $e_1(x)$ be the positive solution of the problems

$$\begin{cases}
-\Delta_p e_1 = 1, & x \in \Omega, \\
e_1 = 0, & x \in \partial \Omega.
\end{cases}$$

(4)

We denote $z = Ae_1(x)$ where the constant $A > 0$ is sufficiently large and to be chosen later. Let $w \in W^{1,p}_0(\Omega)$ with $w \geq 0$. Then we have

$$\int_{\Omega} |\nabla z|^{p-2} \nabla z \cdot \nabla w \, dx = A^{p-1} \int_{\Omega} w \, dx,$$

Let $l = \|e_1\|_\infty$. Since $\alpha < p - 1$, it is easy that there exists positive large constant $A$ such that

$$A^{p-1-\alpha} \geq \lambda l^\alpha.$$ 

These imply that
\[ A^{p-1} \geq \lambda z^\alpha \quad x \in \Omega, \]

then (3) is satisfy and \( z \) is a supersolution. Obviously, we have \( \psi(x) \leq z(x) \) in \( \Omega \) with large \( A \). thus, by comparison principle, there exists a solution \( u \) of (1) with \( \psi \leq u \leq z \). This completes the proof of Theorem 1.

References