

Identification of heat exchange coefficient of the arctic snow and numerical simulation*

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Abstract. This paper establishes a multi-domain coupled distributed parameter system of temperature field and the identification model of heat exchange coefficient of the Arctic snow, based on the characteristics of temperature distribution of the Arctic snow, ice and in suit observation in the Arctic. The weak solution's existence and its continuous dependability in terms of heat exchange coefficient have been discussed. The identification model of heat exchange coefficient of the Arctic snow has been studied. The optimal algorithm for this model has been constructed and the optimum parameter has been found. Furthermore, with the use of the optimum parameter, the numerical simulation about the coupled system has been given.

Keywords: the multi-domain coupled system, distributed parameter, arctic sea ice, heat exchange coefficient

1 Introduction

The study of sea ice conditions in the Arctic Ocean is one of the most important lines of survey in modern polar oceanography and ecology. These investigations are of primary practical importance as well as theoretically significance for the development of the theory of ice formation, mechanisms of pollutant transport and global climate change in the Arctic Ocean. A knowledge of the specific features of sea ice in the Arctic Basin and marginal seas is necessary to resolve the problems of human activity and polar ecology. Since weather and hydrologic conditions in different sea area are variable, the thermodynamic parameters of sea ice, sea water and snow are also different. Especially, the quantitative analysis of thermodynamic parameters is very important. Hibler^[2] brought forward the plastic rheological model of sea ice and plays an important role in the development of the model of the sea ice. Maykut^[1] developed the more self-contained thermodynamic mode MU71. Winton^[6] established the multi-layer thermodynamic model of sea ice, considering the specific heat and melt heat which are variable. Bitz^[7] adopted global coupled model, including the model of the original equation, the aerial model of the energy hydrosphere, and the model about thickness of sea ice. J. Sun, H.D. Wu, S. Bai^{[9][10]} analysed the ice thickness and heat budget at the ice surface according to the energy conversation law. S. R. Wang^[8] established the first dynamic and thermodynamic model to simulate the growth and melting of sea ice. Z. S. Lin, Q. Le, X. Q. Wang, B. Cheng^[5] studied the ice temperature, the range of ice interface and definite conditions via differential equation and partial differential equation. N. U. Ahmed^[11] and Q. F. Wang, D. X. Feng^[13] and H. Gao^[4] discussed the properties of the weak solution and the optimization condition of identification of the distributed parameter system.

According to the data from the monitor buoy installed in the Arctic Ocean, from August, 2003 to September, 2003, the temperature of ice is from -1.9°C to -2.0°C , and the temperature of air is from -5.1°C to -5.4°C

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. In October the temperature of ice is from -2.1°C to -4.6°C , and the temperature of air is from -5.4°C to -20°C . In November the temperature of ice is from -4.1°C to -7.1°C , and the temperature of air is from -12.4°C to -19.4°C . From December, 2003 to March, 2004, the temperature of ice is from -7.1°C to -15.8°C , and the temperature of air is from -11.4°C to -22.8°C . The quantity of heat has a direct influence on the circumfluence, and it plays an important role in global climate. The heat given out has a direct relationship with heat exchange coefficient. The papers about heat exchange coefficient of the Arctic snow are quite few, so the study of it is very essential.

Based on the temperature of air, snow and the thickness of sea ice acquired by a monitor buoy installed in the Arctic Ocean from August, 2003 to April, 2004, we discuss the multi-domain coupled distributed parameter system of temperature field. We discuss the existence, uniqueness and continuity of the weak solution for the coupled system. Moreover, we establish the identification model of heat exchange coefficient and discuss the differentiability of the problem. We establish the optimal algorithm for the identifiability problem, and using it, we get the optimum value of heat exchange coefficient. By using the optimum parameter, we give the numerical simulation about the coupled system.

2 The properties of multi-domain temperature field coupled dynamic system

Based on the temperature distribution characteristic of the Arctic snow, sea ice and sea water, in which the gradient change along depth direction is far greater than along horizontal direction, the heat exchange exists mainly in depth direction and can be ignored in horizontal direction. Hence, the temperature distribution can be described by the one dimensional temperature field equation. Set any point of interface between snow and ice be coordinate origin and axis z represents the depth direction. Fig. 1 shows the definitions.

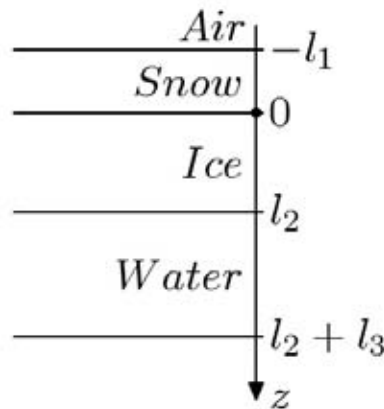


Fig. 1. Position

Let l_1, l_2, l_3 be the thickness of snow, ice and water (m). Set $D_1 = [-l_1, 0]$, $D_2 = [0, l_2]$, $D_3 = [l_2, l_2 + l_3]$, be the domain representing snow, ice and water respectively. $I = [0, t_f] \subset \mathbb{R}$ be the time domain of the coupled dynamic system. Set $Q = D \times I \subset \mathbb{R}^2$, $Q_j = D_j \times I \subset \mathbb{R}^2$, $j = 1, 2, 3$.

By using the energy conversation law and the theory of Fourier heat exchange, the temperature field equation and definite conditions of the Arctic snow, sea ice and water can be given as

$$(c\rho)(z, t) \frac{\partial T(z, t)}{\partial t} = \frac{\partial}{\partial z} \left(K(z, t) \frac{\partial T(z, t)}{\partial z} \right) + g(z, t), (z, t) \in Q \quad (1)$$

$$T(z, 0) = T_0(z), z \in D \quad (2)$$

$$K(z, t) \frac{\partial T(z, t)}{\partial z} = h(T(z, t) - T_c(t)), z = -l_1, t \in I \quad (3)$$

$$T(l_2 + l_3, t) = T_3(t), t \in I \quad (4)$$

Where heat storage capacity (c), density (ρ) and heat conductivity (K) in each domain can be expressed as

$$(c\rho)(z, t) = \begin{cases} c_s \rho_s & (z, t) \in Q_1 \\ c_i \rho_i + \lambda \cdot S_i(z)/(T_i(z, t) - 273.15)^2 & (z, t) \in Q_2 \\ c_w \rho_w & (z, t) \in Q_3 \end{cases} \quad (5)$$

$$K(z, t) = \begin{cases} K_s & (z, t) \in Q_1 \\ K_i + \beta \cdot S_i(z)/(T(z, t) - 273.15) & (z, t) \in Q_2 \\ c_w \rho_w & (z, t) \in Q_3 \end{cases} \quad (6)$$

$c_s, c_i, c_w, \rho_s, \rho_i, \rho_w, K_s, K_i, K_w$ denotes heat storage capacity, density and conduction coefficient of snow, ice and water respectively. And all these parameters are known. Let $S_i(z)$ be the salinity in $z \in D_2$. Heat storage capacity can be expressed as:

$$S_i(z) = \begin{cases} 14.24 - 19.39z & 0 \leq z < 0.57(m) \\ 3.2 & 0.57 \leq z \leq l_2(m) \end{cases} \quad (7)$$

λ and β are positive constants. From above, the multi-domain coupled dynamic system of temperature field system TCS can be defined by equations (1) -(4).

In equation (1), the heat source term $g(z, t)$ can be described by :

$$g(z, t) = \begin{cases} g_1(z, t) & (z, t) \in Q_1 \\ g_2(z, t) & (z, t) \in Q_2 \\ 0 & (z, t) \in Q_3 \end{cases} \quad (8)$$

where

$$g_j(z, t) = q_j(1 - \alpha_j) \cdot \gamma_j \cdot I_{0j} \cdot \exp(-\gamma_j |z|) \quad j = 1, 2. \quad (9)$$

q_j, α_j, I_{0j} and $\gamma_j, j = 1, 2$ is the shortwave radiation, reflectivity, the transmission and extinction coefficient of sun respectively.

In this paper we identify heat exchange coefficient $u = h$ primarily, $h \in U_{ad} = [h_-, h_+] \subset R_+^1$.

Based on the physical properties of the Arctic snow, ice and water, we introduce the assumption:

A₁: In the coupled system TCS, functions $g_1 : Q_1 \rightarrow R, g_2 : Q_2 \rightarrow R, T_3 : I \rightarrow R, P_3 : I \rightarrow R, T_c : I \rightarrow R$ should be bounded, continuously differentiable. The derivatives in terms of all the variables are also bounded.

Based on assumption **A₁** and equation (5)-(9), the coefficients $(c\rho)(z, t), K(z, t)$ and heat source term $g(z, t)$ of coupled dynamic system TCS are all piecewise continuously differentiable.

Let $p_0(t) = K^{-1}(-l_1, t)h(T(-l_1, t) - T_c(t)), \omega(z, t) = T(z, t) - (z - l_2 - l_3)p_0(t) - T_3(t)$, then the system TCS can be expressed as ITCS:

$$\begin{cases} \frac{\partial \omega}{\partial t} - (c\rho)^{-1} \frac{\partial}{\partial z} (k(z, t) \frac{\partial \omega}{\partial z}) = p(z, t; u) \\ \omega(z, 0) = \omega_0(z) \\ \frac{\partial \omega}{\partial z} \Big|_{z=-l_1} = 0 \\ \omega \Big|_{z=l_2+l_3} = 0 \end{cases}$$

where $p(z, t; u) = (c\rho)^{-1}(p_0(t) \frac{\partial k(z, t)}{\partial z} + g(z, t) - c\rho(z - l_2 - l_3)p_0'(t) - c\rho T_3'(t)), \omega_0(z) = T_0(z) - (z - l_2 - l_3)p_0(0) - T_3(0)$.

And we now explain the notations used in this paper. Let H^1 and V be two real Hilbert spaces with norm denoted by $|\cdot|_{H^1}$ and $\|\cdot\|_V$, respectively, here $H^1 = H^1(D) = \{\phi \in L^2(D) \mid \frac{\partial \phi(z, t)}{\partial t}, \frac{\partial \phi(z, t)}{\partial z} \in L^2(D)\}, V = \{\phi \mid \phi \in H^1, \frac{\partial \phi}{\partial z} \Big|_{z=-l_1} = \phi \Big|_{z=l_2+l_3} = 0\}$. The symbol $\langle \cdot, \cdot \rangle_{H^1}$ and $\langle \cdot, \cdot \rangle_V$ denote the inner product

on H^1 and V , respectively. V^* denotes the dual space of V and $\langle \cdot, \cdot \rangle_{V^*, V}$ denotes the dual pairing between V^* and V . Assume that (V, H^1, V^*) is a Gelfand triple space with $V \hookrightarrow H^1 \equiv H^{1*} \hookrightarrow V^*$, which means that the embedding is continuous and V is dense in H^1 .

Based on the above work, we consider a bilinear form on $V \times V$,

$$a(t, u, \omega, \phi) = \int_D K \frac{\partial \omega}{\partial z} \frac{\partial((c\rho)^{-1}\phi)}{\partial z} dz \quad (10)$$

and $a(t, u, \omega, \phi)$ satisfies the following properties:

- (i) $a(t, u, \omega, \phi) = a(t, u, \phi, \omega)$, for all $\omega, \phi \in V$, and $t \in I$.
- (ii) there exists $ac_1(u) > 0$, such that $|a(t, u, \omega, \phi)| \leq c_1(u)\|\omega\|_V\|\phi\|_V$, for all $\omega, \phi \in V$, and $t \in I$.
- (iii) there exists $\lambda_1(u) > 0$ and $\lambda_2(u) \in R$, such that $a(t, u, \omega, \omega) + \lambda_2(u)\|\omega\|_{H^1}^2 \geq \lambda_1(u)\|\omega\|_V^2$, for all $\omega \in V$, and $t \in I$.

Then, we can define an operator $A(t, u) \in L(V, V^*)$, for $t \in I$ via the relation, $a(t, u, \omega, \phi) = \langle A(t, u)\omega, \phi \rangle_{V^*, V}$, for all $\omega, \phi \in V$.

Let $f_0(p(z, t, ; u); \phi) = \int_D \phi p(z, t, ; u) dz$, for each $u \in U_{ad}$ and $\phi \in V$, $f_0(p(z, t, ; u); \phi)$ is bounded linear function on V . According to Risez representation theorem, there exists $f(z, t; u) \in V^*$, for each $\phi \in V$, such that

$$f_0(p(z, t, ; u); \phi) = \langle f(z, t; u), \phi \rangle_{V^*, V} \quad (11)$$

Thus, for each $u \in U_{ad}$, the system TCS can be converted to a nonlinear parabolic evolution equation as follows:

$$\begin{cases} \frac{\partial \omega(z, t; u)}{\partial t} + A(t, u)\omega(z, t; u) = f(z, t; u) \\ \omega(z, 0) = \omega_0(z) \end{cases}$$

We write the above system as IITCS.

Using the definition of the weak solution^{[2] 1} given by R.Dautray, J.L. Lions and the Galerkin method^{[2] 1}, we can prove:

Theorem 1. Suppose that A_1 holds, and $a(t, u, \omega, \phi)$ is defined by (10), then for $u \in U_{ad}$, there exists the unique weak solution $\omega(z, t; u) \in L^2(0, t_f; H^1(D)) \cap C(0, t_f; H^1(D))$, such that $\omega(z, t; u)$ satisfies the system IITCS, and $\omega(z, t; u)$ is continuous for $u \in U_{ad}$.

So the TCS system also has the unique weak solution $T(z, t; u)$, and $T(z, t; u)$ is continuous for $u \in U_{ad}$. Let $S \subset L^2(0, t_f; H^1(D)) \cap C(0, t_f; H^1(D))$ be the set of solutions of TCS, i.e.

$$S := \{T(z, t; u) \in L^2(I; H^1) \cap C(I; H^1) \mid T(z, t; u) \text{ is the weak solution of TCS corresponding to } u \in U_{ad}\}$$

Corollary 1. Let S be defined as above, then S is the compact set of $L^2(I; H^1) \cap C(I; H^1)$.

Proof. From theorem 1, the mapping $u \rightarrow T(z, t; u): U_{ad} \rightarrow S \subset L^2(I; H^1) \cap C(I; H^1)$ is continuous and $U_{ad} \subset R_+^1$ is bounded and closed. Hence, S is the compact set in $L^2(I; H^1) \cap C(I; H^1)$.

3 Identification of heat exchange coefficient of the Arctic snow

For the sake of researching the characteristic and correlativity of the Arctic snow, ice, water and surface atmospheric, monitoring the sea ice freezing and melting process and evaluating its influence on global weather. From 2003. 8 to 2004. 4, satellite tracker-localizer was installed on arctic which can monitor the hydro-meteorological data including wind speed, wind direction, air temperature, air pressure, ice and water temperature at surveying points. Moreover, the thickness of snow, ice, the area of ice and salinity have also been provided. Generally, the recording time interval is from one hour to three hours.

Let $T_d(z_k, t_j), j = 1, 2, \dots, l_t, k = 1, 2, \dots, l_z$. be the observed temperature at $z_k \in D$ and $t_j \in I$. The temperature distribution function $\{T_d(z, t)\}, (z, t) \in Q$, can be fitted by the observed data $T_d(z_k, t_j)$. Because of the continuity of temperature change, the function $T_d(z, t) : Q \rightarrow R$ belongs to $C(Q, R)$ and according to theorem 1, the TCS has the unique solution $T(z, t; u), u \in U_{ad}$. In order to estimate the error between $T(z, t; u)$ and $T_d(z, t)$, the performance criterion is given by

$$J(u) := |T(z, t; u) - T_d(z, t)|_{C(Q,R)}^2 \tag{12}$$

Then, identification model of the coupled dynamic system TCS can be expressed as:

$$\begin{aligned} \text{ITCSP: } \min \quad & J(u) \\ \text{s. t. } \quad & T(z, t; u) \in S \\ & u \in U_{ad} \end{aligned}$$

Because $U_{ad} \subset R_+^1$ is nonempty, bounded and closed, the mapping $u \rightarrow T(z, t; u)$ is continuous. By Corollary1, $S \subset L^2(I; H^1) \cap C(I; H^1)$ is compact. From equation (12), $J(u)$ is continuous for u and $T(z, t; u)$. Therefore, there exists the optimal solution $(u^*, T(z, t; u^*)) \in U_{ad} \times S$ such that

$$J(u^*) \leq J(u) \forall u \in U_{ad}$$

Theorem 2. Suppose that assumption A_1 holds, then there exists the optimal solution $(u^*, T(z, t; u^*)) \in U_{ad} \times S$, such that $J(u^*) \leq J(u), \forall u \in U_{ad}$.

We now consider the first order necessary condition for the optimal parameter u^* . Based on assumption A_1 , and the equation (10), we can obtain the mapping $u \rightarrow T(z, t; u)$ is Gâteaux differentiable with respect to $u \in U_{ad}$, and the Gâteaux differential of $T(z, t; u)$ in the direction $u - u^* \in U_{ad}$ exists. So the necessary condition for the optimal parameter is characterized by the following variational inequality: $J'(u^*)(u - u^*) \geq 0$.

In order to reduce the error and make the calculation convenient, performance criterion (12) can be modified:

$$J_d(u) = \sum_{k=1}^{l_z} \sum_{l=1}^{l_t} (T(z_k, t_l; u) - T_d(z_k, t_l))^2 \tag{13}$$

Thus the practical identification problem ITCSP can be converted to:

$$\begin{aligned} \text{ITCSP}_d : \min \quad & J_d(u) \\ \text{s. t. } \quad & T(z, t; u) \in S \\ & u \in U_{ad} \end{aligned}$$

Similarly, using the theorem 2, the optimal solution $(u^*, T(z, t; u^*)) \in U_{ad} \times S$ exists.

4 Optimal algorithm and numerical simulation

4.1 Optimal algorithm

By using equation (1) and the physical properties of heat exchanger, we know that the temperature $T(z, t; u)$ is monotone in terms of parameters in sub-domain Q . In order to construct the optimal algorithm, the $J_{dj}(u)$ must be modified.

We set

$$\begin{aligned} T_{sj}(u) &= \sum_{Z_k \in D_j} \sum_{l=1}^{l_t} T(z_k, t_l; u)^2; & T_{dj} &= \sum_{Z_k \in D_j} \sum_{l=1}^{l_t} T_d(z_k, t_l)^2 \\ J_{bj}(u) &= (T_{sj} - (u)T_{dj})^2; & J_b(u) &= \sum_{j=1}^3 J_{bj}(u) \quad j = 1, 2, 3. \end{aligned}$$

where $T_{dj}(j = 1, 2, 3)$ is independent of u . So $T_{dj}(j = 1, 2, 3)$ can be considered as constant.

Identification model of coupled system TCS is now expressed by:

$$\begin{aligned} ITCSP_b \quad \min \quad & J_b(u) \\ \text{s. t.} \quad & T(z, t; u) \in S \\ & u \in U_{ad} \end{aligned}$$

Theorem 3. If $u^* \in U_{ad}$ is the optimal solution of $ITCSP_d$, then u^* is also the optimal solution of $ITCSP_b$.

The identification parameter is $u \in U_{ad}$. As analyzing above, we construct the optimal algorithm as follows:

Step 1. Read observed parameters c_j , ρ_j , and K_j , $j = 1, 2, 3$. Select starting points $h_- \in U_{ad}$, and $h_+ \in U_{ad}$, give the precision $\varepsilon > 0$.

Step 2. Obtain the numerical solution $T(z, t; h_-)$ of coupled system TCS by the semi-implicit finite difference scheme. Calculate $J_b(h_-)$ as defined in (12).

Step 3. Obtain the numerical solution $T(z, t; h_+)$ of coupled system TCS by the semi-implicit finite difference scheme. Calculate $J_b(h_+)$ as defined in (12).

Step 4. If $J_b(h_-) > J_b(h_+)$, goto Step 5, Otherwise goto Step 6.

Step 5. Set $h_- = (h_- + h_+)/2$, then goto Step 2.

Step 6. Set $h_+ = (h_- + h_+)/2$, then goto Step 3.

Step 7. When $|h_- - h_+| < \varepsilon$, then stop, and h_- is the optimal solution.

4.2 Numerical results

According to the above optimal algorithm, we calculate the temperature distribution of the Arctic ice from the real data set acquired by monitor buoy installed on the Arctic. The optimal parameters are $h = 20.4$. The average absolute error is defined by:

$$e = \sum_{k=1}^{l_z} \sum_{l=1}^{l_t} (T(z_k, t_l; u) - T_d(z_k, t_l))^2 / (l_z \cdot l_t).$$

And the final average absolute error $e=0.41$.

Fig. 2 is the calculated temperature $T(z, t; u)$ comparing with observed temperature $T_d(z, t)$ at all locations. The time ranges from November, 2003 to February, 2004. Let the horizontal coordinate be the time and the perpendicular coordinate be the temperature $T(z, t; u)$ and $T_d(z, t)$.

Fig. 3 is the calculated temperature comparing with observed temperature at location $z = 1.28m$. The horizontal direction coordinate the ice thickness and the perpendicular coordinate represents the temperature $T(z, t; u)$ and $T_d(z, t)$.

5 Conclusions

This paper studies the multi-domain coupled distributed parameter system of temperature field, and establishes the identification model of heat exchange coefficient, also discusses the existence and continuity of the weak solution for heat exchange coefficient. Moreover, the paper establishes the distributed parameter identification model of this thermodynamic system and discusses the identification problem. Finally, optimal algorithm for this system is presented and the numerical results of the Arctic sea ice temperature distribution are shown in this paper. In this paper we identify only one parameter of the system, and there are many other parameters such as heat storage capacity, density, conduction coefficient and so on, that are very important. We will study the other parameters for the next time.

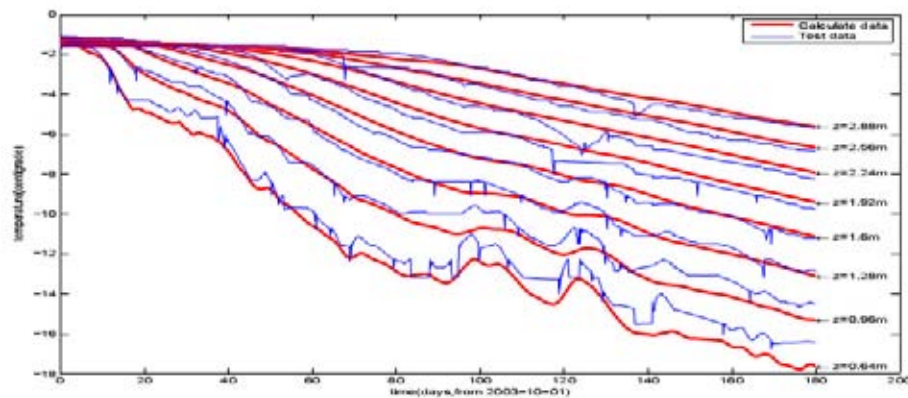


Fig. 2. Calculated temperature $T(z, t; u)$ comparing with observed temperature $T_d(z, t)$ at all locations

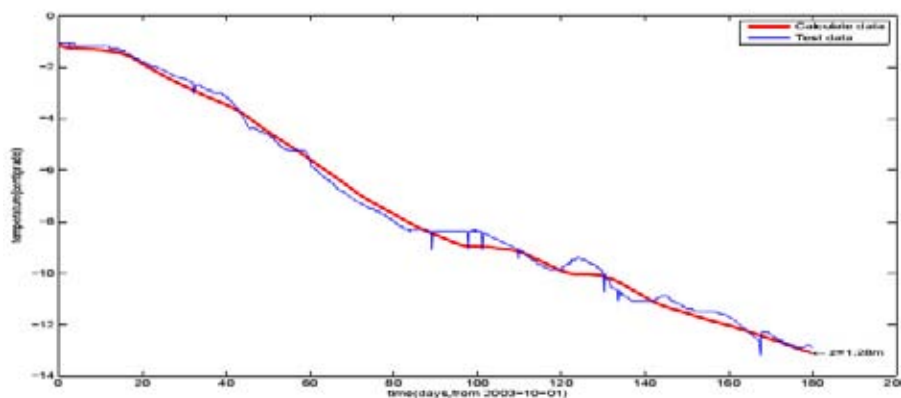


Fig. 3. Calculated temperature $T(z, t; u)$ comparing with observed temperature $T_d(z, t)$ at the location $z=1.28m$.

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