

A numerical method to obtain positive solution for classes of competitive systems

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Abstract. Using a numerical method based on sub-super solution, we will obtain positive solution to the coupled-system of boundary value problems of the form

$$\begin{aligned} -\Delta u(x) &= \lambda f(x, u, v) & x \in \Omega \\ -\Delta v(x) &= \lambda g(x, u, v) & x \in \Omega \\ u(x) = 0 &= v(x) & x \in \partial\Omega \end{aligned}$$

where f, g are C^1 functions with at least one of $f(x_0, 0, 0)$ or $g(x_0, 0, 0)$ being negative for some $x_0 \in \Omega$ (semipositone).

Keywords: positive solutions, sub and super-solutions

1 Introduction

Consider positive solutions to the coupled-system of boundary value problems

$$\begin{cases} -\Delta u(x) = \lambda f(x, u, v) & x \in \Omega \\ -\Delta v(x) = \lambda g(x, u, v) & x \in \Omega \\ u(x) = 0 = v(x) & x \in \partial\Omega \end{cases} \quad (1)$$

where $\lambda > 0$, Δ is the Laplacian operator, Ω is a bounded region in R^n , ($N \geq 1$) with smooth boundary $\partial\Omega$, and f, g are C^1 functions with at least one of $f(x_0, 0, 0)$ or $g(x_0, 0, 0)$ being negative for some $x_0 \in \Omega$ (semipositone).

In this paper, we want to investigate numerically positive solution of (1) by using the method of sub-super solutions. A super solution to (1) is defined as an ordered pair of smooth functions (\bar{u}, \bar{v}) on Ω satisfying

$$\begin{cases} -\Delta \bar{u} \geq \lambda f(x, \bar{u}, \bar{v}) & x \in \Omega \\ -\Delta \bar{v} \geq \lambda g(x, \bar{u}, \bar{v}) & x \in \Omega \\ \bar{u} \geq 0; \bar{v} \geq 0 & x \in \partial\Omega. \end{cases} \quad (2)$$

Sub solutions are similarly defined with inequalities reversed. Let $D = [\underline{\rho}_1, \bar{\rho}_1] \times [\underline{\rho}_2, \bar{\rho}_2]$ where $\underline{\rho}_1 = \inf\{\underline{u}(x) : x \in \partial\Omega\}$, $\bar{\rho}_1 = \sup\{\bar{u}(x) : x \in \partial\Omega\}$, $\underline{\rho}_2 = \inf\{\underline{v}(x) : x \in \partial\Omega\}$, $\bar{\rho}_2 = \sup\{\bar{v}(x) : x \in \partial\Omega\}$.

Theorem 1. Let $(\bar{u}, \bar{v}), (\underline{u}, \underline{v})$ be ordered pairs of smooth functions such that (\bar{u}, \bar{v}) satisfies:

$$-\Delta \bar{u} \geq \lambda f(x, \bar{u}, \bar{v}) \quad x \in \Omega$$

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$$\begin{aligned} -\Delta \underline{v} &\leq \lambda g(x, \bar{u}, \underline{v}) \quad x \in \Omega \\ \bar{u} &\geq 0, \quad \underline{v} \leq 0 \quad x \in \partial\Omega \end{aligned}$$

and (\underline{u}, \bar{v}) satisfies the corresponding reversed inequalities. Suppose that

$$\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \leq 0 \quad \text{on } \bar{\Omega} \times D \quad (\text{competitive system}).$$

If $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$ on $\bar{\Omega}$, then there is solutions (u, v) of (1) such that $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$ on Ω .

In [1] for the first time in the literature, the authors consider a class of semipositone systems. In particular they extend many of the results discussed for the positive solutions of single equation in [2] to semipositone systems. It was shown positive solutions to (1) for either λ near the first eigenvalue λ_1 of the operator $-\Delta$ subject to Dirichlet boundary conditions, or for λ large exists. We consider following assumptions:

f, g are C^1 functions satisfying

$$\text{either } f(x_0, 0, 0) < 0 \text{ or } g(x_0, 0, 0) < 0 \text{ for some } x_0 \in \Omega \quad (3)$$

$$\lim_{u \rightarrow \infty} \frac{f(x, u, v)}{u} = 0 \quad \text{uniformly in } x, v \quad (4)$$

and

$$\lim_{u \rightarrow \infty} \frac{g(x, u, v)}{v} = 0 \quad \text{uniformly in } x, u. \quad (5)$$

To introduce additional hypotheses to prove existence results near λ_1 , first we recall the anti-maximum principal by Clement Pletier (see [4]), namely, if Z_λ is the unique solution of

$$\begin{cases} -\Delta z - \lambda z = -1 & x \in \Omega \\ z = 0 & x \in \partial\Omega \end{cases} \quad (6)$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, where λ_1 is the smallest eigenvalue of the problem

$$\begin{cases} -\Delta \phi(x) = \lambda \phi(x) & x \in \Omega \\ \phi(x) = 0 & x \in \partial\Omega. \end{cases} \quad (7)$$

Let $I = [\alpha, \gamma]$ where $\alpha > \lambda_1$ and $\gamma < \lambda_1 + \delta$, and let

$$\sigma := \max_{\lambda \in I} \|z_\lambda\|$$

where $\|\cdot\|$ denotes the supremum norm. Now assuming that there exists a $m_1 > 0$ such that

$$f(x, u, v) \geq u - m_1 \quad \forall x \in \bar{\Omega}, u \in [0, m_1 \gamma \sigma], v \geq 0 \quad (8)$$

and exists a $m_2 > 0$ such that

$$g(x, u, v) \geq v - m_2 \quad \forall x \in \bar{\Omega}, v \in [0, m_2 \gamma \sigma], u \geq 0. \quad (9)$$

Finally to prove existence results for λ large, in addition to (3)-(5), we assume $\exists f_1(u) \leq f(x, u, v) \forall x \in \bar{\Omega}, u \geq 0, v \geq 0$ such that

$$f_1(r_1) = 0, f_1'(r_1) < 0, \quad \int_0^{r_1} f_1(s) ds > 0 \quad \text{for some } r_1 > 0 \quad (10)$$

and $\exists g_2(v) \leq g(x, u, v) \forall x \in \bar{\Omega}, u \geq 0, v \geq 0$ such that

$$g_2(r_2) = 0, g_2'(r_2) < 0, \quad \int_0^{r_2} g_2(s) ds > 0 \quad \text{for some } r_2 > 0 \quad (11)$$

2 Existence results

Theorem 2. Let $\lambda \in I$ and assume (3)-(5), and (8)-(9) hold, Then (1) has a positive solution.

It was shown in [1] (u, v) is a subsolution of (1), and where $\underline{u}(x) = \gamma m_1 Z_\lambda$ and $\underline{v}(x) = \gamma m_2 Z_\lambda$.

Now let $w(x)$ to be the unique positive solution of

$$\begin{cases} -\Delta w(x) = 1 & x \in \Omega \\ w(x) \leq 0 & x \in \partial\Omega \end{cases} \tag{12}$$

(\bar{u}, \bar{v}) is a supersolution that $\bar{u} = Jw(x)$ and $\bar{v} = \tilde{J}w(x)$ where $J, \tilde{J} > 0$ are sufficiently large such that

$$\frac{1}{\lambda \|w\|} \geq \frac{f(x, J\|w\|, v)}{J\|w\|}, \frac{g(x, u, \tilde{J}\|w\|)}{\tilde{J}\|w\|} \tag{13}$$

and

$$\bar{u}(x) \geq \underline{u}(x) \text{ on } \Omega \text{ and } \bar{v}(x) \geq \underline{v}(x) \text{ on } \Omega$$

Theorem 3. Assume (3)-(5) and (10)-(11) hold. Then there exists a $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, (1) has a positive solutions. Here we give a simple example that satisfies the hypotheses of theorem 2 and 3. Consider

$$f(x, u, v) = m_1 \sqrt{20u + 1} - \frac{3m_1}{2} + e^{\frac{v}{m_1 v + 1}} \forall u \geq 0, v \geq 0 \tag{14}$$

$$g(x, u, v) = m_2 \sqrt{\frac{1}{2}v + 1} - \frac{3m_2}{2} + e^{\frac{2u}{u+1}} \forall u \geq 0, v \geq 0 \tag{15}$$

where $m_1, m_2 > 0$ are constant.

Here $f(x, 0, 0) = 1 - \frac{m}{2} < 0$ for $m > 2$, f is increasing in u, v , and $\lim_{u \rightarrow \infty} \frac{f(x, u, v)}{u} = 0$ uniformly in v . Also, $g(x, 0, 0) = 1 - \frac{m}{2} < 0$ for $m > 2$, g is increasing in u, v , and $\lim_{v \rightarrow \infty} \frac{g(x, u, v)}{v} = 0$ uniformly in u . Now, to show that (8) and (9) are satisfied, it suffices to show that $h_1(u) = m(u + 1)^{\frac{1}{2}} - \frac{3m}{2}$ satisfies (8) since $f(x, u, v) \geq h_1(u) \forall u \geq 0, v \geq 0$. Let $p > 0$ be such that $h_1(p) = p - m$. That is,

$$m(p + 1)^{\frac{1}{2}} - \frac{3m}{2} = p - m, \quad m^2(p + 1) = \left\{p + \frac{m}{2}\right\}^2, \quad p^2 + (m - m^2)p - \frac{3m^2}{4} = 0,$$

and

$$p = \frac{(m^2 - m) + \sqrt{m^4 - 2m^3 + 4m^2}}{2} = \frac{(m^2 - m) + m \sqrt{m(m - 1)^2 + 3}}{2}.$$

Hence in order that (8) be satisfied, we must have

$$\frac{m^2 - m + m \sqrt{(m - 1)^2 + 3}}{2} \geq m(\sigma\alpha),$$

that is,

$$(m - 1) + \sqrt{(m - 1)^2 + 3} \geq 2(\sigma\alpha). \tag{16}$$

Since σ and α are quantities that depend only on Ω , clearly for a given σ and α , there exists and m_0 sufficiently large such that if $m > m_0$, then (16) is satisfied and equivalently (8) will be satisfied. Thus, (9) is also satisfied for $m \geq m_0$.

Note that this example satisfies the hypotheses of theorem 3 also since $f(x, u, v) \geq h_1(u) \forall u \geq 0, v \geq 0$ and one can construct a function $f_1(u) \leq h_1(u)$ satisfying (10).

It can be easily shown that g satisfies the hypotheses of theorem 2 and 3 with $m_2 > m_0^*$ for some $m_0^* > 0$.

3 Numerical results

We see in section 2 that there must always exists a solution for problems such as (1) between a sub-solution $(\underline{u}, \underline{v})$ and a super-solution (\bar{u}, \bar{v}) when $\frac{\partial f}{\partial v} \leq 0, \frac{\partial g}{\partial u} \leq 0$.

Consider the coupled-system boundary value problems

$$\begin{cases} -\Delta u(x) = \lambda f(x, u, v) & x \in \Omega \\ -\Delta v(x) = \lambda g(x, u, v) & x \in \Omega \\ u(x) = 0 = v(x) & x \in \partial\Omega. \end{cases} \tag{17}$$

Since f, g are C^1 functions, there exists positive constants K_1, K_2 such that $\frac{\partial f}{\partial u} \geq -K_1$, and $\frac{\partial g}{\partial v} \geq -K_2$ on $\bar{\Omega} \times D$. Thus we can study the equivalent system

$$\begin{cases} -\Delta u(x) + \lambda K_1 u(x) = \lambda f(x, u, v) + \lambda K_1 u(x) = \lambda \hat{f}(x, u, v) & x \in \Omega \\ -\Delta v(x) + \lambda K_2 v(x) = \lambda g(x, u, v) + \lambda K_2 v(x) = \lambda \hat{g}(x, u, v) & x \in \Omega \\ u(x) = 0 = v(x) & x \in \partial\Omega. \end{cases} \tag{18}$$

The mapping

$$T : (u_1, v_1) \rightarrow (u_2, v_2), \quad (u_2, v_2) = T(u_1, v_1)$$

$$(u_1, v_1) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}] \quad \forall x \in \bar{\Omega}$$

where (u_2, v_2) is the unique solution of the coupled-system

$$\begin{cases} -\Delta u_2(x) + \lambda K_1 u_2(x) = \lambda f(x, u_1, v_1) + \lambda K_1 u_1(x) & x \in \Omega \\ -\Delta v_2(x) + \lambda K_1 v_2(x) = \lambda g(x, u_1, v_1) + \lambda K_1 v_1(x) & x \in \Omega \\ u_2(x) = 0 = v_2(x) & x \in \partial\Omega \end{cases} \tag{19}$$

satisfied the hypotheses of Schauder fixed point theorem, and then we can conclude that

$$\exists (u, v) \in D \quad T(u, v) = (u, v)$$

so (u, v) is a solution of (1) (see [3]).

By letting $\hat{f}(x, u, v) = \lambda f(x, u, v) + \lambda K_1 u$ and $\hat{g}(x, u, v) = \lambda g(x, u, v) + \lambda K_2 v$, we use the following iteration to obtain solution:

$$\begin{cases} u_0(x) = \underline{u}, v_0(x) = \bar{v} \\ (\Delta - \lambda K_1)u_{n+1} = -\hat{f}(x, u_n, v_n) & x \in \Omega \quad n = 0, 1, 2, \dots \\ (\Delta - \lambda K_2)v_{n+1} = -\hat{g}(x, u_{n+1}, v_n) & x \in \Omega \\ u_{n+1} = 0 = v_{n+1} & x \in \partial\Omega \end{cases} \tag{20}$$

We can also use $u_0(x) = \bar{u}, v_0(x) = \underline{v}$ as initial guesses. we use following algorithm:

Sub- and super-solution algorithm

1. Find $u_0 = \underline{u}$ and $v_0 = \bar{v}$. Choose numbers $k_1, k_2 > 0$;
 2. Solve the boundary value system (20);
 3. If $\|u_{n+1} - u_n\| < \epsilon$ and $\|v_{n+1} - v_n\| < \epsilon$, output and stop. Else go to step 2.
- Now we want to apply the algorithm for:

$$\begin{cases} -\Delta u(x) = \lambda(m_1 \sqrt{20u + 1} - \frac{3m_1}{2} + e^{\frac{v}{m_1 v + 1}}) & x \in \Omega \\ -\Delta v(x) = \lambda(m_2 \sqrt{\frac{1}{2}v + 1} - \frac{3m_2}{2} + e^{\frac{2u}{u + 1}}) & x \in \Omega \\ u(x) = 0 = v(x) & x \in \partial\Omega. \end{cases} \tag{21}$$

For doing step 1, we solve the problem

Table 1. approximation of z_λ for $\lambda = 15$

x / y	0.2	0.4	0.6	0.8
0.2	-0.268	0.423	-0.431	-0.283
0.4	-0.447	-0.701	-0.718	-0.493
0.6	-0.513	-0.753	-0.778	-0.636
0.8	-0.505	-0.528	-0.497	-0.345

Table 2. approximation of z_λ for $\lambda = 17$

x / y	0.2	0.4	0.6	0.8
0.2	1.895	3.067	3.130	2.022
0.4	3.266	5.199	5.341	3.625
0.6	3.789	5.626	5.818	4.712
0.8	3.727	3.912	3.676	2.514

Table 3. approximation of z_λ for $\lambda = 30$

x / y	0.2	0.4	0.6	0.8
0.2	0.002	0.017	0.021	0.008
0.4	0.028	0.062	0.068	0.041
0.6	0.050	0.087	0.093	0.070
0.8	0.054	0.060	0.056	0.033

Table 4. approximation of z_λ for $\lambda = 36$

x / y	0.2	0.4	0.6	0.8
0.2	-0.001	0.012	0.016	0.005
0.4	0.024	0.056	0.063	0.038
0.6	0.048	0.084	0.091	0.068
0.8	0.053	0.060	0.056	0.033

$$\begin{cases} -\Delta z - \lambda z = -1 & x \in \Omega \\ z = 0 & x \in \partial\Omega \end{cases} \tag{22}$$

to obtain \underline{u} . We know from section 2 that problem (22) has a positive solution for $\lambda \in (\lambda_1, \lambda_1 + \delta)$. The obtained results shows there is an array of positive solution for $\lambda \in (17, 35)$ so λ_1 is around 17. For brevity we express just some of those numerical results.

Let $\underline{u} = \gamma m_1 z_\lambda(x)$ where γ and m obtained from section 1, 2 and to obtain \bar{v} for $\lambda \in I$ ($I = [\alpha, \gamma]$) where $\alpha > \lambda_1$ and $\gamma < \lambda_1 + \delta$) we solve

$$\begin{cases} -\Delta v(x) = 1 & x \in \Omega \\ v(x) = 0 & x \in \partial\Omega \end{cases} \tag{23}$$

by finite difference (see [5, 6]). We choose J such that (13) is satisfied.

We execute algorithm for $\lambda \in I = [17.1, 34.9]$. For brevity we express just some of those numerical results

Table 5. approximations of u, v for $\lambda = 17.1$ respectively

x / y	0.2	0.4	0.6	0.8
0.2	0.514×10^5	0.769×10^5	0.769×10^5	0.514×10^5
0.4	0.769×10^5	1.159×10^5	1.159×10^5	0.769×10^5
0.6	0.769×10^5	1.159×10^5	1.159×10^5	0.769×10^5
0.8	0.514×10^5	0.769×10^5	0.769×10^5	0.514×10^5
x / y	0.2	0.4	0.6	0.8
0.2	0.492×10^4	0.737×10^4	0.737×10^4	0.492×10^4
0.4	0.737×10^4	1.112×10^4	1.112×10^4	0.737×10^4
0.6	0.737×10^4	1.112×10^4	1.112×10^4	0.737×10^4
0.8	0.492×10^4	0.737×10^4	0.737×10^4	0.492×10^4

Our numerical results (in following tables) show that there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, (21) has a positive solution. In this case $\lambda^* = 166.696$ with decimal accuracy.

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Table 6. approximations of u, v for $\lambda = 25$ respectively

x / y	0.2	0.4	0.6	0.8
0.2	1.101×10^5	1.647×10^5	1.647×10^5	1.101×10^5
0.4	1.647×10^5	2.481×10^5	2.481×10^5	1.647×10^5
0.6	1.647×10^5	2.481×10^5	2.481×10^5	1.647×10^5
0.8	1.101×10^5	1.647×10^5	1.647×10^5	1.101×10^5
x / y	0.2	0.4	0.6	0.8
0.2	1.068×10^4	1.600×10^4	1.600×10^4	1.068×10^4
0.4	1.600×10^4	2.412×10^4	2.412×10^4	1.600×10^4
0.6	1.600×10^4	2.412×10^4	2.412×10^4	1.600×10^4
0.8	1.068×10^4	1.600×10^4	1.600×10^4	1.068×10^4

Table 7. approximations of u, v for $\lambda = 30$ respectively

x / y	0.2	0.4	0.6	0.8
0.2	1.587×10^5	2.373×10^5	2.373×10^5	1.587×10^5
0.4	2.373×10^5	3.575×10^5	3.575×10^5	2.373×10^5
0.6	2.373×10^5	3.575×10^5	3.575×10^5	2.373×10^5
0.8	1.587×10^5	2.373×10^5	2.373×10^5	1.587×10^5
x / y	0.2	0.4	0.6	0.8
0.2	1.547×10^4	2.317×10^4	2.317×10^4	1.547×10^4
0.4	2.317×10^4	3.493×10^4	3.493×10^4	2.317×10^4
0.6	2.317×10^4	3.493×10^4	3.493×10^4	2.317×10^4
0.8	1.547×10^4	2.317×10^4	2.317×10^4	1.547×10^4

Table 8. approximations of u, v for $\lambda = 34.9$ respectively

x / y	0.2	0.4	0.6	0.8
0.2	2.148×10^5	3.213×10^5	3.213×10^5	2.148×10^5
0.4	3.213×10^5	4.840×10^5	4.840×10^5	3.213×10^5
0.6	3.213×10^5	4.840×10^5	4.840×10^5	3.213×10^5
0.8	2.148×10^5	3.213×10^5	3.213×10^5	2.148×10^5
x / y	0.2	0.4	0.6	0.8
0.2	2.102×10^4	3.147×10^4	3.147×10^4	2.102×10^4
0.4	3.147×10^4	4.744×10^4	4.744×10^4	3.147×10^4
0.6	3.147×10^4	4.744×10^4	4.744×10^4	3.147×10^4
0.8	2.102×10^4	3.147×10^4	3.147×10^4	2.102×10^4

Table 9. approximations of u, v for $\lambda = 170$ respectively

x / y	0.2	0.4	0.6	0.8
0.2	0.510×10^7	0.763×10^7	0.763×10^7	0.510×10^7
0.4	0.763×10^7	1.150×10^7	1.150×10^7	0.763×10^7
0.6	0.763×10^7	1.150×10^7	1.150×10^7	0.763×10^7
0.8	0.510×10^7	0.763×10^7	0.763×10^7	0.510×10^7
x / y	0.2	0.4	0.6	0.8
0.2	0.508×10^6	0.760×10^6	0.760×10^6	0.508×10^6
0.4	0.760×10^6	1.145×10^6	1.145×10^6	0.760×10^6
0.6	0.760×10^6	1.145×10^6	1.145×10^6	0.760×10^6
0.8	0.508×10^6	0.760×10^6	0.760×10^6	0.508×10^6

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Table 10. approximations of u, v for $\lambda = 500$ respectively

x / y	0.2	0.4	0.6	0.8
0.2	4.419×10^7	6.609×10^7	6.609×10^7	4.419×10^7
0.4	6.609×10^7	9.956×10^7	9.956×10^7	6.609×10^7
0.6	6.609×10^7	9.956×10^7	9.956×10^7	6.609×10^7
0.8	4.419×10^7	6.609×10^7	6.609×10^7	4.419×10^7
x / y	0.2	0.4	0.6	0.8
0.2	4.413×10^6	6.600×10^6	6.600×10^6	4.413×10^6
0.4	6.600×10^6	9.942×10^6	9.942×10^6	6.600×10^6
0.6	6.600×10^6	9.942×10^6	9.942×10^6	6.600×10^6
0.8	4.413×10^6	6.600×10^6	6.600×10^6	4.413×10^6