The effect of different arrival rates on $\text{Geom}^\xi/G/1$ queue with multiple adaptive vacations and server setup/closedown times

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Abstract. In this paper, we discuss the batch arrival $\text{Geom}/G/1$ queue with different arrival rates during the multiple adaptive vacations and server setup/closedown times. We derive the probability generating function (PGF) of the steady-state queue length immediately after a service completion by the embedded Markov chain and the PGF of the queue length immediately after an arbitrary slot boundary. We also get the PGF of the stationary waiting times under first come first service (FCFS) service discipline. From the results, we obtain the conclusion that the steady-state queue length and waiting times have the property of stochastic decomposition. And we also get the generating function of busy period and give several special cases. Many discussed models about $\text{Geom}/G/1$ queue are special cases of our model. Finally, we give some numerical examples in order to verify the effect of different arrival rates on the model.

Keywords: multiple adaptive vacations, embedded Markov chain, stochastic decomposition, arrival rates, setup/closedown times

1 Introduction

Modern telecommunication systems have become more of digital systems these days than analog. It is therefore more appropriate to develop vacation models which are applicable to these systems using discrete time approach. There have been fewer models developed for vacation systems in discrete time compared to literature on continuous time models (see Doshi\textsuperscript{[3]}, Lee\textsuperscript{[4]} and Minh\textsuperscript{[7]}). As to the discrete-time $\text{Geom}/G/1$ queue with vacations, most works were concentrated on the study for models of multiple vacations and single vacation (see Takagi\textsuperscript{[10]}). Tian and Zhang\textsuperscript{[12, 14]} adopted the policy of multiple adaptive vacations. Based on this, we get a fully new model and generalize the results given by Zhang and Tian\textsuperscript{[14]}.

There have been some research results about the discrete-time batch arrival $\text{Geom}/G/1$ queue. Atencia and Moreno\textsuperscript{[2]} analyzed a discrete-time $\text{Geom}^\xi/G_H/1$ (where $\xi$ represents the random batch size) retrial queue with Bernoulli feedback. Artalejo\textsuperscript{[1]} considered a discrete-time $\text{Geom}^\xi/G/1$ retrial queue with control of admission. Concerning the batch arrival queue with vacations, Lee\textsuperscript{[5]} analyzed the $M^S/G/1$ queueing system with $N$-policy whose arrival process is compound Poisson by means of the supplementary variable method. As for the batch arrival $\text{Geom}/G/1$ queue under various vacation policies, Takagi\textsuperscript{[10]} analyzed the batch arrival $\text{Geom}/G/1$ queue under multiple vacations and single vacation policies, while we consider the model under multiple adaptive vacations and server setup/closedown times.

Almost all researches on discrete-time models, including those mentioned above, have studied the systems under the assumption that the arrival rate of the customers into the system is fixed. Actually, however,
there are many cases in which the customers do not have a constant arrival rate at an arbitrary time point. It may depend on time, system state, or server’s status. In these cases about $\text{Geom}^p/G/1$ queue, the arrival processes are no longer invariable compound Bernoulli processes. Sun [8] has considered the effect of different arrival rates on the $N$-policy of $M/G/1$ with server setup. Tang and Mao [11] analyzed the $M/G/1$ queue with $p$-entering discipline during server vacations and gave out the stochastic decomposition results. In this paper, we consider the case when the arrival rate $p(0 \leq p \leq 1)$ varies according to the server’s status: vacation, setup/closedown times and busy (idle) states.

In Section 2, the model under investigation is described. In section 3, we discuss the property of stochastic decomposition. We get the PGF of the steady-state queue length immediately after the service completion by the method of embedded Markov chain and we also get the PGF of the queue length immediately after an arbitrary slot boundary in section 3.1. In section 3.2, we concentrate on the waiting times. The PGF of the stationary waiting times of an arbitrary customer in the queue is given. And we also get the PGF of busy period in section 4. In section 5, we consider several special cases. Finally, we give some numerical examples in order to verify the effect of different arrival rates during vacation and server setup/closedown periods on the model.

2 Model description and notations

We consider a discrete-time batch arrival $\text{Geom}/G/1$ queueing system, where arrival process during the multiple adaptive vacations and server setup/closedown times has different arrival rates compared with that during busy (idle) period. The service time, denoted by $B$, is a generally distributed random variable with probability distribution $P\{B = j\} = b(j)$, the PGF $B(z)$, and the finite mean $E[B] = \mu^{-1}$ and second factorial moment $E[B^2] = b(2)$. We assume that this queueing model is an exhaustive service system in which the server continues serving the queue until the queue is empty.

One cycle begins right after the system is empty and the server starts a closedown time $C$ which has the probability distribution $P\{C = j\} = c(j)$, the PGF $C(z)$, and the finite mean $E[C]$ and second moment $E[C(C - 1)]$, the customers arrive at the system according to compound Bernoulli process with arrival rate $p_1$, this is the closedown period. Service begins after the closedown period on condition that there is no less than one customer arrival during the closedown time. Otherwise, the server starts vacations.

The server will take a random maximum number, denoted by $H$, of vacations on condition that there is no customer arrival during the closedown period. The probability distribution of $H$ is $P\{H = j\} = h_j$, $j = 1, 2, \ldots$. The random variable $H$ may represent the maximum number tasks or jobs available for server to work on if the system is empty. The consecutive vacations, denoted by $V_k$, $k = 1, 2, \ldots, H$, are i.i.d. random variables with the probability distribution $P\{V = j\} = v(j)$, the PGF $V(z)$, and the finite mean $E[V]$ and second moment $E[V(V - 1)]$. At each vacation completion instant, the server checks the system to see if there is any customer waiting and decides the action to take according to the state of the system. There are three cases:

**Case 1:** if there is any customer waiting, the server will no longer take another vacation and begin setup.

**Case 2:** if there is no customer waiting and the total number of vacations taken is still less than $H$, the server will take another vacation.

**Case 3:** if there is no customer waiting and the number of vacations taken is equal to $H$, the server will stay idle and wait for the next arrival.

Let $J$ represent the actual number of vacations taken by the server, then $J = \min\{H, k : V^{(k-1)} < T < V^{(k)}\}$, where $V^{(k)}(V^{(0)} \equiv 0)$ and $T$ stand for the sum of $k$ vacations and the inter-arrival times respectively. This type of limited number of vacations policy reflects the trade-off between the benefit of working on the queue and the benefit of doing other jobs represented by the vacations. In addition, we assume that the random variables $T, B, H$ and $V$ are mutually independent. Note that multiple vacations and single vacation policies are special cases of multiple adaptive vacations policy when $H = \infty$ and $H = 1$, respectively.

During the vacations, the customers arrive at the system according to compound Bernoulli process with arrival rate $p_2$, this is the vacation period. Let $A_I$ and $A_V$ denote the event that the first batch of customers arrive at an empty system occurs in an idle state and a server’s vacation state, respectively. We have
\[ P(A_l) = \sum_{v=1}^{\infty} h_v \sum_{k=0}^{\infty} P(V^{(v)} = k)(\overline{p}_2)^k = H(V(\overline{p}_2)), \]  

\[ P(A_V) = 1 - H(V(\overline{p}_2)), \]  

where \( \overline{p}_2 = 1 - p_2 \) and \( H(z) \) is the PGF of \( H \).

In case 1, the server begins his setup after the \( k \)th vacation time, in case 3, he begins his setup as soon as customer arrives. Now the arrival stream changes to compound Bernoulli process with arrival rate \( p_3 \). The length of the setup time \( U \) which has the probability distribution \( P\{U = j\} = u_j \), the PGF \( U(z) \), and the finite mean \( E[U] \) and second moment \( E[U(U - 1)] \) is generally distributed and independent of all others. This is the setup period. The customers arriving during the setup period as well as those arriving during the idle period are not served yet since the server is under setup. Now right after the server finishes setup period, customers begin to be served under the FCFS service discipline, because the customers of the same batch are random, the service order is arbitrary. This is the busy period. It ends when there are no customers remaining in the queue. The arrivals during the busy period form a compound Bernoulli process with arrival rate \( p \). In general, we assume \( p_1 \neq p_2 \neq p_3 \neq p \), and arrival interval, service time, vacation time and setup/closedown times are mutually independent.

We use the following notations throughout the paper:

- \( \xi; E[\xi] = r; D[\xi] = \sigma^2; \xi(z) \): random batch size and its mean, variance and the PGF.
- \( p; p_1; p_2; p_3 \): arrival rates during busy (idle), closedown, vacation, setup period, respectively.
- \( Q_b; Q_3; Q_n \): number of customers and batches when a busy period begins and queue length when the \( n \)th customer leaves.
- \( Q_b(z); Q_3(z); Q_n(z) \): PGFs of the number of customers and batches when a busy period begins and queue length when the \( n \)th customer leaves.
- \( Y_n; B_n \): the number of customers arriving during the \( n \)th service time and service time of the \( n \)th customer.
- \( X(B); N(B) \): the number of customers and batches arriving during a service time.
- \( \Theta, \Theta(z); \theta, \theta(z) \): busy period caused by a batch and a customer and their PGFs.
- \( W_{Q_b}, W_{Q_3}(z); W_{Q_1}, W_{Q_2}(z) \): batch and inter-batch waiting times of a customer and their PGFs.

### 3 Stochastic decomposition property

From the results below, we obtain the conclusion that the steady-state queue length and waiting times of an arbitrary customer have the stochastic decomposition property.

#### 3.1 The steady-state queue length and embedded markov chain

First, we consider the case where \( \xi \equiv 1 \). Let \( P_b \) represent the number of customers when a busy period begins, we give the distribution of additional queue length in Lemma 1.

**Lemma 1.** When \( \rho = rp_3\mu^{-1} < 1 \), the PGF of additional queue length, denoted by \( P_d \), is as

\[ P_d(z) = \frac{1 - P_b(z)}{E[P_b](1 - z)}, \]  

where

\[ P_b(z) = C(1 - p_1(1 - z)) - c(\overline{p}_1) + c(\overline{p}_1) \times \left\{ H(v(\overline{p}_2))zU(1 - p_3(1 - z)) + \frac{[1 - H(v(\overline{p}_2))]U(1 - p_3(1 - z))(V(1 - p_2(1 - z)) - v(\overline{p}_2))}{1 - v(\overline{p}_2)} \right\} \]  

\[ E[P_b] = p_1 E[C] + c(\overline{p}_1) \left\{ H(v(\overline{p}_2))(p_3E[U] + 1) + \frac{1 - H(v(\overline{p}_2))}{1 - v(\overline{p}_2)} [(1 - v(\overline{p}_2))p_3E[U] + p_2E[V]] \right\} \]
Based on the characteristics of our model, we can easily observe that there are three cases resulting in \( P_b = j (j \geq 1) \):

**Case 1:** There are \( j \) customer arrivals during closedown period and busy period begins immediately after the closedown time.

**Case 2:** There is no customer arrival during closedown period and \( H \) times vacations, but one customer arrival when the server is idle and \( j - 1 \) customer arrivals during setup period.

**Case 3:** There is no customer arrival during closedown period, but \( j \) customer arrivals during the \( k \)th vacation and setup period.

Combining the three cases, we can derive the distribution of \( P_b \) as

\[
b_j = P\{P_b = j\} = c_j + c(p_1) \left[ H(v(p_2))u_{j-1} + \frac{1 - H(v(p_2))}{1 - v(p_2)} \sum_{i=1}^{j} v_i u_{j-i} \right], \quad j \geq 1, \tag{6}
\]

where

\[
c_j = \sum_{n=j}^{\infty} c(n) \binom{n}{j} p_1^{n-j}, \quad v_j = \sum_{n=j}^{\infty} v(n) \binom{n}{j} p_2^{n-j}, \quad u_j = \sum_{n=j}^{\infty} u(n) \binom{n}{j} p_3^{n-j}.
\]

So we obtain the PGF \( P_b(z) \) of \( P_b \) as

\[
P_b(z) = \sum_{j=1}^{\infty} b_j z^j = C(1 - p_1(1 - z)) - c(p_1) + c(p_1)
\times \left\{ H(v(p_2))zU(1 - p_3(1 - z)) + \frac{[1 - H(v(p_2))U(1 - p_3(1 - z))(V(1 - p_2(1 - z)) - v(p_2))]}{1 - v(p_2)} \right\},
\]

\[
E[P_b] = p_1 E[C] + c(p_1) \left\{ H(v(p_2))(p_3E[U] + 1) + \frac{1 - H(v(p_2))}{1 - v(p_2)} \left[ (1 - v(p_2))p_3E[U] + p_2E[V] \right] \right\}.
\]

We assume that \( P_n \) is the number of customers in the system when the \( n \)th customer completes his service and leaves the system, so we can obtain the embedded Markov chain \( \{P_n, n \geq 1\} \) as

\[
P_{n+1} = \begin{cases} 
P_n - 1 + A & \text{if } P_n > 0, \\
P_b - 1 + A & \text{if } P_n = 0,
\end{cases} \tag{7}
\]

where \( A \) denotes the number of customers arriving during the service time of the \( n + 1 \)th customer, and the transition probability matrix is as

\[
\tilde{P} = \begin{bmatrix}
h_0 & h_1 & h_2 & h_3 & \cdots \\
k_0 & k_1 & k_2 & k_3 & \cdots \\
k_0 & k_1 & k_2 & \cdots \\
k_0 & k_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \tag{8}
\]

where \( \{k_j, j \geq 0\} \) represents the probability that there are \( j \) customer arrivals during the service time of the \( n + 1 \)th customer, and

\[
h_j = P\{P_b - 1 + A = j\} = \sum_{i=1}^{j+1} b_i k_{j-i+1}
= \sum_{i=1}^{j+1} \left\{ c_i + c(p_1) \left[ H(v(p_2))u_{i-1} + \frac{1 - H(v(p_2))}{1 - v(p_2)} \sum_{k=1}^{i} v_k u_{i-k} \right] \right\} k_{j-i+1}. \tag{9}
\]

Let \( \{\pi_k, k \geq 0\} \) be the steady-state distribution of the Markov chain \( \{P_n, n \geq 1\} \) on condition that \( \rho = r \lambda \mu^{-1} < 1 \). Because \( \{\pi_k, k \geq 0\} \) satisfies the equilibrium equation \( \Pi \tilde{P} = \Pi \) and the normalization condition \( \Pi \mathbf{e} = 1 \) where \( \mathbf{e} \) is the unit vector and \( \Pi = \{(\pi_0, \pi_1, \pi_2, \cdots)\} 
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Let $L_v(z)$ represent the PGF of the steady-state distribution $\{\pi_k, k \geq 0\}$. Multiplying (10) by $z^j$ and summing over $j = 0, 1, \cdots$, we have

$$L_v(z) = \pi_0 \sum_{j=0}^{\infty} h_j z^j + \sum_{j=0}^{\infty} z^j \sum_{i=1}^{j+1} \pi_i k_{j-i+1} = \pi_0 R(z) + \frac{1}{z} A(z) (L_v(z) - \pi_0),$$

where

$$A(z) = \sum_{j=0}^{\infty} k_j z^j = B(1 - p(1 - z)),$$

$$R(z) = \sum_{j=0}^{\infty} h_j z^j = \sum_{j=0}^{\infty} z^j \sum_{i=1}^{j+1} c_i k_{j-i+1}$$

$$+ c(\overline{\rho}_1) \left[ H(v(\overline{\rho}_2)) \sum_{j=0}^{\infty} z^j \sum_{i=1}^{j+1} u_{i-1} k_{j-i+1} + \frac{1 - H(v(\overline{\rho}_2))}{1 - v(\overline{\rho}_2)} \sum_{j=0}^{\infty} z^j \sum_{i=1}^{j+1} k_{j-i+1} \sum_{k=1}^{i} u_k u_{i-k} \right]$$

$$= A(z) \left\{ \frac{1}{z} (C(1 - p_1(1 - z)) - c(\overline{\rho}_1)) + c(\overline{\rho}_1) \left[ H(v(\overline{\rho}_2)) U(1 - p_3(1 - z)) + \frac{1 - H(v(\overline{\rho}_2))}{1 - v(\overline{\rho}_2)} U(1 - p_3(1 - z)) (V(1 - p_2(1 - z)) - v(\overline{\rho}_2)) \right] \right\}.$$

Substituting $R(z)$ and $A(z)$ into $L_v(z)$, we get

$$L_v(z) = \frac{\pi_0 B(1 - p(1 - z))}{B(1 - p(1 - z)) - z} \left\{ 1 + c(\overline{\rho}_1) - C(1 - p_1(1 - z)) - c(\overline{\rho}_1) \right\} \left[ \frac{1 - H(v(\overline{\rho}_2))}{1 - v(\overline{\rho}_2)} U(1 - p_3(1 - z)) \right.$$  

$$\times V(1 - p_2(1 - z)) - \frac{v(\overline{\rho}_2)(1 - H(v(\overline{\rho}_2)))}{1 - v(\overline{\rho}_2)} U(1 - p_3(1 - z)) + H(v(\overline{\rho}_2)) z U(1 - p_3(1 - z)) \left\} \right\}.$$

Based on the normalization condition $L_v(1) = 1$ and the L’Hospital rule, we get

$$\pi_0 = \frac{1 - \rho}{E[P_b]}.$$  

Substituting (11) into $L_v(z)$, we get

$$L_v(z) = \frac{(1 - \rho)(1 - z) B(1 - p(1 - z))}{B(1 - p(1 - z)) - z} \frac{1}{E[P_b](1 - z)}$$

$$\times \left\{ 1 + c(\overline{\rho}_1) - C(1 - p_1(1 - z)) - c(\overline{\rho}_1) \right\} \left[ \frac{1 - H(v(\overline{\rho}_2))}{1 - v(\overline{\rho}_2)} U(1 - p_3(1 - z)) \right.$$  

$$\times V(1 - p_2(1 - z)) - \frac{v(\overline{\rho}_2)(1 - H(v(\overline{\rho}_2)))}{1 - v(\overline{\rho}_2)} U(1 - p_3(1 - z)) + H(v(\overline{\rho}_2)) z U(1 - p_3(1 - z)) \left\} \right\}.$$  

$$= L_0(z) P_d(z),$$

where

$$L_0(z) = \frac{(1 - \rho)(1 - z) B(1 - p(1 - z))}{B(1 - p(1 - z)) - z}$$

is the PGF of steady-state queue length in the classical $Geom/G/1$ queueing system, and

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According to the different cases of customer arrival time, we can observe that 
\[ Q \] period with probability \( c \) So the relationship of PGFs of the number of customers and batches when a busy period begins is that 
\[ H \] (1


So the relationship of PGFs of the number of customers and batches when a busy period begins is that \( Q_b(z) = P_b(\xi(z)) \), we give the results as follows

From Lemma 1, the PGF \( Q_d(z) \) of the additional queue length can be decomposed into four parts, we give

which yields

According to the different cases of customer arrival time, we can observe that \( Q_d \), which equals to the number of customers who arrive during the residual closedown period with probability \( p_1E[C]P_b \), or the number of customers who arrive during the residual setup period with probability \( p_3E[\xi(z)]P_b \), or the number of customers who arrive during the residual vacation period and a whole setup period with probability \( (1 - H(v(\bar{p_2}))p_2E[V]E[P_b](1 - v(\bar{p_2}))) \), or the number of customers who arrive during a whole setup period with probability \( c(\bar{p_1})H(v(\bar{p_2})E[P_b] \) is diverse with different probabilities.

Theorem 1. When \( \rho = r\mu_1^{-1} < 1 \), the steady-state queue length immediately after the service completions, denoted by \( Q_{\nu} \), can be decomposed into the sum of two independent random variables: \( Q_{\nu} = Q_0 + Q_d \), where \( Q_0 \) has the PGF as

and the additional queue length \( Q_d \) has the PGF as

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Proof. From the notations, we can easily observe that \( \{Q_n, n \geq 1\} \) is an embedded Markov chain, then it has steady distribution when \( \rho = r \mu^{-1} < 1 \). First we note the sign

\[
\varepsilon(x) = \begin{cases} 
1, & \text{if } x > 0, \\
0, & \text{if } x \leq 0.
\end{cases}
\]

Because

\[
Q_{n+1} = Q_n - \varepsilon(Q_n) + Y_{n+1} + (Q_b - 1)\varepsilon(1 - Q_n),
\]

where \( Y_{n+1} \) represents the number of customers who arrive during the service time of the \( n + 1 \) th customer, we can easily derive that

\[
E[Y_{n+1}] = E[X(B_{n+1})] = rE[N(B_{n+1})] = \rho,
\]

\[
E[z^{Y_{n+1}}] = E[z^{X(B_{n+1})}] = E[z^{\xi_1 + \xi_2 + \cdots + \xi_N(B_{n+1})}] = B(1 - p(1 - \xi(z))).
\]

Taking expect on both sides of the equation, we notice that

\[
E[Q_v] \equiv \lim_{n \to \infty} E[Q_{n+1}] = \lim_{n \to \infty} E[Q_n], \quad E[X(B)] = E[Y] \equiv \lim_{n \to \infty} E[Y_{n+1}].
\]

Therefore,

\[
E[\varepsilon(Q_v)] = E[Y] + E[Q_b - 1]E[\varepsilon(1 - Q_v)].
\]

Substituting (23) into (25), we obtain

\[
P\{Q_v > 0\} = \rho + (E[Q_b] - 1)P\{Q_v = 0\} = 1 - P\{Q_v = 0\},
\]

\[
P\{Q_v = 0\} = \frac{1 - \rho}{E[Q_b]},
\]

and we also give

\[
E[z^{Q_n - \varepsilon(Q_n) + (Q_b - 1)\varepsilon(1 - Q_n)}] = \sum_{k=0}^{\infty} E[z^{k - \varepsilon(k) + (Q_b - 1)\varepsilon(1-k)}]
\]

\[
= E[z^{Q_b-1}]P\{Q_n = 0\} + \sum_{k=1}^{\infty} z^{k-1}P\{Q_n = k\}
\]

\[
= \frac{1 - \rho}{E[Q_b]}z^{Q_b}(z) + \frac{1}{z}[Q_n(z) - P\{Q_n = 0\}].
\]

Substituting (24) and (26) into (27), we get

\[
Q_v(z) = \lim_{n \to \infty} Q_{n+1}(z) = E[z^Y] \lim_{n \to \infty} E[z^{Q_n - \varepsilon(Q_n) + (Q_b - 1)\varepsilon(1 - Q_n)}]
\]

\[
= B[1 - p(1 - \xi(z))] \left[ \frac{1 - \rho}{E[Q_b]}z^{Q_b(z) - 1} + \frac{1}{z}Q_v(z) \right].
\]

Then we can easily get

\[
Q_v(z) = \frac{(1 - \rho)(Q_b(z) - 1)B(1 - p(1 - \xi(z)))}{E[Q_b](z - B(1 - p(1 - \xi(z)))))}
\]

\[
= \frac{(1 - \rho)(\xi(z) - 1)B(1 - p(1 - \xi(z)))}{r(z - B(1 - p(1 - \xi(z))))} \frac{1 - Q_b(z)}{E[Q_b](1 - \xi(z))},
\]

which yields
\[ E[Q_v] = \rho + \frac{\sigma_r^2 + r^2 - r + p^2 b(2) - \mu r \rho^2}{2(1 - \rho)} + E[Q_d]. \]

Now we utilize the relationship of the PGFs of the steady-state queue length immediately after a service completion and the queue length immediately after an arbitrary slot boundary, denoted by \( P_v(z) \), which is given as (see Takagi [10])

\[ Q_v(z) = P_v(z) \frac{1 - \xi(z)}{r(1 - z)}. \]

then we get

\[ P_v(z) = \frac{(1 - \rho)(z - 1)B(1 - p(1 - \xi(z)))}{z - B(1 - p(1 - \xi(z)))} \frac{1 - Q_b(z)}{E[P_b](1 - \xi(z))}, \]

where the first part is the PGF of the queue length immediately after an arbitrary slot boundary in the classical \( Geom^C/G/1 \) queueing system.

From Theorem 1, we can obviously derive that the expected stationary queue length is as

\[ E[P_v] = \rho + \frac{\sigma_r^2 + r^2 - r + p^2 b(2) - \mu r \rho^2}{2(1 - \rho)} - \frac{\sigma_r^2 + r^2 - r}{2r} + E[Q_d]. \]

which is composed by two parts: One is the stationary queue length of the \( Geom^C/G/1 \) queue system without vacations and setup/closedown times, the other is the additional queue length.

### 3.2 Stationary waiting times

Let \( W_v \) represent the stationary waiting times of an arbitrary customer. Because the waiting times of an arbitrary customer is composed by two parts \( W_{Q_b} \) and \( W_{Q_s} \), we can decompose \( W_v = W_{Q_b} + W_{Q_s} \) by the method of stochastic decomposition. Since \( W_{Q_b} \) and \( W_{Q_s} \) are independent, therefore

\[ W_v(z) = W_{Q_b}(z)W_{Q_s}(z). \]

Let \( W_{Q_d} \) denote the additional delay due to the vacations and server setup/closedown times. Obviously, the additional queue length is equal to the number of customer arrivals during the additional delay. According to Lemma 1 as well as Theorem 2 and Theorem 3 which are given by Ma et al [6], we can easily obtain the following results.

**Lemma 2.** When \( \rho = r p \mu^{-1} < 1 \), the stationary batch waiting times, denoted by \( W_{Q_b} \), can be decomposed into the sum of two independent random variables: \( W_{Q_b} = W_{Q_{b0}} + W_{Q_{da}} \), where \( W_{Q_{b0}} \) is the stationary batch waiting times without vacations and setup/closedown times and has the PGF as

\[ W_{Q_{b0}}(z) = \frac{(1 - \rho)(1 - z)}{(1 - z) - p(1 - \xi(B(z)))}, \]

and the additional delay \( W_{Q_{da}} \) has the PGF as

\[ W_{Q_{da}}(z) = \frac{1}{E[P_b](1 - z)} \{ p_1(1 - C(z)) + c(\overline{P_1})p_3 - (p_3 - p_2)U(z) \}

\[ - H(v(\overline{P_2}))(z - \overline{P_2})U(z) - \frac{1 - H(v(\overline{P_2}))}{1 - v(\overline{P_2})}p_2U(z)(V(z) - v(\overline{P_2})). \}

Corresponding with \( Q_d \), the PGF of \( W_{Q_d} \) also can be decomposed into four parts, we give

\[ W_{Q_d}(z) = \frac{p_1E[C]}{E[P_b]} \frac{1 - C(z)}{E[C](1 - z)} + \frac{p_2c(\overline{P_1})E[U]}{E[P_b]} \frac{1 - U(z)}{E[U](1 - z)} + \frac{(1 - H(V(\overline{P_2})))p_2c(\overline{P_1})E[V]}{E[P_b](1 - v(\overline{P_2}))} \frac{1 - V(z)U(z)}{E[V](1 - z)} + \frac{c(\overline{P_1})H(V(\overline{P_2}))}{E[P_b]}U(z). \]
which yields
\[
E[W_{Q_d}] = \frac{p_1E[C]}{E[P_b]} + \frac{(1 - H(v(\bar{p}_2)))p_2c(\bar{p}_1)E[V]}{2E[C]} \left( E[U] + \frac{E[V(V - 1)]}{2E[V]} \right)
\]

Parallel to the analysis of \(Q_d\), we can also observe that \(W_{Q_d}\), which equals to the residual closedown period with probability \(p_1E[C]E[P_b]^{-1}\), or the residual setup period with probability \(p_3c(\bar{p}_1)E[U]E[P_b]^{-1}\), or the residual vacation period and a whole setup period with probability \((1 - H(v(\bar{p}_2)))p_2c(\bar{p}_1)E[V](E[P_b](1 - v(\bar{p}_2)))^{-1}\), or a whole setup period with probability \(c(\bar{p}_1)H(v(\bar{p}_2))E[P_b]^{-1}\), is diverse with different probabilities.

**Theorem 2.** If \(\rho = rp\mu^{-1} < 1\), the stationary waiting times, denoted by \(W_v\), can be decomposed into the sum of two independent random variables: \(W_v = W_{Q_b} + W_{Q_s}\), where \(W_{Q_b}\) has the PGF as
\[
W_{Q_b}(z) = \frac{(1 - \rho)(1 - z)}{(1 - z) - p(1 - \xi(B(z)))} \left\{ \frac{p_1(1 - C(z)) + c(\bar{p}_1)(p_3 - (p_3 - p_2)U(z)}{E[P_b](1 - z)} - H(v(\bar{p}_2))(z - \bar{p}_2)U(z) - \frac{1 - H(v(\bar{p}_2))}{1 - v(\bar{p}_2)}p_2U(z)(V(z) - v(\bar{p}_2)) \right\},
\]

and the inter-batch waiting times \(W_{Q_s}\) has the PGF as
\[
W_{Q_s}(z) = \frac{(1 - \xi(B(z)))}{r(1 - B(z))}.
\]

Corresponding with the stationary queue length, we also can easily derive from Theorem 2 that the expected stationary waiting times is as
\[
E[W_v] = \frac{pr^2\mu^2(2) - pr^2 + \sigma_r^2 + r^2 - r}{2r\mu(1 - \rho)} + E[W_d],
\]
which is composed of two parts: One is the stationary waiting times of the \(\text{Geom}^\xi/G/1\) queue system without vacations and setup/closedown times, the other is the additional delay.

## 4 Analysis of busy period

As for the batch arrival \(\text{Geom}/G/1\) system, we have known that there are two kinds of busy period caused by a single customer and customers of a batch, respectively. Let \(\Theta_b\) represent the length of the busy period. Due to the vacations and the setup/closedown times, the busy period starts with \(Q_b\) customers present. From the notations, we can easily get that
\[
\Theta_v = \sum_{j=1}^{Q_b} \theta_j, \quad \Theta = \sum_{i=1}^{\xi} \theta_i,
\]
where we assume that \(\theta_1, \theta_2, \theta_3, \cdots\) are i.i.d random variables and have the same distribution with \(\theta\). So we can get
\[
\Theta_v(z) = Q_b(\theta(z)), \quad \Theta(z) = \xi(\theta(z)).
\]

Moreover, since
\[
\theta = B + \Theta_1 + \Theta_2 + \cdots + \Theta_{N(B)}, \quad \Theta = U_\xi + \Theta_1 + \Theta_2 + \cdots + \Theta_{N(U_\xi)},
\]

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where we assume $\Theta_1, \Theta_2, \Theta_3, \ldots$ are i.i.d random variables and have the same distribution with $\Theta$ and $U_\xi = \sum_{i=1}^{\xi} B_i(B_1, B_2, B_3, \ldots$ are i.i.d random variables and have the same distribution with $B$) respectively, we obtain

$$\theta(z) = E[z^\theta] = E[z^{(B+\Theta_1+\Theta_2+\ldots+\Theta_{N(B)})}] = \sum_{n=1}^{\infty} z^n E[z^{(\Theta_1+\Theta_2+\ldots+\Theta_{N(n)})}] P\{B = n\}$$

$$= \sum_{n=1}^{\infty} z^n \sum_{j=1}^{n} (Q(z))^j \left( \frac{n}{j} \right) p^j(1-p)^{n-j} P\{B = n\} = \sum_{n=1}^{\infty} [z - zp + zp\theta(z)]^n P\{B = n\}$$

which yields

$$E[\Theta_v] = E[\theta] E[Q_b] = \frac{E[Q_b]}{\mu(1 - \rho)}.$$

5 Several special cases

- **Case 1:** If we consider the case $P\{V = 0\} = P\{U = 0\} = P\{C = 0\} = 1$ and $\xi \equiv 1$, the model changes into the classical $Geom/G/1$ queue system with no vacation. The results have been given by Sun and Li [9].

- **Case 2:** If we consider the case $P\{V = 0\} = P\{U = 0\} = P\{C = 0\} = 1$, the model changes into the batch arrival $Geom/G/1$ queue with no vacation. The results also have been given by Sun and Li and Takagi [10].

- **Case 3:** If we consider the case $P\{U = 0\} = P\{C = 0\} = 1, p_2 = p$ and $\xi \equiv 1$, the model changes into the $Geom/G/1$ queue with multiple adaptive vacations. The results have been given by Zhang and Tian [14].

- **Case 4:** If we consider the case $P\{C = 0\} = 1, p_2 = p_3 = p$ and $\xi \equiv 1$, the model changes into the $Geom/\gamma/G/1$ queue with multiple adaptive vacations and setup time. The continuous-time results have been given by Wei et al [13].

- **Case 5:** If we consider the case $p_1 = p_2 = p_3 = p^* \neq p$, the model changes into the $Geom/\xi/G/1$ queue system with uniform arrival rates on multiple adaptive vacations and setup/closedown times, we can get the PGF of the steady-state queue length immediately after a service completion and the queue length immediately after an arbitrary slot boundary, waiting times and busy period in steady state on condition that $\rho = r\mu^{-1} < 1$, respectively. ($p^* = 1 - p^*$).

1. **queue length**

   $$Q_b(z) = C(1 - p^*(1 - \xi(z))) - c(p^*) + c(p^*) H(v(p^*)) \xi(z) U(1 - p^*(1 - \xi(z)))$$

   $$+ \frac{c(p^*)[1 - H(v(p^*))] U(1 - p^*(1 - \xi(z))) (V(1 - p^*(1 - \xi(z))) - v(p^*))}{1 - v(p^*)},$$

   $$E[P_b] = p^* E[C] + c(p^*) \left[ H(v(p^*)) (p^* E[U] + 1) + \frac{1 - H(v(p^*))}{1 - v(p^*)} p^* [(1 - v(p^*)) E[U] + E[V]] \right],$$

   $$Q_v(z) = \frac{(1 - \rho)[Q_b(z) - 1] B(1 - p(1 - \xi(z)))}{E[Q_b](z - B[1 - p(1 - \xi(z))])} = \frac{(1 - \rho)[\xi(z) - 1] B[1 - p(1 - \xi(z))]}{r(z - B[1 - p(1 - \xi(z))])},$$

   $$P_v(z) = \frac{(1 - \rho)(z - 1) B(1 - p(1 - \xi(z)))}{z - B(1 - p(1 - \xi(z)))} E[P_b](1 - \xi(z)).$$

2. **waiting times**


\[ W_{Q_b}(z) = \frac{(1 - \rho)(1 - z)}{(1 - z) - p(1 - \xi(B(z)))} E[P_b](1 - z) \]

\[ \times \left\{ p^*(1 - C(z)) + c(\bar{p}) \left[ p^* - H(v(\bar{p}^*)) (z - \bar{p}^*) U(z) - \frac{1 - H(v(\bar{p}^*))}{1 - v(\bar{p}^*)} p^* U(z)(V(z) - v(\bar{p}^*)) \right] \right\} , \]

\[ W_{Q_b}(z) = \frac{1 - \xi(B(z))}{r(1 - B(z))}, \quad W_v(z) = W_{Q_b}(z)W_{Q_v}(z). \]

(3) busy period

\[ \theta(z) = B(z - pz(1 - \xi(\theta(z)))), \quad \Theta(z) = \xi(B(z - pz(1 - \xi(\theta(z))))) , \]

\[ \Theta_v(z) = Q_b(\theta(z)) = Q_b(B(z - pz(1 - \xi(\theta(z))))) . \]

6 Numerical examples

Through analyzing the model we present and contrasting with the queue in which \( p_1 = p_2 = p_3 = p \), we notice that our model is more general. Actually, there may be some customers losing their patience and leaving the system when they are waiting in the queue during the vacation or server setup/closedown period, which lead to the actual arrival rates decrease. Therefore, the situation that \( p_1 \leq p, p_2 \leq p \) and \( p_3 \leq p \) is more reasonable.

Different systems may have different parameters in the practical problems. Specially, we assume all of the setup/closedown times and vacation times as well as the maximum number of vacations the server may follow geometric distributions with parameters \( p_u, p_c, p_v \) and \( p_h \), respectively, and the mean batch size \( r = 1 \) and variance \( \sigma_v^2 = 1 \), the arrival rate in busy or idle period \( p = 0.5 \), the service rate \( \mu = 0.8 \). Obviously, we have the positive recurrent condition \( \rho = rp\mu^{-1} = 0.625 < 1 \). From the above analysis, we obtain the steady-state expected queue length \( E[Q_v] \) and waiting times \( E[W_v] \). Now we begin to observe the effect of different arrival rates during the vacation and server setup/closedown period on the queue model, and we can verify that our results obtained above are correct in this special case and explain that our model is reasonable in practical problems by presenting numerical examples in some situations.

Fig. 1. \( p_i \)'s effect on expected queue length \( (i = 1, 2, 3) \)

In Fig. 1, we show the effect of \( p_i \) \( (i = 1, 2, 3) \) on the steady-state expected queue length when \( p_c, p_v, p_h \) and \( p_u \) are all fixed. In the first figure of Fig. 1, we consider four situations with different values of \( p_2 \) and \( p_3 \). Through changing the big and small order of the values of \( p_2 \) and \( p_3 \), we observe the mutative trend of the expected queue length along with the increase of \( p_1 \). In the case of \( p_2 = 0.0, p_3 = 0.0 \), i.e. no customer arrival during vacation and setup period, the expected queue length gradually increases along with the the increase of \( p_1 \), which is obvious. While in the last three cases when \( p_2 \neq 0.0, p_3 \neq 0.0 \), the expected queue length gradually decreases. Because we assume that busy period begins after the closedown times if there is no less than one customer arrival during the closedown times, once that occurs, there is no need to take the vacation and setup period, which can obviously decrease the expected queue length. Hence the decrease trend in the last three cases is reasonable. Similar to the analysis above, the second figure of Fig. 1 shows that the
expected queue length first decreases when $p_2$ is very small and then increases along with the increase of $p_2$ on condition that $p_1 \neq 0, p_3 \neq 0$. The initial decrease trend, which is caused by $p_2 \ll p_1$, is reasonable. Moreover, we confirm that the shorter expected vacation time $E[V] = 1/p_v$ results in the bigger proportion occupied by the decrease part. But the expected queue length begins to increase as soon as the value of $p_2$ is bigger than some threshold value. In the third figure of Fig. 1, the four pieces of curve increase by and large, because the mean vacation and setup period is longer than the mean closedown times. And the smaller $p_1$ results in the steeper curve. If the value of $p_1$ is big enough (but smaller than $p$), the expected queue length will first decrease and then increase. In Fig. 2, we show the mutative trend of the expected queue length along with the increase of $p_c, p_v$ and $p_u$, respectively, on condition that $p_i (i = 1, 2, 3)$ are endowed with different values in eight situations.

Parallel to the analysis process above, in Fig. 3, we show the effect of $p_i (i = 1, 2, 3)$ on the steady-state expected waiting time when $p_c, p_v, p_h$ and $p_u$ are all fixed. In the first figure of Fig. 3, the expected waiting times in all of the situations decrease along with the increase of $p_1$. The second figure of Fig. 3 shows that the expected waiting times first decreases when $p_2$ is very small and then increases along with the increase of $p_2$ on condition that $p_1 \neq 0, p_3 \neq 0$, which is similar to the second figure of Fig. 1. And the expected waiting times reaches its asymptotic value of 16 when $p_2 > 0.2$ approximately and $p_1 = 0.0, p_3 = 0.0$. In the third figure of Fig. 3, all of the four pieces of curve decrease, which indicates that no matter which situation, the expected waiting times will always decrease along with the increase of $p_3$. In Fig. 4, we show the mutative trend of the expected waiting times along with the increase of $p_c, p_v$ and $p_u$, respectively, on condition that $p_i (i = 1, 2, 3)$ are also endowed with different values in eight situations, which are the same with Fig. 2. By analyzing, all of them, including Fig. 2, are consistent with the practical situations.

According to the analysis of the effect of different arrival rates on the queue model by numerical examples, we find that the performance measures obtained in our model are reasonable in this special case, and of
course, we can replace the general distributions of $B, C, V, H$ and $U$ with arbitrary distributions to obtain the corresponding expected queue length and waiting times and verify that the results are correct by numerical analysis. So, on the basis of the specific problems with different situations, the service departments can design the reasonable vacation rate and service rate and other parameters in order to balance the auxiliary work in the vacation period and the normal service work in busy period, which can avoid too many customers waiting in the system.

References