

Multiplicity theorem for a Dirichlet boundary value problem in N-dimensional case

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Abstract. In this paper, the existence of at least three weak solutions for Dirichlet problem

$$\begin{cases} -\Delta_p u + \lambda f(x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$ and $f : \Omega \times R \rightarrow R$ is a negative Carathéodory function, is established. The results is based on a recent three critical points theorem.

Keywords: three solutions, critical point, multiplicity results, Dirichlet problem

1 Introduction

We consider the boundary value problem

$$\begin{cases} -\Delta_p u + \lambda f(x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$ and $f : \Omega \times R \rightarrow R$ is a negative Carathéodory function.

In this paper, under novel assumptions, we are interested in ensuring the existence of at least three weak solutions for the problem (1).

Let us recall that a weak solution of problem (1) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega} (|\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x)) dx + \lambda \int_{\Omega} f(x, u(x)) v(x) dx = 0, \quad \forall v \in W_0^{1,p}(\Omega).$$

Multiplicity results for the problem of the type (1) have been broadly investigated in recent years (see, for example, [1, 3]); for instance, in [1], using variational methods, the authors ensure the existence of a sequence of arbitrarily small positive solutions for problem

$$\begin{cases} \Delta_p u + \lambda f(x, u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

when the function f has a suitable oscillating behaviour at zero.

In particular, in [2], the author proves multiplicity results for the problem

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$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

which for each $\lambda \in [0, +\infty)$, admits at least three solutions in $W_0^{1,2}([0, 1])$ when f is a continuous function.

In the present paper, under novel assumptions, we are interested in ensuring the existence of at least three weak solutions for the problem (1). Our approach is based on a three critical points theorem proved in [4], recalled below for the reader's convenience (Theorem 1), and on technical lemma that allow us to apply it. Theorem 2 which is our main result, under novel assumptions ensures the existence of an open interval $\Lambda \subseteq [0, \infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (3) admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega)$ are less than q .

As a consequence of Theorem 2, we obtain Theorem 3 and, in turn, Theorem 4.

Theorem 3 ensures the existence of three weak solutions for the problem

$$\begin{cases} -\Delta_p u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

when $f : R \rightarrow R$ is a negative continuous function.

Theorem 4 deals with the case $N = 1, p = 2$ and it ensures that, for any negative continuous function $f : R \rightarrow R$, there exists an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -u''(x) + \lambda f(u(x)) = 0 & x \in (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{5}$$

admits at least three solutions whose norms in $W_0^{1,2}([0, 1])$ are less than q , as Example 2.5 shows.

The aim of the present paper is to extend the main result of [2] to the problem (1).

Finally, we here recall for the reader's convenience the three critical points theorem of [4], Proposition 1 of [4]:

Theorem 1. *Let X be a separable and reflexive real Banach space; $\Phi : X \rightarrow R$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* ; $\Psi : X \rightarrow R$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\lim_{\|u\| \rightarrow +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty]$, and that there exists a continuous concave function $h : [0, \infty] \rightarrow R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty]$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ has at least three solutions in X whose norms are less than q .

Proposition 1. *Let X be a non-empty set and Φ, J two real function on X . Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = J(x_0) = 0, \Phi(x_1) > r,$$

$$\sup_{x \in \Phi^{-1}((-\infty, r])} J(x) < r \frac{J(x_1)}{\Phi(x_1)}.$$

Then, for each ρ satisfying

$$\sup_{x \in \Phi^{-1}((-\infty, r])} J(x) < \rho < r \frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho - J(x))).$$

2 Main results

Here and in the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$\|u\| = \left(\int_{\Omega} |\nabla u(x)|^p dx \right)^{1/p}.$$

Assume that for each $\varrho > 0$ there exists $H_{1\varrho}, H_{2\varrho} \in L^1(\Omega)$ such that $H_{1\varrho}(x) \leq f(x, \xi) \leq H_{2\varrho}(x)$ for each $\xi \in [-\varrho, \varrho]$ and almost everywhere in Ω and put

$$g(x, t) = \int_0^t f(x, \xi) d\xi$$

for each $(x, t) \in \Omega \times \mathbb{R}$.

Now, fix $x^0 \in \Omega$ and pick r_1, r_2 with $0 < r_1 < r_2$ such that

$$S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega.$$

Put

$$k_1 = \frac{1}{r_2 - r_1} \left((r_2^N - r_1^N) \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \right)^{1/p} c m(\Omega)^{\frac{1}{N} - \frac{1}{p}} \quad (6)$$

where Γ denotes the Gamma function, $c = c(N, p)$ is a positive constant and $m(\Omega)$ is the Lebesgue measure of the set Ω .

Our main results fully depend on the following lemma:

Lemma 1. Assume that there exist two positive constants θ and τ with $k_1\tau > \theta$ such that

$$m(\Omega) k_1^p \frac{\inf_{(x,t) \in \Omega \times [-\theta, \theta]} g(x, t)}{\theta^p} > \frac{\int_{S(x^0, r_1)} g(x, \tau) dx}{\tau^p},$$

where k_1 is given in (6).

Then, there exist $r > 0$ and $w \in X$ such that $\|w\|^p > pr$ and

$$m(\Omega) \inf g(x, t) > pr \frac{\int_{\Omega} g(x, w(x)) dx}{\|w\|^p}$$

where $(x, t) \in \Omega \times [-cm(\Omega)^{\frac{1}{N} - \frac{1}{p}} \sqrt[p]{pr}, cm(\Omega)^{\frac{1}{N} - \frac{1}{p}} \sqrt[p]{pr}]$.

Proof. We put

$$w(x) = \begin{cases} 0, & x \in \Omega \setminus S(x^0, r_2) \\ \frac{\tau}{r_2 - r_1} [r_2 - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}], & x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ \tau, & x \in S(x^0, r_1) \end{cases}$$

and $r = \frac{\theta^p}{pc^p m(\Omega)^{\frac{p}{N} - 1}}$. It is easy to see that $w \in X$ and, in particular, one has

$$\|w\|^p = (r_2^N - r_1^N) \frac{\pi^{N/2}}{\Gamma(1 + N/2)} \left(\frac{\tau}{r_2 - r_1} \right)^p.$$

Hence, taking into account that $k_1\tau > \theta$, one has

$$pr < \|w\|^p.$$

Since f is a negative function it follows for each $x \in \Omega$,

$$\int_{\Omega \setminus S(x^0, r_2)} g(x, w(x)) dx + \int_{S(x^0, r_2) \setminus S(x^0, r_1)} g(x, w(x)) dx \leq 0.$$

Moreover, owing to our assumptions, we have

$$m(\Omega) \inf g(x, t) > \left(\frac{\theta}{k_1 \tau}\right)^p \int_{S(x^0, r_1)} g(x, \tau) dx \geq pr \frac{\int_{\Omega} g(x, w(x)) dx}{\|w\|^p}$$

where $(x, t) \in \Omega \times [-cm(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}, cm(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}]$.

So, the Proof is complete.

Now, we state our main result:

Theorem 2. Assume that there exist three positive constants θ, τ, γ with $k_1 \tau > \theta, \gamma < p$ and a function $a \in L^1(\Omega)$ such that

(i) $m(\Omega) k_1^p \frac{\inf_{(x,t) \in \Omega \times [-\theta, \theta]} g(x,t)}{\theta^p} > \frac{\int_{S(x^0, r_1)} g(x, \tau) dx}{\tau^p}$, (ii) $g(x, t) \leq a(x)(1 + |t|^\gamma)$ almost everywhere in Ω and for each $t \in R$, where k_1 is given in (6).

Then, there exists an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions in X whose norms are less than q .

Proof. For each $u \in X$, we put

$$\begin{aligned} \Phi(u) &= \frac{\|u\|^p}{p}, \\ \Psi(u) &= \int_{\Omega} g(x, u(x)) dx. \end{aligned}$$

Of course, Φ is a continuously *Gâteaux* differentiable and sequentially weakly lower semi continuous functional whose *Gâteaux* derivative admits a continuous inverse on X^* and Ψ is a continuously *Gâteaux* differentiable functional whose *Gâteaux* derivative is compact. In particular, for each $u, v \in X$ one has

$$\begin{aligned} \Phi'(u)(v) &= \int_{\Omega} (|\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x)) dx, \\ \Psi'(u)(v) &= \int_{\Omega} f(x, u(x)) v(x) dx. \end{aligned}$$

Hence, the weak solutions of (1) exactly the solutions of the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0.$$

Thanks to (ii), for each $\lambda > 0$ one has that

$$\lim_{\|u\| \rightarrow +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u \in \Phi^{-1}([-\infty, r])} (-\Psi(u)) < r \frac{(-\Psi(w))}{\Phi(w)}.$$

Now, taking into account that for every $u \in X$, one has

$$\sup_{x \in \Omega} |u(x)| \leq cm(\Omega)^{\frac{1}{N}-\frac{1}{p}} \|u\|$$

for each $u \in X$, it follows that

$$\inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) = \inf_{\|u\|^p \leq pr} \int_{\Omega} g(x, u(x)) dx \geq m(\Omega) \inf g(x, t)$$

where $(x, t) \in \Omega \times [-cm(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}, cm(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}]$. Now, thanks to Lemma 1, there exist $r > 0$ and $w \in X$ such that

$$m(\Omega) \inf g(x, t) > pr \frac{\int_{\Omega} g(x, w(x)) dx}{\|w\|^p}$$

where $(x, t) \in \Omega \times [-cm(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}, cm(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}]$. So,

$$\inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) > r \frac{\Psi(w)}{\Phi(w)}.$$

Namely

$$\sup_{u \in \Phi^{-1}((-\infty, r])} (-\Psi(u)) < r \frac{(-\Psi(w))}{\Phi(w)}.$$

Fix ρ such that

$$\sup_{u \in \Phi^{-1}((-\infty, r])} (-\Psi(u)) < \rho < r \frac{(-\Psi(w))}{\Phi(w)}$$

and define $h(\lambda) = \lambda\rho$ for every $\lambda \geq 0$, from Proposition 1, with $x_0 = 0, x_1 = w, J = -\Psi$ we obtain

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 1.

Now, we put

$$k_2 = \frac{1}{r_2 - r_1} \left(\frac{r_2^N - r_1^N}{r_1^N} \right)^{1/p} cm(\Omega)^{\frac{1}{N}-\frac{1}{p}}. \tag{7}$$

Then, with use the Theorem 2, we have the following result:

Theorem 3. Let $f : R \rightarrow R$ be a negative continuous function. Put $g(t) = \int_0^t f(\xi) d\xi$ for each $t \in R$ and assume that there exist four positive constants θ, τ, γ and σ with $k_1\tau > \theta$ and $\gamma < p$ such that

(j) $m(\Omega)k_2^p \frac{g(\theta)}{\theta^p} > \frac{g(\tau)}{\tau^p}$,

(jj) $g(t) \leq \sigma(1 + |t|^\gamma)$ for each $t \in R$, where k_1 is given in (6) and k_2 by (7).

Then, there exists an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (4) admits at least three solutions in X whose norms are less than q .

Proof. From (jj) and since $\int_{S(x^0, r_1)} g(\tau) dx = r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\tau)$, we have

$$\begin{aligned} \min_{t \in [-\theta, \theta]} g(t) = g(\theta) &> \frac{\theta^p}{\left(\frac{\tau}{r_2-r_1}\right)^p (r_2^N - r_1^N) c^p m(\Omega)^{\frac{p}{N}-1} \frac{\pi^{N/2}}{\Gamma(1+N/2)}} r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\tau) \\ &= \left(\frac{\theta}{k_1\tau}\right)^p \int_{S(x^0, r_1)} g(\tau) dx. \end{aligned}$$

Now, our conclusion follows from Theorem 2.

We now want to point out a simple consequence of Theorem 3 in the case where $N = 1$ and $p = 2$. For simplicity, we fix $\Omega = [0, 1]$ and consider a negative continuous function $f : R \rightarrow R$. Moreover, put $g(t) = \int_0^t f(\xi) d\xi$ for all $t \in R$.

Taking into account that, in this situation, $c = \frac{1}{2}, k_1 = \sqrt{\frac{1}{2(r_2-r_1)}}$ and $k_2 = \frac{1}{2} \sqrt{\frac{1}{r_1(r_2-r_1)}}$, we have the following result:

Theorem 4. Assume that there exist four positive constants θ, τ, γ and σ with $\sqrt{\frac{1}{2(r_2-r_1)}} \tau > \theta$ and $\gamma < 2$ such that

$$(k) \frac{1}{4r_1(r_2-r_1)} \frac{g(\theta)}{\theta^2} > \frac{g(\tau)}{\tau^2},$$

$$(kk) g(t) \leq \sigma(1 + |t|^\gamma) \text{ for each } t \in R.$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (5) admits at least three solutions in $W_0^{1,2}([0, 1])$ whose norms are less than q .

Example. Consider the problem

$$\begin{cases} -u'' + \lambda(-2e^u u^2) = 0, \\ u(0) = u(1) = 0, \end{cases} \quad (8)$$

and with $r_1 = \frac{1}{3}, r_2 = \frac{2}{3}$ and so that $k_1 = \sqrt{\frac{3}{2}}$ and $k_2 = \frac{3}{2}$, all the assumptions of Theorem 2.4, are satisfied by choosing, for instance $\theta = 1, \tau = 3, \gamma = 1$ and σ sufficiently large. So there exists an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (8) admits at least three solutions in $W_0^{1,2}([0, 1])$ whose norms are less than q .

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