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On the minimax inequality for a special class of functionals

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Abstract. In this paper, we establish an equivalent statement of minimax inequality for a special class of functionals, also some conditions that imply minimax inequality are pointed out and equivalent formulations are proved.

Keywords: minimax inequality, critical point, multiplicity results, dirichlet problem

1 Introduction

Throughout the sequel, $\Omega \subset R^N (N \geq 1)$ is nonempty bounded open set with a boundary $\partial\Omega$ of class C^1 and $p > N$.

Given two $G\hat{a}teaux$ differentiable functionals Φ and Ψ on a real Banach space X, the minimax inequality

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + \lambda\rho) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + \lambda\rho), \ \rho \in R, \tag{1}$$

plays a fundamental role for establishing the existence of at least three critical points for the functional $\Phi(u) + \lambda\Psi(u)$, as the theorem of B. Ricceri below ensures (see [4]).

In recent years, many authors have studied multiple solutions from several point of view and with different approaches (see, for example, [1–3]); for instance, in [2], the author proves multiplicity results for the problem

$$\begin{cases} u'' + \lambda f(x, u) = 0, \\ u(a) = u(b) = 0, \end{cases} \tag{2}$$

which for each $\lambda \in [0, +\infty]$, admits at least three solutions in $W_0^{1,2}([a, b])$ when $f : [a, b] \times R \to R$ is a continuous function.

In this paper some conditions that imply minimax inequality (1) are pointed out and equivalent formulations are proved. Moreover, the aim of this paper is to establishes an equivalent statement of minimax inequality (1) for a special class of functionals.

2 Main results

In the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$|| \, u \, || = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p},$$

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$f : \Omega \times R \to R$ is a positive Caratheodory function, $a(x) \in C(\overline{\Omega})$ is a positive weight function,

$$g(x,y) = \int_0^y f(x,\xi)d\xi$$

for each (x,y)$\in \Omega \times R$ and

$$k = \sup_{u \in X \setminus \{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{||u||},$$

since $p > N$, one has $k < \infty$, (see [[5], formula (6b)]). Assume that there exists continuous function $b(x) \geq 1$ on $[a_1, a_2]$, $(a_1, a_2 \in R)$ such that

$$g(x, u(x)) \geq pk^p b(x)|\nabla u(x)|^p \tag{3}$$

for each $u \in X$. We define $|| \ u \ ||_I = \left(\int_\Omega (|\nabla u(x)|^p + a(x)|u(x)|^p)dx\right)^{1/p}$, such that there exist positive suitable constants c_1 and c_2 :

$$c_1||u|| \leq ||u||_I \leq c_2||u|| (\text{i.e., the above norms are equivalent}). \tag{4}$$

We now introduce two special functionals on the Sobolev space X as follows

$$\Phi(u) = \frac{||u||_I^p}{p}$$

and

$$\Psi(u) = \int_\Omega g(x, u(x))dx$$

for every $u \in X$. Let $r, \rho \in R$, $w \in X$ be such that $0 < r < \Phi(w)$ and $0 < \rho < \Psi(w)$, we put

$$A_1(r,w) = r\frac{\Psi(w)}{\Phi(w)},\ A_2(\rho,w) = \rho\frac{\Phi(w)}{\Psi(w)}\ A_3(r,w) = (\frac{1}{p}A_1(r,w))^{1/p}.$$

Clearly, $A_1(r,w)$, $A_2(\rho,w)$ and $A_3(r,w)$ are positive. In this work, $m(\Omega)$ is Lebesgue measure on Ω set. From (3) and since $b(x) \geq 1$ on $[a_1, a_2]$, we have

$$\int_\Omega g(x,u(x))dx \geq pk^p||u||^p. \tag{5}$$

Now, we put

$$\alpha_1 = \inf\{k \ || \ u \ || \in R^+; \Phi(u) \leq r\},$$
$$\alpha_2 = \inf\{k \ || \ u \ || \in R^+;\ \frac{m(\Omega)}{p} \inf_{|t| \leq k||u||} (|\nabla t|^p + a(x)|t|^p) \leq r\}$$

for every $t \in X$ and

$$\alpha_r = \alpha_1 - \alpha_2.$$

Since

$$\sup_{x\in\Omega} \mid u(x) \mid \leq k \ || \ u \ ||$$

for every $x \in \Omega$ and for every $u \in X$, we have

$$m(\Omega) \inf_{|t| \leq k||u||} (|\nabla t|^p + a(x)|t|^p) \leq \int_\Omega (|\nabla u(x)|^p + a(x)|u(x)|^p)dx$$

for every $u \in X$, namely

$$\frac{m(\Omega)}{p}\inf_{|t|\leq k||u||}(|\nabla t|^p + a(x)|t|^p) \leq \Phi(u)$$

for every $u \in X$; therefore

$$\{k \mid\mid u \mid\mid\in R^+; \Phi(u) \leq r\} \subseteq \{k \mid\mid u \mid\mid\in R^+; \frac{m(\Omega)}{p}\inf_{|t|\leq k||u||}(|\nabla t|^p + a(x)|t|^p) \leq r\}.$$

Hence $\alpha_1 \geq \alpha_2$, namely $\alpha_r \geq 0$. Now, the main result:

Theorem 1. *Assume that there exist* $r \in R, w \in X$ *such that*
(i) $0 < r < \Phi(w)$,
(ii) $\frac{m(\Omega)}{p}\inf_{|t|\leq A_3(r,w)-\alpha_r}(|\nabla t|^p + a(x)|t|^p) > r.$
Then, there exists $\rho \in R$ *such that*

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \lambda\rho) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + \lambda\rho)$$

Proof. From (ii), we obtain

$$A_3(r,w) - \alpha_r \notin \inf\{l \in R^+;\ \frac{m(\Omega)}{p}\inf_{|t|\leq l}(|\nabla t|^p + a(x)|t|^p) \leq r\}.$$

Moreover

$$\inf\{l \in R^+; \frac{m(\Omega)}{p}\inf_{|t|\leq l}(|\nabla t|^p + a(x)|t|^p) \leq r\} \geq A_3(r,w) - \alpha_r.$$

In fact, arguing by contradiction, we assume that there is $l_1 \in R^+$ such that

$$\frac{m(\Omega)}{p}\inf_{|t|\leq l_1}(|\nabla t|^p + a(x)|t|^p) \leq r$$

and

$$l_1 < A_3(r,w) - \alpha_r,$$

so

$$\frac{m(\Omega)}{p}\inf_{|t|\leq A_3(r,w)-\alpha_r}(|\nabla t|^p + a(x)|t|^p) \leq \frac{m(\Omega)}{p}\inf_{|t|\leq l_1}(|\nabla t|^p + a(x)|t|^p) \leq r$$

and this is a contradiction. So

$$\inf\{l \in R^+; \frac{m(\Omega)}{p}\inf_{|t|\leq l}(|\nabla t|^p + a(x)|t|^p) \leq r\} \geq A_3(r,w) - \alpha_r.$$

Therefore,

$$\inf\{k \mid\mid u \mid\mid\in R^+;\ \frac{m(\Omega)}{p}\inf_{|t|\leq k||u||}(|\nabla t|^p + a(x)|t|^p) \leq r\} + \alpha_r > A_3(r,w),$$

namely

$$\inf\{k \mid\mid u \mid\mid\in R^+; \Phi(u) \leq r\} > A_3(r,w).$$

So, we have

$$\inf\{pk^p \mid\mid u \mid\mid^p\in R^+; \Phi(u) \leq r\} > A_1(r,w).$$

Using of (5), one has

$$\inf\{\int_{\Omega} g(x,u(x))dx; \Phi(u) \leq r\} > A_1(r,w),$$

namely

$$\inf\{\Psi(u); \Phi(u) \leq r\} > r\frac{\Psi(w)}{\Phi(w)}$$

or

$$-\inf\{\Psi(u); \Phi(u) \leq r\} < r\frac{(-\Psi(w))}{\Phi(w)}$$

and with $-\Psi = T$, we have

$$\sup\{T(u); \Phi(u) \leq r\} < r\frac{T(w)}{\Phi(w)}.$$

Now, we claim for each ρ satisfying

$$\sup\{T(u); \Phi(u) \leq r\} < \rho < r\frac{T(w)}{\Phi(w)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda(\rho - T(u)) < \inf_{u \in X} \sup_{\lambda \geq 0}(\Phi(u) + \lambda(\rho - T(u)).$$

From $\sup\{T(u); \Phi(u) \leq r\} < \rho$, we obtain $r \leq \inf\{\Phi(u);\ T(u) \geq \rho\}$ and $\rho > 0$.

Moreover, from $\rho < r\frac{T(w)}{\Phi(w)}$ and $\rho > 0$, one has $\rho < T(w)$ and $\rho\frac{\Phi(w)}{T(w)} < r$. Hence,

$$\rho\frac{\Phi(w)}{T(w)} < \inf\{\Phi(u);\ T(u) \geq \rho\},$$

namely

$$\frac{\inf\{\Phi(u);\ T(u) \geq \rho\}}{\rho} > \frac{\Phi(w) - \inf\{\Phi(u);\ T(u) \geq \rho\}}{T(w) - \rho}.$$

Now, there exists $\lambda \in R$ such that

$$\lambda > \frac{\Phi(w) - \inf\{\Phi(u);\ T(u) \geq \rho\}}{T(w) - \rho}$$

and

$$\lambda < \frac{\inf\{\Phi(u);\ T(u) \geq \rho\}}{\rho}.$$

Namely $\Phi(w) + \lambda(\rho - T(w)) < \inf\{\Phi(u);\ T(u) \geq \rho\}$ and $\lambda\rho < \inf\{\Phi(u);\ T(u) \geq \rho\}$.

So, by choose $\Phi(0) = T(0) = 0$ and thanks of $0 < \rho < T(w)$, we obtain

$$\inf_{u \in X} (\Phi(u) + \lambda(\rho - T(u)) < \inf\{\Phi(u);\ T(u) \geq \rho\},$$

and with respect to

$$\inf_{u \in X} (\Phi(u) + \lambda(\rho - T(u)) < (\Phi(0) + \lambda(\rho - T(0)) = \lambda\rho,$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda(\rho - T(u)) < \inf\{\Phi(u);\ T(u) \geq \rho\}.$$

In other hand, for $\{u \in X;\ T(u) \geq \rho\}$ we have

$$\inf_{u \in X} \sup_{\lambda \geq 0}(\Phi(u) + \lambda(\rho - T(u)) = \inf\{\Phi(u);\ T(u) \geq \rho\}$$

and since $-\Psi = T$, we have

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\lambda\rho) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\lambda\rho).$$

Remark 1. If in theorem 1, $A_3(r,w)-\alpha_r \leq 0$; the theorem holds again. Because, $A_3(r,w) \leq \alpha_1-\alpha_2 \leq \alpha_1$, and by arguing as in the proof of theorem 1, the results holds.

If instead of condition (ii) in theorem 1, we put $\frac{m(\Omega)}{p}\inf_{|t|\leq A_3(r,w)}(|\nabla t|^p + a(x)|t|^p) > r$, then the result holds. Because $\frac{m(\Omega)}{p}\inf_{|t|\leq A_3(r,w)-\alpha_r}(|\nabla t|^p + a(x)|t|^p) \geq \frac{m(\Omega)}{p}\inf_{|t|\leq A_3(r,w)}(|\nabla t|^p + a(x)|t|^p) > r$. So, we have:

Theorem 2. *Assume that there exist* $r\in R, w\in X$ *such that*
(i) $0<r<\Phi(w)$,
(ii) $\frac{m(\Omega)}{p}\inf_{|t|\leq A_3(r,w)}(|\nabla t|^p + a(x)|t|^p) > r$.
Then, there exists $\rho\in R$ *such that*

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\lambda\rho) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\lambda\rho).$$

Now, If we give $\rho = A_1(r,w)$, *then we have* $r = A_2(\rho,w)$ *and* $A_3(r,w) = \sqrt[p]{\frac{\rho}{p}}$. *So, we have the following result:*

Proposition 1. *The following assertions are equivalent:*
(a) *there are* $r\in R, w\in X$ *such that*

(i) $0<r<\Phi(w)$,
(ii) $\frac{m(\Omega)}{p}\inf_{|t|\leq A_3(r,w)}(|\nabla t|^p + a(x)|t|^p) > r$.
(b) *there are* $\rho\in R, w\in X$ *such that*

(j) $0<\rho<\Psi(w)$,
(jj) $\frac{m(\Omega)}{p}\inf_{|t|\leq \sqrt[p]{\frac{\rho}{p}}}(|\nabla t|^p + a(x)|t|^p) > A_2(\rho,w)$.

Finally, by using of the theorem 2 and proposition 1, we have:

Theorem 3. *Assume that there exist* $\rho\in R, w\in X$ *such that*
(j) $0<\rho<\Psi(w)$,
(jj) $\frac{m(\Omega)}{p}\inf_{|t|\leq \sqrt[p]{\frac{\rho}{p}}}(|\nabla t|^p + a(x)|t|^p) > A_2(\rho,w)$.
Then, there exists $\rho\in R$ *such that*

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+\lambda\rho) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda\Psi(u)+\lambda\rho).$$

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