ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 3 (2007) No. 2, pp. 106-112

Wavelet analysis of brownian bridge

Xuewen Xia[1] , Kai Liu[2]*

[1] Hunan Institute of Engineering, Xiangtan, Hunan, 411104, China
[2] Department of Mathematical Sciences, University of Liverpool, Liverpool, L69 7ZL, UK

(*Received October 28 2006, Accepted January 8 2007*)

Abstract. In this paper, we shall use the methods of wavelet analysis to study the fundamental stochastic process, Brownian Motion. We shall obtain some properties under wavelet alternation. These results are important and will motivate further investigation of general stochastic system.

Keywords: wavelet analysis, wavelet alternation, stochastic system, Brownian Motion, Brownian bridge, drift, wavelet representation

1 Introduction

Wavelet transform is a basic tool for decomposing functions in various mathematical and engineering applications. It can be particularly viewed as a synthesis of ideas that have emerged since the 60-s in fields as diverse as mathematics, physics and electrical engineering. With the rapid development of computerized scientific instruments come a wide variety of interesting problems for data analysis and signal processing. In fields ranging from Extragalactic Astronomy to Molecular Spectroscopy, Medical Imaging and Computer Vision, one must recover a signal, curve, image, spectrum or density from incomplete, indirect and noisy data. Wavelets have contributed to this which has been the field intensely developed and rapidly advanced.

Wavelet analysis consists of a versatile collection of tools for the analysis and manipulation of signals such as sound and images as well as more general digital data sets such as speech, electrocardiograms and images. Wavelet analysis is a remarkable tool for the analyzing function of one or several variables that appear in mathematics or in signal and image processing. With hindsight the wavelet transform can be viewed as diverse as mathematics, physics and electrical engineering.

There are two important aspects to wavelets, which shall be called "mathematically" and "algorithmically". Numerical algorithms using the wavelet bases are similar to other transform methods in which vectors and operators are expanded into a basis and the computations take place in the new system of coordinates.

Recently, some researchers have studied wavelet problems for some stochastic systems. A number of papers have addressed the topic of wavelet decomposition of random processes[1, 2], Houdre and Hamid krim have derived other fundamental results on the wavelet transform of stochastic processes with stationary increments of an arbitrary order and nonstationary process[2, 5]. Xia Xuewen has obtained also some results about wavelet properties of stochastic system[6, 8], Zhang and Walter studied the wavelet expansion for wide-sense stationary processes[9], Flandrin studied the problem of fractional brown motion[3], Haobo studied wavelet estimation for jumps in regressian model[4].

In this paper, wavelet transform approach was adopted and derived some results of a sorts of stochastic system-"Brownian Motion". We obtain correlative degree of wavelet transform of Brownian Motion and Zero Density Degree and wavelet expansion.

* Corresponding author.
E-mail address: xxw1234567@163.com, K. Liu@liverpool.ac.uk.

2 Wavelet

Definition 1. *Let $\phi \in L' \bigcap L^2$, and $\hat{\phi}(0) = 0$, then*

$$\phi_{a,b}(t) = |a|^{-\frac{1}{2}\phi(\frac{t-b}{a}),b\in R,a\in R-\{0\}} \tag{1}$$

be called Analysis wavelet.

Definition 2. *Let $f \in L^2(R, dt)$, the wavelet transform of function f is*

$$Wf(s,x) = \frac{1}{s}\int_R f(t)\phi(\frac{x-t}{s})dt \tag{2}$$

Definition 3. *Let H be the Hilbert space that consists of processes with 2-order moments, and for $\forall x(t) \in H$,the wavelet transform of $x(t)$ is*

$$WX(s,x) = \frac{1}{s}\int_R X(t)\phi(\frac{x-t}{s})dt \tag{3}$$

Where $\phi(t)$ is Mother wavelet.

Then, we have

$$R(\tau) = E\big[WX(s,x)WX(s,x+\tau)\big] = \frac{1}{s^2}\int\int_{R^2} E\big[X(u)X(v)\big]\phi(\frac{x-u}{s})\phi(\frac{x+\tau-v}{s})dudv \tag{4}$$

where $E\big[WX(s,x)\big] = \frac{1}{s}\int_R E\big[X(t)\big]\phi(\frac{x-t}{s})dt$.

3 Problem of brownian bradge

Let $B(t), (t > 0)$ be a standard Brown motion, and $B(0) = 0, B_{00}(t) \triangleq B(t) - tB(1)$, we call $B_{00}(t), (0 \leqslant t \leqslant 1)$ be Brown Bridge.

We know $E\big[B_{00}(t)\big] = 0$, $E\big[B_{00}^2(t)\big] = E\big[B(t) - tB(1)\big]^2 = t(1-t)$ and

$$\begin{aligned} cov\big[B_{00}(s), B_{00}(t)\big] &= E\big[B_{00}(s)B_{00}(t)\big] - E\big[B_{00}(s)\big]E\big[B_{00}(t)\big] \\ &= E\big[B(s) - sB(1)\big]\big[B(t) - tB(1)\big] - 0 \\ &= E\big[B(s) - B(t)\big] - tE\big[B(s)B(1)\big] - sE\big[B(1)B(t)\big] + stE\big\lfloor B^2(1)\big\rfloor \\ &= s \wedge t - st \end{aligned}$$

Let $s \leq t$, then

$$E\big[B_{00}(s)B_{00}(t)\big] = cov\big[B_{00}(s), B_{00}(t)\big] = s(1-t) \tag{5}$$

Then, we have

$$\begin{aligned} R(\tau) &= \frac{1}{s^2}\int\int_{R^2} E\big[X(u)X(v)\big]\phi\Big(\frac{x-u}{s}\Big)\phi\Big(\frac{x+\tau-v}{s}\Big)dudv \\ &= \frac{1}{s^2}\int\int_{R^2} E\big[B_{00}(u)B_{00}(v)\big]\phi\Big(\frac{x-u}{s}\Big)\phi\Big(\frac{x+\tau-v}{s}\Big)dudv \\ &= \frac{1}{s^2}\int\int_{R^2} u(1-v)\phi(\frac{x-u}{s})\phi(\frac{x+\tau-v}{s})dudv \end{aligned} \tag{6}$$

Because Haar wavelet

$$\phi(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & otherwise \end{cases} \tag{7}$$

Then we have

$$\phi(fracx - us) = \begin{cases} 1, & x - fracs2 \leq u < x \\ -1, & x - s < x \leq x - \frac{s}{2} \\ 0, & otherwise \end{cases} \tag{8}$$

$$\phi\left(\frac{x+\tau-v}{s}\right) = \begin{cases} 1, & x+\tau-\frac{s}{2} < u \leq x+\tau \\ -1, & x+\tau-s < x \leq x-\frac{s}{2}+\tau \\ 0, & otherwise \end{cases} \tag{9}$$

Then

$$\begin{aligned}
R(\tau) &= \frac{1}{s^2}\Big[\int_{x+\tau-s}^{x+\tau-\frac{s}{2}} dv \int_{x-s}^{x-\frac{s}{2}} u(1-v)du + \int_{x+\tau-s}^{x+\tau-\frac{s}{2}} dv \int_{x-\frac{s}{2}}^{x} -u(1-v)du \\
&\quad + \int_{x+\tau-\frac{s}{2}}^{x+\tau} dv \int_{x-s}^{x-\frac{s}{2}} -u(1-v)du + \int_{x+\tau-\frac{s}{2}}^{x+\tau} dv \int_{x-\frac{s}{2}}^{x} u(1-v)du\Big] \\
&= \frac{1}{s^2}\Big[\int_{x+\tau-s}^{x+\tau-\frac{s}{2}} (1-v)dv \int_{x-s}^{x-\frac{s}{2}} udu + \int_{x+\tau-s}^{x+\tau-\frac{s}{2}} (v-1)dv \int_{x-\frac{s}{2}}^{x} udu \\
&\quad + \int_{x+\tau-\frac{s}{2}}^{x+\tau} (v-1)dv \int_{x-s}^{x-\frac{s}{2}} udu + \int_{x+\tau-\frac{s}{2}}^{x+\tau} (1-v)dv \int_{x-\frac{s}{2}}^{x} udu\Big] \\
&= \frac{1}{s^2}\Big\{\Big[\frac{s}{2} - \frac{s}{4}(2x+2\tau-\frac{3}{2}s)\Big]\Big[\frac{s}{4}(2x-\frac{3}{2}s)\Big] \\
&\quad + \Big[\frac{s}{4}(2x+2\tau-\frac{3}{2}s) - \frac{s}{2}(2x+2\tau-\frac{3}{2}s)\Big]\Big[\frac{s}{4}(2x-\frac{s}{2})\Big] \\
&\quad + \Big[\frac{s}{4}(2x+2\tau-\frac{3}{2}s) - \frac{s}{2}\Big]\cdot\Big[\frac{s}{4}(2x-\frac{3}{2}s)\Big] \\
&\quad + \Big[\frac{s}{2} - \frac{s}{4}(2x+2\tau-\frac{s}{2})\Big]\Big[\frac{1}{2}s(x-\frac{x}{4})\Big]\Big\} \\
&= \Big[\frac{1}{2} - \frac{1}{4}(2x+2\tau-\frac{3}{2}s)\Big]\Big[\frac{1}{4}(2x-\frac{3}{2}s)\Big] + \Big[\frac{1}{4}(2x+2\tau-\frac{3}{2}s) \\
&\quad -\frac{1}{2}(2x+2\tau-\frac{3}{2}s)\Big]\Big[\frac{1}{4}(2x-\frac{s}{2})\Big] \\
&\quad + \Big[\frac{1}{4}(2x+2\tau-\frac{3}{2}s) - \frac{1}{2}\Big]\cdot\Big[\frac{1}{4}(2x-\frac{3}{2}s)\Big] + \Big[\frac{1}{2} - \frac{1}{4}(2x+2\tau-\frac{s}{2})\Big]\Big[\frac{1}{2}s(x-\frac{x}{4})\Big] \\
&= (\frac{1}{2} - \frac{x}{2} - \frac{\tau}{2} + \frac{3s}{8})(\frac{x}{2} - \frac{3s}{8}) + (-\frac{x}{2} - \frac{\tau}{2} + \frac{3s}{8})(\frac{x}{2} - \frac{s}{8}) \\
&\quad +(\frac{x}{2} + \frac{\tau}{2} - \frac{s}{8} - \frac{1}{2})(\frac{x}{2} - \frac{3s}{8}) + (\frac{1}{2} - \frac{x}{2} - \frac{\tau}{2} + \frac{3s}{8})(\frac{x}{2} - \frac{s}{8}) \\
&= -\frac{x^2}{2} + \frac{1}{2}sx - \frac{x}{32}s^2 - \frac{1}{2}\tau x - \frac{s}{16} + \frac{1}{8}\tau s
\end{aligned} \tag{10}$$

Then $R'(\tau) = -\frac{1}{2}x + \frac{1}{8}s, R''(\tau) = 0$, then zero density of wavelet alternatiive of $B_{00}(t)$:

$$\sqrt{\left|\frac{R''(0)}{\pi^2 R(0)}\right|} = 0$$

Because

$$\begin{aligned} cov\big[B(s),B(t)|B(1)=0\big] &= E\big[B(s)B(t)|B(1)=0\big]-E\big[B(s)|B(1)=0\big]E\big[B(t)|B(1)=0\big] \\ &= E\big\{E\big[B(s)B(t)|B(t),B(1)=0\big]|B(1)=0\big\}-0 \\ &= E\big\{B(t)E\big[B(s)B(t)|B(t)),B(1)=0\big]|B(1)=0\big\} \\ &= E\big\{B(t)\big[0+\frac{B(t)(s-0)}{t-0}\big]|B(1)=0\big\} \\ &= E\big\{B^2(t)\frac{s}{t}|B(1)=0\big\} \\ &= \frac{s}{t}E\big[B^2(t)|B(1)=0\big] \\ &= \frac{s}{t}\frac{(1-t)(t-0)}{1-0} \\ &= s(1-t) \end{aligned} \tag{11}$$

Then, Wavelet alternative of Brownian Motion $B(t)$ have the same result as Brown Bridge $B_{00}(t)$.

4 Analysis of brownian motion with drift

Let $B(t)$ be Brownian Motion, we call

$$y(t)=\mu t+\sigma B(t) \tag{12}$$

be the Brownian Motionian with drift μ. We know $dB(t)=W(t)dt$, where, $W(t)$ be winner processes, and we have

$$E\big[W(t)W(s)\big]=Q(t)\delta(t-s) \tag{13}$$

where $Q(t)\geq 0$.

We have

$$\begin{aligned} E\big[y(t)y(s)\big] &= E\big[\mu t+\sigma B(t)\big]\big[\mu s+\sigma B(s)\big] \\ &= \mu^2 ts+\mu t\sigma E\big(B(s)\big)+\mu t\sigma SE\big(B(s)\big)+\sigma^2E\big[B(t)B(s)B(s)\big)\big] \\ &= \mu^2 ts\sigma^2E\big[B(t)B(s)\big)\big] \\ &= \mu^2 ts+\int_{t_0}^{t}Q(t)\delta(t-s)dtds \\ &= \mu^2 ts \end{aligned} \tag{14}$$

Then

$$\begin{aligned} R(\tau) &= \frac{1}{s^2}\Big[\int_{x-\frac{s}{2}}^{x}\mu^2udu\int_{x+\tau-\frac{s}{2}}^{x+\tau}u(1-v)vdv-\int_{x-\frac{s}{2}}^{x}\mu^2udu\int_{x+\tau-s}^{x+\tau-\frac{s}{2}}vdv \\ &\quad -\int_{x-s}^{x-\frac{s}{2}}\mu^2udu\int_{x+\tau-\frac{s}{2}}^{x+\tau}u(1-v)vdv+\int_{x-s}^{x-\frac{s}{2}}\mu^2udu\int_{x+\tau-s}^{x+\tau-\frac{s}{2}}vdv\Big] \\ &= \frac{\mu^2}{4s^2}\big[(sx-\frac{s^2}{4})(2x+2\tau-\frac{s}{2})\frac{s}{2}-(sx-\frac{s^2}{4})(2x+2\tau-\frac{s}{2})\frac{s}{2} \\ &\quad -(2x-\frac{3}{2}s)\frac{s}{2}(2x+2\tau-\frac{s}{2})\frac{s}{2}+(2x-\frac{3}{2}s)\frac{s}{2}(2x+2\tau-\frac{3s}{2})\frac{s}{2}\big] \\ &= \frac{\mu^2}{4}\big[\frac{1}{2}(x-\frac{s}{4})+\frac{1}{4}(2x-\frac{3s}{2})(-s)\big] \\ &= \frac{\mu^2 s}{16}\big[2(x-\frac{s}{4})+(2x-\frac{3s}{2})(-s)\big]=\frac{\mu^2 s}{16}(4x-2s)=\frac{\mu^2 s}{8}(2x-s) \end{aligned} \tag{15}$$

Hence, we know, $R(\tau)$ has nothing to do with τ . Because $R'(t) = R''(t) = 0$, then zero density of wavelet alternative of $y(t)$:

$$\sqrt{\left|\frac{R''(0)}{\pi^2 R(0)}\right|} = 0$$

5 Wavelet representation

We know to exist $X_m(t) \in H$, have

$$E\big[X(t) - X_m(t)\big]^2 \to 0, \quad m \to \infty, \quad t \in R$$

and

$$X_m(t) = \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} b_{kn}\phi_{kn}(t) \tag{16}$$

Where,$b_{kn} = \int_R x(t)\phi_{kn}(t)da$, we have

$$E\big[b_{mm}b_{kj}\big] = \int\int_{R^2} E\big[x(t)x(s)\big]\phi(2^m t - n)\phi(2^k s - j)2^{\frac{m}{2}}2^{\frac{k}{2}}dtds$$

We can compute $E\lfloor b_{mm}b_{kj}\rfloor$ using equation (5) and (7) and (11) and (14). For example, we use (7), have

$$\phi(2^m t - n) = \begin{cases} 1, & \frac{n}{2^m} \le x < \frac{1}{2^m}(n + \frac{1}{2}) \\ -1, & \frac{1}{2^m}(n + \frac{1}{2}) \le x < \frac{1}{2^m}(n + 1) \\ 0, & otherwise \end{cases}$$

$$\phi(2^k s - j) = \begin{cases} 1, & \frac{j}{2^k} \le s < \frac{1}{2^k}(j + \frac{1}{2}) \\ -1, & \frac{1}{2^k}(j + \frac{1}{2}) \le s < \frac{1}{2^k}(j + 1) \\ 0, & otherwise \end{cases}$$

Then, use equation (5), we have

$$
\begin{aligned}
E\big[b_{mm}b_{kj}\big] &= \int\int_{R^2} s(1-t)\phi(2^m t-n)\phi(2^k s-j)2^{\frac{m+k}{j}}dtds \\
&= 2^{\frac{m+k}{j}}\Big[\int_{2^{-m}n}^{2^{-m}(n+\frac{1}{2})}(1-t)dt\int_{2^{-k}j}^{2^{-k}(j+\frac{1}{2})}sds \\
&\quad -\int_{2^{-m}n}^{2^{-m}(n+\frac{1}{2})}(1-t)dt\int_{2^{-k}(j+\frac{1}{2})}^{2^{-k}(j+1)}sds \\
&\quad -\int_{2^{-m}(n+\frac{1}{2})}^{2^{-m}(n+1)}(1-t)dt\int_{2^{-k}j}^{2^{-k}(j+\frac{1}{2})}sds \\
&\quad +\int_{2^{-m}(n+\frac{1}{2})}^{2^{-m}(n+1)}(1-t)dt\int_{2^{-k}(j+\frac{1}{2})}^{2^{-k}(j+1)}sds\Big] \\
&= 2^{\frac{m+k}{j}}\Big\{\Big[2^{-m-1}-2^{-2m-1}(n+\frac{1}{4})\Big]2^{-2k-1}(j+\frac{1}{4}) \qquad (17)\\
&\quad -\Big[2^{-m-1}-2^{-2m-1}(n+\frac{1}{4})\Big]2^{-2k-1}(j+\frac{3}{4}) \\
&\quad -\Big[2^{-m-1}-2^{-2m-1}(n+\frac{3}{4})\Big]2^{-2k-1}(j+\frac{1}{4}) \\
&\quad -\Big[2^{-m-1}-2^{-2m-1}(n+\frac{3}{4})\Big]2^{-2k-1}(j+\frac{3}{4})\Big\} \\
&= 2^{\frac{m+k}{j}}\Big\{\Big[2^{-m-1}-2^{-2m-1}(n+\frac{1}{4})\Big](-\frac{1}{2}) \\
&\quad +\Big[2^{-m-1}-2^{-2m-1}(n+\frac{3}{4})\Big]2^{-2k-1}\frac{1}{2} \\
&= 2^{\frac{m+k}{j}}\Big[2^{-2k-2}(-2^{-2m-2})\frac{1}{2}\Big] = 2^{\frac{m+k}{j}}(-2^{-2k-2m-4}) = -\frac{1}{16}2^{-\frac{3}{2}(k+m)}
\end{aligned}
$$

Use equation (14), we have

$$
\begin{aligned}
E\big[b_{mm}b_{kj}\big] &= \int\int_{R^2}\mu^2 st\phi(2^m t-n)\phi(2^k s-j)2^{\frac{m+k}{j}}dtds \\
&= 2^{\frac{m+k}{j}}\Big[\int_{2^{-m}n}^{2^{-m}(n+\frac{1}{2})}tdt\int_{2^{-k}j}^{2^{-k}(j+\frac{1}{2})}sds-\int_{2^{-m}n}^{2^{-m}(n+\frac{1}{2})}tdt\int_{2^{-k}(j+\frac{1}{2})}^{2^{-k}(j+1)}sds \\
&\quad -\int_{2^{-m}(n+\frac{1}{2})}^{2^{-m}(n+1)}tdt\int_{2^{-k}j}^{2^{-k}(j+\frac{1}{2})}sds+\int_{2^{-m}(n+\frac{1}{2})}^{2^{-m}(n+1)}tdt\int_{2^{-k}(j+\frac{1}{2})}^{2^{-k}(j+1)}sds\Big] \qquad (18)\\
&= \mu^2 2^{\frac{m+k}{2}}\frac{1}{4}\Big[2^{-2m-2k}(n+\frac{1}{4})(j+\frac{1}{4})-2^{-2m-2k}(n+\frac{1}{4})(j+\frac{3}{4}) \\
&\quad -2^{-2m-2k}(n+\frac{3}{4})(j+\frac{1}{4})+2^{-2m-2k}(n+\frac{3}{4})(j+\frac{3}{4})\Big] \\
&= \frac{\mu^2}{4}2^{\frac{m+k}{2}}\Big[2^{-2m-2k}(n+\frac{1}{4})(-\frac{1}{2})-2^{-2m-2k}(n+\frac{3}{4})\frac{1}{2}\Big] \\
&= \frac{\mu^2}{4}2^{\frac{m+k}{2}}2^{-2m-2k-2}=\frac{\mu^2}{16}2^{-\frac{3}{2}(k+m)}
\end{aligned}
$$

References

[1] M. Basseville. Modeling and estimation of multiresolution stochastic processes. *IEEE Trans. Inform. Theory*, 1992, **38**: 529–532.

[2] S. Cambanis, E. Masry. Wavelet approximations of deteministic and random signals. *IEEE Trans. Inform. Theory*, 1994, **40**: 1003–1012.

[3] P. Flandrin. Wavelet analysis and aynthesis of fractional brownian motion. *IEEE Tran. On Information Theory*, 1992, **38**(2): 910–916.

[4] R. Haobo. Wavelet estimation for jumps in a heteroscedastic regressian model. *Acta Mathematica Scientia*, 2002, **22**(2): 269–276.

[5] H. Krim. Multiresolution analysis of a class of nonstationary process. *IEEE Trans. Inform. Theory*, 1995, **41**: 1010–1019.

[6] X. Xia. Some properties of wavelet with random system. *J. of Biomathematics*, 1998, **13**: 249–253.

[2] X. Xia. Wavelet properties of linear stochastic system. *Acta Mathematia scientia*, 1998, **18**: 144–149.

[8] X. Xia. Wavelet analysis of the stochastic system with coular stationary noise. *Engineering Science*, 2005, **1**: 43–46.

[9] J. Zhang, G. walter. A wavelet based k-l-like expansion for widesense stationary processes. *IEEE Trans sig, Proc*, 1994, **38**(2): 814–823.