ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 3 (2007) No. 2, pp. 83-89

On the minimax inequality and its application to existence of three solutions for elliptic equations with Dirichlet boundary condition

G. A. Afrouzi* , S. Heidarkhani

Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar

(*Received August 14 2006, Accepted December 23 2006*)

Abstract. In this paper, we establish an equivalent statement of minimax inequality for a special class of functionals. As an application, a result for the existence of three solutions to the Dirichlet problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u & in\ \Omega, \\ u = 0 & on\partial\ \Omega, \end{cases}$$

where $\Delta_p u$ =div$(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$, $f : \Omega \times R \to R$ is a positive continuous function and positive function $a(x) \in C(\overline{\Omega})$, is emphasized.

Keywords: minimax inequality, critical point, three solutions, multiplicity results, Dirichlet problem

1 Introduction

Throughout the sequel, $\Omega \subset R^N (N \geq 1)$ is nonempty bounded open set with smooth boundary $\partial\Omega$ and $p > N$.

Given two $G\hat{a}teaux$ differentiable functionals Φ and Ψ on a real Banach space X, the minimax inequality

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u))),\ \rho \in R, \tag{1}$$

plays a fundamental role for establishing the existence of at least three critical points for the functional $\Phi(u) - \lambda\Psi(u)$.

The main result of this paper (Theorem 2) establishes an equivalent statement of minimax inequality (1) for a special class of functionals, while its consequences (Theorem 3 and Theorem 5) guarantee some conditions so that minimax inequality holds.

Finally, we apply Theorem 1 to elliptic equations, by using an immediate consequence of Theorem 2, and we consider the boundary value problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{2}$$

where $\Delta_p u$ =div$(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\lambda > 0$, $f : \Omega \times R \to R$ is a positive continuous function and positive function $a(x) \in C(\overline{\Omega})$, and we establish some conditions on f so that problem (2) admits at least three weak solutions. We say that u is a weak solution to (2) if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \lambda\int_\Omega f(x,u(x))v(x)dx = -\int_\Omega a(x)|u(x)|^{p-2}u(x)v(x)dx$$

* *E-mail address*: afrouzi@umz.ac.ir.

for every $v \in W_0^{1,p}(\Omega)$.

Also by a similar arguments as in the problem (2), we will have the existence of at least three weak solutions for the problem

$$\begin{cases} \Delta_p u + \lambda h_1(x) h_2(u) = a(x)|u|^{p-2}u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{3}$$

where $h_1 \in C(\Omega)$ and $h_2 \in C(R)$ are two positive functions, and for the problem

$$\begin{cases} \Delta_p u + \lambda f(u) = a(x)|u|^{p-2}u & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{4}$$

where $f : R \to R$ is a positive continuous function.

In recent years, many authors have studied multiple solutions from several points of view and with different approaches and we refer to [1]-[6] and the references therein for more details, for instance, in their interesting paper [3], the authors studied problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{5}$$

(independent of λ, in the case), where $f : R \to R$ is a continuous function and λ is a real parameter, by using a multiple fixed-point theorem to obtain three symmetric positive solutions under growth conditions on f, and, in [4], the author proves multiplicity results for the problem (5) which for each $\lambda \in [0, +\infty]$, admits at least three solutions in $W_0^{1,2}([0,1])$ where f is a continuous function.

Also, in [6], the authors, established the existence of three positive solutions for classes of nondecreasing, p-sublinear function f belonging to $C^1([0,\infty))$ for a p- Laplacian version of [3], i.e., the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{6}$$

where $p > 1$, $\lambda > 0$ is a parameter and Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected.

In [2], using variational methods, the authors ensure the existence of a sequence of arbitrarily small positive solutions for problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{7}$$

when the function f has a suitable oscillating behaviour at zero.

In particular, in [1] we obtained the existence of an interval $\Lambda \subseteq [0, +\infty]$ and a positive real number q such that for each $\lambda \in \Lambda$ the problem (2) where $\Omega \subset R^N(N \geq 2)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$, $f : \Omega \times R \to R$ is a continuous function and positive weight function $a(x) \in C(\overline{\Omega})$, admits at least three weak solutions in whose norms in $W_0^{1,p}(\Omega)$ are less than q.

We now recall the three critical points theorem of B. Ricceri[7] by choosing $h(\lambda) = \lambda\rho$:

Theorem 1. *Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semicontinuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^*; $T : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact.*

Assume that $\lim_{||u|| \to +\infty}(\Phi(u) + \lambda T(u)) = +\infty$ for all $\lambda \in [0, +\infty]$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda T(u) + \lambda\rho) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda T(u) + \lambda\rho).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty]$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation $\Phi'(u) + \lambda T'(u) = 0$ has at least three solutions in X whose norms are less that than q.

2 Main results

In the sequel, let $f : \Omega \times R \to R$ be a positive continuous function and $g : \Omega \times R \to R$ be the function defined as follows $g(x,t) = \int_0^t f(x,\xi)d\xi$ for each $(x,t) \in \Omega \times R$. X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm $|| \, u \, || := \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}$,

Now, we define $||u||_* := \left(\int_\Omega (|\nabla u|^p + a(x)|u|^p)dx\right)^{1/p}$, such that there exist positive suitable constants c_1 and c_2 :

$$c_1||u|| \leq ||u||_* \leq c_2||u|| \tag{8}$$

(i.e., the above norms are equivalent).

We now introduce two positive special functionals on the Sobolev space X as follows $\Phi(u) := \frac{||u||_*^p}{p}$ for every $u \in X$, and $\Psi(u) := \int_\Omega g(x,u(x))dx$ for every $u \in X$.

Let $\rho, r \in R$, $w \in X$ be such that $0 < \rho < \Psi(w)$ and $0 < r < \Phi(w)$. We put

$$\beta_1 = \beta_1(\rho, w) := \rho \frac{\Phi(w)}{\Psi(w)} \tag{9}$$

$$\beta_2 = \beta_2(r, w) := r \frac{\Psi(w)}{\Phi(w)} \tag{10}$$

$$\beta_3 = \beta_3(\rho, w) := \frac{k}{c_1} m(\Omega)^{\frac{1}{N} - \frac{1}{p}} (p \, \beta_1(\rho, w))^{1/p} \tag{11}$$

and

$$\sigma = \sigma(c_1, u) = \frac{k}{c_1} m(\Omega)^{\frac{1}{N} - \frac{1}{p}} \, || \, u \, ||_* \tag{12}$$

for every $u \in X$, where $k = k(N,p)$ is a positive constant and $m(\Omega)$ is the Lebesgue measure of the set Ω.

Clearly, β_1, β_2, β_3 and σ are positive. Now, we put $\delta_1 := \inf\{\sigma \in R^+; \Psi(u) \geq \rho\}$, $\delta_2 := \inf\{\sigma \in R^+; \; m(\Omega) \max_{x \in \overline{\Omega}} g(x,\sigma) \geq \rho\}$ and

$$\delta_\rho := \delta_1 - \delta_2 \tag{13}$$

Clearly, $\delta_1 \geq \delta_2$.

Taking into account that for every $u \in X$, one has $\sup_{x\in\Omega} |u(x)| \leq km(\Omega)^{\frac{1}{N}-\frac{1}{p}}||u||$ for each $u \in X$, namely $\sup_{x\in\Omega} |u(x)| \leq \frac{k}{c_1} m(\Omega)^{\frac{1}{N}-\frac{1}{p}}||u||_*$ for each $u \in X$, so that $\Psi(u) = \int_\Omega g(x,u(x))dx \leq m(\Omega)\max_{x\in\overline{\Omega}} g(x,\sigma)$.

Namely $\Psi(u) \leq m(\Omega)\max_{x\in\overline{\Omega}} g(x,\sigma)$; therefore, $\{\sigma \in R^+; \Psi(u) \geq \rho\} \subseteq \{\sigma \in R^+; m(\Omega)\max_{x\in\overline{\Omega}} g(x,\sigma) \geq \rho \}$.

So, we have $\inf\{\sigma \in R^+; \Psi(u) \geq \rho\} \geq \inf\{\sigma \in R^+; m(\Omega)\max_{x\in\overline{\Omega}} g(x,\sigma) \geq \rho \}$.

Hence $\delta_\rho \geq 0$.

The main result of this paper is the following theorem:

Theorem 2. *Assume that there exist $\rho \in R$, $w \in X$ such that*

(i) $0 < \rho < \Psi(w)$,

(ii) $m(\Omega)\max_{x\in\overline{\Omega}} g(x, \beta_3 - \delta_\rho) < \rho$;

where β_3 is given by (11) and δ_ρ by (13).

Then,

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda(\rho - \Psi(u))).$$

Proof. From (ii), we obtain $\beta_3 - \delta_\rho \notin \{\theta \in R^+; \; m(\Omega)\max_{x\in\overline{\Omega}} g(x,\theta) \geq \rho\}$. Moreover $\inf\{\theta \in R^+; m(\Omega)\max_{x\in\overline{\Omega}} g(x,\theta) \geq \rho\} \geq \beta_3 - \delta_\rho$; in fact, arguing by contradiction, we assume that there is $\vartheta \in R^+$ such that $m(\Omega)\max_{x\in\overline{\Omega}} g(x,\vartheta) \geq \rho$ and $\vartheta < \beta_3 - \delta_\rho$, so $m(\Omega)\max_{x\in\overline{\Omega}} g(x, \beta_3 - \delta_\rho) \geq$

$m(\Omega)\max_{x\in\overline{\Omega}} g(x,\vartheta) \geq \rho$ and this is a contradiction. So $\inf\{\theta \in R^+; m(\Omega)\max_{x\in\overline{\Omega}} g(x,\theta) \geq \rho\} > \beta_3 - \delta_\rho$. Therefore, $\inf\{\sigma \in R^+; m(\Omega)\max_{x\in\overline{\Omega}} g(x,\sigma) \geq \rho\} > \beta_3 - \delta_\rho$; namely $\beta_3 < \delta_1$. So, we have $\inf\left\{\frac{||u||_*^p}{p} \in R^+; \Psi(u) \geq \rho\right\} > \beta_1$, namely $\inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u) > \rho\frac{\Phi(w)}{\Psi(w)}$, and, taking in to account that (i) holds, one has $\frac{\inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)}{\rho} > \frac{\Phi(w)-\inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)}{\Psi(w)-\rho}$. Now, let $\lambda \in R$, and taking into account the previous inequality, one has either $\lambda > \frac{\Phi(w)-\inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)}{\Psi(w)-\rho}$ or $\lambda < \frac{\inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)}{\rho}$. Namely $\inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u) > \Phi(w) + \lambda(\rho - \Psi(w))$ or $\lambda\rho < \inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)$. Therefore, thanks to the $0 < \rho < \Psi(w)$, we obtain $\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)$, and then, one has $\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)$. Therefore, thanks to the $\inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u))) = \inf_{u\in\Psi^{-1}([\rho,+\infty])}\Phi(u)$, we have the $\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u)))$.

Remark 1. $\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda(\rho-\Psi(u)))$ is well define, because $\lambda \to \inf_{u\in X}(\Phi(u)+\lambda(\rho-\Psi(u)))$ is upper semicontinuous in $[0,+\infty]$ and tends to $-\infty$ as $\lambda \to +\infty$.

Remark 2. If in Theorem 2, $\beta_3 - \delta_\rho \leq 0$; the Theorem holds again. Because, $\beta_3 \leq \delta_1 - \delta_2 \leq \delta_1$. Arguing as before, proof Theorem 2, result holds.

If instead of condition (ii) in Theorem 2, we put $m(\Omega)\max_{x\in\overline{\Omega}} g(x,\beta_3) < \rho$, then the result holds, because $m(\Omega)\max_{x\in\overline{\Omega}} g(x,\beta_3 - \delta_\rho) \leq m(\Omega)\max_{x\in\overline{\Omega}} g(x,\beta_3) < \rho$.

So, we have the following result:

Theorem 3. *Assume that there exist* $\rho \in R,\ w \in X$ *such that*

(i) $0 < \rho < \Psi(w)$,

(ii) $m(\Omega)\max_{x\in\overline{\Omega}} g(x,\beta_3) < \rho$.

where β_3 *is given by (11).*

Then,

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - \Psi(u))).$$

Now, we point out the following result:

Proposition 1. *The following assertions are equivalent:*

(***a***) *there are* $\rho \in R,\ w \in X$ *such that*

(i) $0 < \rho < \Psi(w)$,

(ii) $m(\Omega)\max_{x\in\overline{\Omega}} g(x,\beta_3) < \rho$; *where* β_3 *is given by (11).*

(***b***) *there are* $r \in R,\ w \in X$ *such that*

(j) $0 < r < \Phi(w)$,

(jj) $m(\Omega)\max_{x\in\overline{\Omega}} g(x, \frac{k}{c_1}m(\Omega)^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}) < \beta_2$; *where* β_2 *is given by (10).*

Proof. $(a) \Rightarrow (b)$. First we note that $0 < \Phi(w)$, because if $0 \geq \Phi(w)$, from (i) one has $\rho\frac{\Phi(w)}{\Psi(w)} \geq \Phi(w)$, namely $\beta_3 \geq \frac{k}{c_1}m(\Omega)^{\frac{1}{N}-\frac{1}{p}}||w||_*$. Hence, taking into account (ii), one has

$$\Psi(w) \leq m(\Omega)\max_{x\in\overline{\Omega}} g(x, \frac{k}{c_1}m(\Omega)^{\frac{1}{N}-\frac{1}{p}} \mid\mid w \mid\mid_*) \leq m(\Omega)\max_{x\in\overline{\Omega}} g(x,\beta_3) < \rho,$$

and that is in contradiction to (i). We now put $\beta_1 = r$. We obtain $\rho = \beta_2$ and $\beta_3 = \frac{k}{c_1}m(\Omega)^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}$. Therefore, from (i) and (ii), one has $0 < r < \Phi(w)$ and

$$m(\Omega)\max_{x\in\overline{\Omega}} g(x, \frac{k}{c_1}m(\Omega)^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}) < \beta_2.$$

$(b) \Rightarrow (a)$. First we note that $0 < \Psi(w)$, because if $0 \geq \Psi(w)$, from (j) one has $r\frac{\Psi(w)}{\Phi(w)} \leq 0$; namely, $\beta_2 \leq 0$. Hence, from (jj) one has

$$0 = \Psi(0) \leq m(\Omega) \max_{x\in\overline{\Omega}} g(x, \frac{k}{c_1} m(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}) < 0,$$

and this is a contradiction. We now put $\beta_2 = \rho$. We obtain $r = \beta_1$ and $\frac{k}{c_1} m(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr} = \beta_3$. Therefore, from (j) and (jj), we have the conclusion.

The following Theorem is another consequence of Theorem 2.

Theorem 4. *Assume that there exist* $r \in R,\ w \in X$ *such that*

(j) $0 < r < \Phi(w)$,

(jj) $m(\Omega) \max_{x\in\overline{\Omega}} g(x, \frac{k}{c_1} m(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}) < \beta_2$ *where* β_2 *is given by (10).*

Then, there exists $\rho \in R$ *such that*

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda(\rho - \Psi(u))) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda(\rho - \Psi(u))).$$

Proof. It follows from Theorem 3 and Proposition 1.

Finally, we interested in ensuring the existence of at least three weak solutions for the Dirichlet problem (2). Now, we have the following result:

Theorem 5. *Assume that there exist* $\rho \in R$, $b_1 \in L^1(\Omega)$, $w \in X$ *and a positive constant* γ *with* $\gamma < p$ *such that*

(i) $0 < \rho < \int_\Omega g(x, w(x))dx$,

(ii) $m(\Omega) \max_{x\in\overline{\Omega}} g(x, \beta_3) < \rho$.

(iii) $g(x,t) \leq b_1(x)(1 + |t|^\gamma)$ *almost everywhere in* Ω *and for each* $t \in R$.

where β_3 *is given by (11).*

Then, there exists an open interval $\Lambda \subseteq [0, +\infty]$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *problem (2) admits at least three solutions in* X *whose norms are less than* q.

Proof. For each u ∈X, we put $\Phi(u) = \frac{||u||_*^p}{p}$, $T(u) = -\int_\Omega g(x, u(x))dx$ and $J(u) = \Phi(u) + \lambda T(u)$. In particular, for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_\Omega (|\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) + a(x)|u(x)|^{p-2} u(x)v(x))dx$$

and

$$T'(u)(v) = -\int_\Omega f(x, u(x))v(x)dx.$$

It is well known that the critical points of J are the weak solutions of (2), and our goal is to prove that Φ and T satisfy the assumptions of Theorem 1. Clearly, Φ is a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semi continuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^* and T is a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact.
Thanks to (iii), for each $\lambda > 0$ one has that $\lim_{||u||\to+\infty}(\Phi(u) + \lambda T(u)) = +\infty$ for all $\lambda \in [0, +\infty]$.

Furthermore, thanks to Theorem 3, from (i) and (ii) we have

$$\sup_{\lambda\geq 0} \inf_{u\in X} (\Phi(u) + \lambda T(u) + \lambda\rho) < \inf_{u\in X} \sup_{\lambda\geq 0} (\Phi(u) + \lambda T(u) + \lambda\rho).$$

Therefore, we can apply Theorem 1. It follows that there exists an open interval $\Lambda \subseteq [0, +\infty]$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (2) admits at least three solutions in X whose norms are less than q.

We also have the following existence result:

Theorem 6. *Assume that there exist $r \in R$, $b_2 \in L^1(\Omega)$, $w \in X$ and a positive constant γ with $\gamma < p$ such that*

(j) $0 < r < \frac{\|w\|_*^p}{p}$,

(jj) $m(\Omega) \max_{x\in\overline{\Omega}} g(x, \frac{k}{c_1} m(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}) < \beta_2$;

(jjj) $g(x,t) \leq b_2(x)(1+|t|^{\gamma})$ *almost everywhere in Ω and for each $t \in R$. where β_2 is given by (10). Then, there exists an open interval $\Lambda \subseteq [0,+\infty]$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (2) admits at least three solutions in X whose norms are less than q.*

Proof. It follows from Theorem 4 and Theorem 5.

Let $h_1 \in C(\Omega)$ and $h_2 \in C(R)$ be two positive functions. Put $f(x,t) = h_1(x)h_2(t)$ for each $(x,t) \in \Omega \times R$, $H(t) = \int_0^t h_2(\xi)d\xi$ for all $t \in R$, and $b_3(x) = \frac{b_1(x)}{h_1(x)}$ for almost every $x \in \Omega$. Then, with use the Theorem 5, we have the following result:

Theorem 7. *Assume that there exist $\rho \in R$, $b_3 \in L^1(\Omega)$, $w \in X$ and a positive constant γ with $\gamma < p$ such that*

(i') $0 < \rho < \int_\Omega (h_1(x)H(w(x)))dx$,

(ii') $m(\Omega) \max_{x\in\overline{\Omega}} h_1(x) < \frac{\rho}{H(\beta_3)}$.

(iii') $H(t) \leq b_3(x)(1+|t|^{\gamma})$ *almost everywhere in Ω and for each $t \in R$. where β_3 is given by (11).*

Then, there exists an open interval $\Lambda \subseteq [0,+\infty]$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (3) admits at least three solutions in X whose norms are less than q.

Put $b_4(x) = \frac{b_2(x)}{h_1(x)}$ for almost every $x \in \Omega$. Then, with use the Theorem 6, we have the following existence result:

Theorem 8. *Assume that there exist $r \in R$, $b_4 \in L^1(\Omega)$, $w \in X$ and a positive constant γ with $\gamma < p$ such that*

(j') $0 < r < \frac{\|w\|_*^p}{p}$,

(jj') $m(\Omega) \max_{x\in\overline{\Omega}} h_1(x) < \frac{\beta_2}{H(\frac{k}{c_1} m(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr})}$;

(jjj') $H(t) \leq b_4(x)(1+|t|^{\gamma})$ *almost everywhere in Ω and for each $t \in R$. where β_2 is given by (10). Then, there exists an open interval $\Lambda \subseteq [0,+\infty]$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (3) admits at least three solutions in X whose norms are less than q.*

We now want to point out two simple consequences of Theorem 5 and Theorem 6, respectively. Let $f : R \to R$ be a positive continuous function. Put $g(t) = \int_0^t f(\xi)d\xi$ for each $t \in R$.

So we have the following results:

Theorem 9. *Assume that there exist $\rho \in R$, $w \in X$ and two positive constants γ and η_1 with $\gamma < p$ such that*

(i'') $0 < \rho < \int_\Omega g(w(x))dx$,

(ii'') $m(\Omega)g(\beta_3) < \rho$.

(iii'') $g(t) \leq \eta_1(1+|t|^{\gamma})$ *for each $t \in R$. where β_3 is given by (11).*

Then, there exists an open interval $\Lambda \subseteq [0,+\infty]$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (4) admits at least three solutions in X whose norms are less than q.

Theorem 10. *Assume that there exist $r \in R$, $w \in X$ and two positive constants γ and η_2 with $\gamma < p$ such that*

(j'') $0 < r < \frac{\|w\|_*^p}{p}$,

(jj'') $m(\Omega)g(\frac{k}{c_1} m(\Omega)^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}) < \beta_2$;

(jjj'') $g(t) \leq \eta_2(1+|t|^{\gamma})$ *for each $t \in R$. where β_2 is given by (10).*

Then, there exists an open interval $\Lambda \subseteq [0,+\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (4) admits at least three solutions in X whose norms are less than q.

Now, in the case $N = 1$ and $p = 2$, we present two simple consequence of Theorem 9 and Theorem 10, respectively..

For simplicity, we fix $\Omega = [0, 1]$ and consider a positive continuous function $f : R \to R$. Moreover, put $g(t) = \int_0^t f(\xi)d\xi$ for all $t \in R$.

Taking into account that, in this situation, $k = \frac{1}{2}$ and $\beta_3 = \frac{1}{c_1}\sqrt{\frac{\beta_1}{2}}$, we have the following results:

Theorem 11. *Assume that there exist* $\rho \in R$*,* $w \in X$ *and two positive constants* γ *and* η_3 *with* $\gamma < 2$ *such that*

(i''') $0 < \rho < \int_0^1 g(w(x))dx$,

(ii''') $g(\frac{1}{c_1}\sqrt{\frac{\beta_1}{2}}) < \rho$.

(iii''') $g(t) \le \eta_3(1 + |t|^\gamma)$ *for each* $t \in R$*. where* β_1 *is given by (9).*

Then, there exists an open interval $\Lambda \subseteq [0, +\infty)$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$*, problem*

$$\begin{cases} u''(x) + \lambda f(u(x)) = a(x)u(x), \ x \in [0, 1] \\ u(0) = u(1) = 0, \end{cases} \tag{14}$$

admits at least three solutions in $W_0^{1,2}([0, 1])$ *whose norms are less than* q*.*

Theorem 12. *Assume that there exist* $r \in R$*,* $w \in X$ *and two positive constants* γ *and* η_4 *with* $\gamma < 2$ *such that*

(j''') $0 < r < \frac{||w||_*^2}{2}$,

(jj''') $g(\frac{1}{c_1}\sqrt{\frac{r}{2}}) < \beta_2$;

(jjj''') $g(t) \le \eta_4(1 + |t|^\gamma)$ *for each* $t \in R$*. where* β_2 *is given by (10).*

Then, there exists an open interval $\Lambda \subseteq [0, +\infty)$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$*, problem (14) admits at least three solutions in* $W_0^{1,2}([0, 1])$ *whose norms are less than* q*.*

Example 1. Consider the problem

$$\begin{cases} u'' + \lambda(e^u u^2(3 + u)) = e^x u, \\ u(0) = u(1) = 0. \end{cases} \tag{15}$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty)$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (15) admits at least three solutions in $W_0^{1,2}([0, 1])$ whose norms are less than q. In fact, by choosing $\rho = \frac{1}{4}$ and $w(x) = x$ so that $\beta_1(\rho, w) = \frac{e-1}{48-16e}$, all assumptions of Theorem 11, are satisfied with $\gamma = 1$, c_1 is positive constant such that the inequality (8) hold for $m(x) = e^x$ and η_3 sufficiently large, also with choose $r = \frac{1}{2}$ and $w(x) = x$ so that $\beta_2(r, w) = \frac{6-2e}{e-1}$, all assumptions of Theorem 12, are satisfied with $\gamma = 1$, c_1 is positive constant such that the inequality (8) hold for $m(x) = e^x$ and η_4 sufficiently large.

References

[1] G. A. Afrouzi, S. Heidarkhani. Three solutions for a dirichlet boundary value problem involving the p-laplacian. *Nonlinear Analysis*. To appear.

[2] G. Anello, G. Cordaro. Positive infinitely many arbitrarily small solutions for the Dirichlet problem involving the p-Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A*, 2002, **132**(511-519).

[3] R. I. Avery, J. Henderson. Three symmetric positive solutions for a second-order boundary value problem. *Appl. Math. lett.*, 2000, **13**(1-7).

[4] G. Bonanno. Existence of three solutions for a two point boundary value problem. *Appl. Math. Lett.*, 2000, **13**(53-57).

[5] P. K, T. Ouyang. Exact multiplicity results for two classes of boundary value problem. *Diff. Integral Equations*, 1993, **6**(1507-1517).

[6] M. , R. Shivaji. Multiple positive solutions for classes of p-Laplacian equations. *Differential and Integral Equations*, 2004, **17**(11-12): 1255–1261.

[7] B. Ricceri. On a three critical points theorem. *Arch. Math. (Basel)*, 2000, **75**: 220–226.