ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 3 (2007) No. 1, pp. 51-57

Nonclassical symmetries of a class of nonlinear partial differential equations and compatibility

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(*Received May 23 2006, Accepted August 3 2006*)

Abstract. Symmetries play an important role in solving partial differential equations. In this paper, the determining equations for a class of nonlinear partial differential equations with arbitrary order are considered. It is shown that the determining equations for the nonclassical reduction can be obtained by requiring the compatibility between the original equation and the invariant surface condition. A simple partial differential equation and BBM equation serve as examples to illustrating the feasibility of this method.

Keywords: nonclassical symmetry, compatibility, BBM equation, determine equation, governing equation

1 Introduction

Sophus Lie established the concept of Lie group in the study of the invariance of differential equations. It is important to study the property and exact solutions of the nonlinear partial differential equations by the invariance of differential equations. The nonclassical method of reduction was devised originally by Bluman and Cole, in 1969, to find new exact solutions of the heat equation in [4] in terms of vector fields and their prolongations. The nonclassical method could be used for an arbitrary system of differential equations, but for the purposes of this paper, we restrict ourselves to one nth-order PDE of (1 + 1)-dimension as follows:

$$\Delta(x, t, u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0 \tag{1}$$

The infinitesimal generators of (1) as follows:

$$\begin{aligned} \bar{x} &= x + X(t, x, u)\varepsilon + O(\varepsilon^2) \\ \bar{t} &= t + T(t, x, u)\varepsilon + O(\varepsilon^2) \\ \bar{u} &= u + U(t, x, u)\varepsilon + O(\varepsilon^2) \end{aligned} \tag{2}$$

Suppose the (1) is invariant under the action of infinitesimal transformations if and only if it satisfies:

$$\Gamma^{(n)}(\Delta)\Big|_{\Delta=0} = 0 \tag{3}$$

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u} \tag{4}$$

where $\Gamma^{(n)}$ is the nth extension of the Γ.

Solving (3) leads to the infinitesimals X, T and U for the classical Lie point symmetry. This is so called classical Lie method.

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The invariant surface condition about (1) as follows:

$$Tu_t + Xu_x = U \tag{5}$$

If we denote (5) by $\varDelta_0$, then

$$\varDelta_0 = Xu_x + Tu_t - U \tag{6}$$

The main idea of the nonclassical method is to seek the invariance of the original equation augmented with the invariant surface condition.

The nonclassical symmetries are determined by the following governing equations:

$$\varGamma^{(n)}\varDelta\Big|_{\varDelta=0,\varDelta_0=0} = 0 \tag{7}$$

Solving this governing equation leads to a set of the determining equations for the infinitesimals X, T and U. When the determining equations are solved, that gives rise to the nonclassical symmetries of (1).

Substitute solutions of (5) into (1) to reduce the original equation. When the reduced equation are solved, We obtain the invariant solutions under group (2) of the original (1). Nonclassical method can get more solutions than the classical method, so the method of nonclassical reduction has been used to find new exact solutions. Recently, in [2, 6] Broadbridge and Arrigo have shown that all solutions of standard symmetric linear partial differential equations have classical Lie symmetry. Usually, the determining equations for nonclassical method unlike the determining equations for classical method. So the properties and relationships of the nonclassical reduction, the determining equations of it and the invariant surface condition are worthwhile to study. Arrigo and Beckham in [2] show that the determining equations for the nonclassical method can be derived as a consequence of the compatibility for the evolutionary partial differential equations. In this paper, we show that for a class of the nonlinear PDE with arbitrary order instead of the nonlinear evolution equations, the determining equations for the nonclassical symmetries can also be derived by the compatibility between the original equation and the invariant surface condition.

This paper is organized as follows. For motivation, we consider a simple equation. In Section 2. We will show that the determining equations for the nonclassical symmetries of a simple equation are quickly and easily recovered. In Section 3, we will prove that for a class of nonlinear PDE with arbitrary order, compatibility with the invariant surface condition can sententiously leads to the governing equation for their nonclassical symmetries. The determining equations can be obtained by solving the governing equation. In Section 4, we will consider the BBM equation illustrating the feasibility of this method.

2 Nonclassical of a simple partial differential equation

In this section, we derive the governing equations for the nonclassical symmetries of a simple equation via compatibility are the same to the governing equation using the vector fields and their prolongations.

First, we use vector fields and its prolongations to obtain the determining equations of the nonclassical symmetries.

A simple PDE as follows:

$$u_t = u_{xt} + u_{xx} \tag{8}$$

If we denote (8) by $\varDelta_1$, and the invariant surface condition (5) with $T = 1$ (without loss of generality, see [4]) by $\varDelta_2$ then

$$\varDelta_1 = u_t - u_{xt} - u_{xx} \tag{9}$$

$$\varDelta_2 = u_t + Xu_x - U \tag{10}$$

The determining equations for the nonclassical symmetries for (8) are obtained by requiring that

$$\varGamma^{(2)}\varDelta_1\Big|_{\varDelta_1=0,\varDelta_2=0}=0 \tag{11}$$

where the infinitesimal generator $\varGamma$ is given by $\varGamma = T\frac{\partial}{\partial T}+X\frac{\partial}{\partial x}+U\frac{\partial}{\partial u}$ with the first and second extensions as

$$\varGamma^{(1)}=\varGamma+U_{[t]}\frac{\partial}{\partial u_t}+U_{[x]}\frac{\partial}{\partial u_x} \tag{12}$$

$$\varGamma^{(2)}=\varGamma^{(1)}+U_{[tt]}\frac{\partial}{\partial u_{tt}}+U_{[tx]}\frac{\partial}{\partial u_{tx}}+U_{[xx]}\frac{\partial}{\partial u_{xx}} \tag{13}$$

The coefficients of the operators in (12) and (13) are given by

$$U_{[t]}=D_t(U-Xu_x-Tu_t)+Xu_{tx}+Tu_{tt}=D_t(U-Xu_x)+Xu_{tx} \tag{14}$$
$$U_{[x]}=D_x(U-Xu_x-Tu_t)+Xu_{xx}+Tu_{tx}=D_x(U-Xu_x)+Xu_{xx} \tag{15}$$
$$U_{[tt]}=D_{tt}(U-Xu_x-Tu_t)+Xu_{ttx}+Tu_{ttt}=D_{tt}(U-Xu_x)+Xu_{ttx} \tag{16}$$
$$U_{[tx]}=D_{tx}(U-Xu_x-Tu_t)+Xu_{txx}+Tu_{txt}=D_{tx}(U-Xu_x)+Xu_{txx} \tag{17}$$
$$U_{[xx]}=D_{xx}(U-Xu_x-Tu_t)+Xu_{xxx}+Tu_{xxt}=D_{xx}(U-Xu_x)+Xu_{xxx} \tag{18}$$

Invariance of the (8) is given by (11), which by (9) and (13), gives

$$\varGamma^{(2)}\varDelta_1\Big|_{\varDelta_1=0,\varDelta_2=0}=U_{[t]}-U_{[xt]}-U_{[xx]} \tag{19}$$

Substituting (14), (17), (18) into (19) gives the governing equation for the infinitesimals X, T, U. Solving this governing equation leads to a set of the determining equations for X, T, U of nonclassical symmetries.

Next we will make use of the compatibility between the (9) and the invariant surface condition 10 derives (19). Total differentiation D_t of the nonlinear wave equation (8) gives
For $u_t=u_{xt}+u_{xx}$ we have

$$D_t(u_t)=D_t(u_{xt}+u_{xx}) \tag{20}$$

So $D_t(u_t)=u_{xtt}+u_{xxt}=D_{xt}(u_t)+D_{xx}(u_t)$ By $u_t=U-Xu_x$ we get

$$D_t(U-Xu_x)=D_{xt}(U-Xu_x)+D_{xx}(U-Xu_x) \tag{21}$$

Adding $Xu_{tx}+Xu_{xxx}$ to both sides of (21) and regrouping give $D_t(U-Xu_x)+Xu_{tx}=D_{xx}(U-Xu_x)+Xu_{xxx}+D_{xt}(U-Xu_x)+Xu_{txx}-Xu_{txx}+Xu_{tx}-Xu_{xxx}$.

For $u_t=u_{xt}+u_{xx}\Rightarrow D_x(u_t)=D_x(u_{xt}+u_{xx})\Rightarrow u_{tx}=u_{txx}+u_{xxx}$ so it gives the governing equation (19) $U_{[t]}=U_{[tx]}+U_{[xx]}$.

Following (20), (21) and using $\varDelta_2=0$, $D_x(u_t)=D_x(u_{xt}+u_{xx})$ we can obtain the governing equation, then the determining equations for the nonclassical symmetries of the (8) are:

$$\begin{aligned}
&U_{tx}+U_{xx}-U_t+U(U_{ux}-X_x)=0\\
&U_{tu}-X_{tx}+2U_{xu}-X_{xx}+X_t-X(U_{ux}-X_x)+U(U_{uu}-X_{ux}-X_u)=0\\
&U_{uu}-X_{tu}-2X_{xu}-X(U_{uu}-X_{ux}-X_u)-X_{uu}U=0\\
&X_t+X_x+UX_u=0\\
&XX_{uu}-X_{uu}=0\\
&XX_u-X_u=0
\end{aligned}$$

The determining equation above is obtained by the compatibility, it is the same to the determining equations obtained by using the vector fields and their prolongations. In the next section, we will prove the determining equations for the nonclassical symmetries of a class of more general PDE also can be derived by this method.

3 Compatibility of a class of nonlinear partial differential equations

Consider the partial differential equation

$$
\begin{aligned}
u_t = F\left(t, x, u_{tx}, u_{txx}, \cdots, u_{tx(m)}, u, u_x, u_{xx}, \cdots, u_{x(n-1)}\right) u_{x(n)} \\
+G\left(t, x, u_{tx}, u_{txx}, \cdots, u_{tx(m)}, u, u_x, u_{xx}, \cdots, u_{x(n-1)}\right)
\end{aligned} \tag{22}
$$

Where $u_{x(n)} = \partial_x^n u, u_{tx(m)} = \partial_x^m \partial_t u$ and where F and G are smooth functions of their arguments. If we denote (23) by $\varDelta_1$ and the invariant surface condition (5) with $T=1$ by $\varDelta_2$ then

$$
\varDelta_1 = u_t - F u_{x(n)} - G \tag{23}
$$

$$
\varDelta_2 = u_t + X u_x - U \tag{24}
$$

The determining equations for the nonclassical symmetries of (23) are obtained by requiring that

$$
\varGamma^{(k)} \varDelta_1 \Big|_{\varDelta_1=0, \varDelta_2=0} = 0 \tag{25}
$$

where $k = \max(m+1, n)$, the infinitesimal generator $\varGamma$ is given in (4), and its Kth extension is given recursively as

$$
\varGamma^{(k)} = \varGamma^{(k-1)} + \sum_{i=0}^{k} U_{[t(k-i)x(i)]} \frac{\partial}{\partial u_{[t(k-i)x(i)]}} \tag{26}
$$

where $u_{[t(k-i)x(i)]} = \partial_t^{k-i} \partial_x^i u$,the coefficients of the operators in (26) are given by

$$
U_{[t]} = F U_{[x(n)]} + \varGamma^{(k)} F u_{x(n)} + \varGamma^{(k)} G \tag{27}
$$

Solving this governing equation (27) leads to a set of the determining equations for the nonclassical symmetries of the nonlinear PDE (23).Before we establish the main result of the paper, it is important to prove an important relationship between the extended infinitesimal generator $\varGamma^{(k)}$ and the total derivative operators D_x and D_t.

Lemma 1. *If $\varGamma^{(k)}$ is the extended infinitesimal generator, and D_x and D_t are total derivative operators, then for any smooth function $F(t, x, u_{tx}, u_{txx} \cdots, u_{tx(m)}, u, u_x, \cdots, u_{x(n)}$,*

$$
\varGamma^{(k)} F = D_t F + X D_x(F) \tag{28}
$$

Provided $u_t = U - X u_x$.

Proof. From the definition of $\varGamma^{(k)}$ it is clear that

$$
\varGamma^{(k)} F = F_t + X F_x + U F_u + \sum_{j=1}^{m} U_{[tx(j)]} F_{u_{tx(j)}} + \sum_{i=1}^{n} U_{x(i)} F_{u_{x(i)}} \tag{29}
$$

$$
\begin{aligned}
D_t(F) + X D_x F = F_t + F_u u_t + \sum_{j=1}^{m} u_{ttx(j)} \cdot F_{u_{tx(j)}} + \sum_{i=1}^{n} u_{tx(i)} F_{u_{x(i)}} \\
+ X(F_x + F_u u_x + \sum_{j=1}^{m} F_{u_{tx(j)}} u_{tx(j+1)} + \sum_{i=1}^{n} F_{u_{x(i)}} u_{x(i+1)})
\end{aligned} \tag{30}
$$

Because $U_{[t(n-i)x(i)]} = D_t^{n-i} D_x^i (U - X u_x) + X u_{t(n-i)x(i+1)}$, we get

$$
U_{[tx(j)]} = u_{ttx(j)} + X u_{tx(j+1)}, \quad U_{[x(i)]} = u_{tx(i)} + X u_{x(i+1)} \tag{31}
$$

$$
U = u_t + X u_x \tag{32}
$$

We use (31), (32) denote $U, U_{[x(i)]}, U_{[tx(j)]}$ in (30), we get

$$\begin{aligned}\Gamma^{(k)}F = F_t + XF_x + u_tF_u + XF_uu_x + \sum_{j=1}^{m} F_{u_{tx(j)}}u_{ttx(j)} + \sum_{j=1}^{m} XF_{u_{tx(j)}}u_{tx(j+1)} \\ + \sum_{i=1}^{n} F_{u_{x(i)}}u_{tx(i)} + \sum_{i=1}^{n} XF_{u_{x(i)}}u_{x(j+1)} \qquad (33)\end{aligned}$$

It is easy to know that (33)=(30). So (29)=(30).

As a result $\Gamma^{(k)}F = D_tF + XD_x(F)$. Then the lemma holds true.

With the result of this lemma we are now ready to give and prove the main result of the paper.

Theorem 1. *The determining equations for the nonclassical symmetries of the partial differentia equation*

$$\begin{aligned}u_t = F(t, x, u_{tx}, u_{txx}, \cdots, u_{tx(m)}, u, u_x, u_{xx}, \cdots, u_{x(n-1)})u_{x(n)} \\ + G(t, x, u_{tx}, u_{txx}, \cdots, u_{tx(m)}, u, u_x, u_{xx}, \cdots, u_{x(n-1)}) \qquad (34)\end{aligned}$$

can be obtained through compatibility with the invariant surface condition.

$$u_t = U - Xu_x \qquad (35)$$

where $X = X(x, t, u)$ and $U = U(X, T, U)$ are smooth functions.

Proof. Suppose that the two equations are compatible. Subtracting (34) and (35) gives

$$\begin{aligned}U - Xu_x = F(t, x, u_{tx}, u_{txx}, \cdots, u_{tx(m)}, u, u_x, u_{xx}, \cdots, u_{x(n-1)})u_{x(n)} \\ + G(t, x, u_{tx}, u_{txx}, \cdots, u_{tx(m)}, u, u_x, u_{xx}, \cdots, u_{x(n-1)} \qquad (36)\end{aligned}$$

Total differentiation D_t of (36) gives

$$D_t(U - Xu_x) = D_t(F)u_{x(n)} + Fu_{tx(n)} + D_tG \qquad (37)$$

where as repeated total differentiation D_x^n of (35) gives

$$u_{tx(n)} = D_x^n(U - Xu_x) \qquad (38)$$

Eliminating $u_{tx(n)}$ from (37) gives

$$D_t(F)u_{x(n)} + FD_x^n(U - Xu_x) + D_tG = D_t(U - Xu_x) \qquad (39)$$

Adding $Xu_{tx} + FXu_{x(n+1)}$ to both sides and regrouping gives $D_t(U - Xu_x) + Xu_{tx} = FD_x^n(U - Xu_x) + FXu_{x(n+1)} + D_tG + D_t(F)u_{x(n)} + Xu_{tx} - FXu_{x(n+1)}$ By the definition of $U_{[t]}$ and $U_{[x(n)]}$ it follows that

$$U_{[t]} = FU_{x(n)} + D_t(G) + D_t(F)u_{x(n)} + Xu_{tx} - FXU_{x(n+1)} \qquad (40)$$

Total differentiation D_x of (34) gives

$$u_{tx} = D_x(F)u_{x(n)} + Fu_{x(n+1)} + D_xG \qquad (41)$$

Eliminating u_{tx} from (40) gives

$$\begin{aligned}U_{[t]} &= FU_{[x(n)]} + D_t(G) + D_t(F)u_{x(n)} + X(D_x(F)u_{x(n)} + Fu_{x(n+1)} + D_x(G)) - FXU_{x(n+1)} \\ &= FU_{x[n]} + D_t(G) + D_t(F)u_{x(n)} + XD_xFu_{x(n)} + XD_x(G) \qquad (42)\end{aligned}$$

so

$$U_{[t]} = FU_{[x(n)]} + (D_t(F) + XD_x(F)u_{x(n)} + (D_t(G) + XD_x(G)) \qquad (43)$$

Through the above lemma, (43) becomes

$$U_{[t]} = FU_{[x(n)]} + \Gamma^{(k)}Fu_{x(n)} + \Gamma^{(k)}G \qquad (44)$$

According the theorem, we know (37) together with the original equation (34) and the invariant surface condition (35) is equivalent to (44). So following the above theorem and the process of the proof we can obtain the determining equations for the nonclassical symmetries of (34) through directly solving (37) with the substitutions (38) and (41) instead of solving the governing equation (44).

In the next section BBM equation serves as examples.

4 Examples

In this section, we consider the BBM equation to show that compatibility can lead to the determining equations for nonclassical symmetries. The BBM equation:

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{45}$$

First, we use vector fields and its prolongations to obtain the determining equations of the nonclassical symmetries.

We denote the BBM equation by $\varDelta_1$ and the governing equation (25) for X, T, U is given by

$$\left.\varGamma^{(3)}\varDelta_1\right|_{\varDelta_1=0,\varDelta_2=0} = U_{[t]} + U_{[x]} + uU_{[x]} + Uu_x - U_{[xxt]}. \tag{46}$$

Eliminating $U_{[t]}, U_{[x]}, U_{[xxt]}$ from (46) and substituting $u_t = U - Xu_x$ give rise to the governing equation for X, T, U.

On the other hand, we use compatibility to obtain the determining equations, according to the above theorem, taking total differentiation D_t of (45) and substituting $u_{xxxt} = u_{xt} + u_{xx} + u_x^2 + uu_{xx}, u_t = U - Xu_x$ also gives rise to the governing equation for X, T, U. Comparing the two governing equation, They are same. The determining equations for the same governing equation are same. The determining equations are

$$\begin{aligned}
&U_x = 0 \qquad (47)\\
&-3XX_u = 0 \qquad (48)\\
&2XX_{uu} + 3X_u^2 = 0\\
&X_t + X_uU - XX_x = 0\\
&U_t + U_x - U_{txx} + (X_x - U_{uxx})U + (X_{xx} - 2U_{ux})U_x + X_xU_{xx} = 0\\
&U - X_t - 2U_{txu} + X_{txx} - (X_x - U_{uxx})X + (X_{uxx} - 2U_{uux})U + (X_{uxx} - 2U_{uux})U\\
&-(X_{xx} - 2U_{ux})X_x + (4X_{ux} - 2U_{uu})U_x + 2X_xU_{xu} - X_xX_{xx} + 2X_uU_{xx} = 0\\
&2X_{txu} - U_{tuu} - (X_{uxx} - 2U_{uux})X + (2X_{uux} - U_{uuu})U - (X_{xx} - 2U_{ux})X_{ux} + (4X_{ux} - 2U_{uu})U_u\\
&+3X_{uu}U_x + X_xU_{uu} - 2X_xX_{xu} - 4X_uU_{xu} - 2X_uX_{xx} - (4X_{ux} - 2U_{uu})X_x = 0\\
&X_{uuu}U - (4X_{ux} - 2U_{uu})X_u + 3X_{uu}U_u - 4X_{uu}X_x + 2X_uU_{uu} - 4X_uX_{ux}\\
&+X_{tuu} - (2X_{uux} - U_{uuu})X = 0\\
&-XX_{uuu} - 5X_uX_{uu} = 0\\
&-U_{tu} + 2X_{tx} + (2X_{ux} - U_{uu})U - (X_{xx} - 2U_{ux})X + 3X_uU_x + X_xU_u - 2X_x^2 = 0\\
&3X_{tu} - (2X_{ux} - U_{uu})X + 3X_{uu}U - (4X_{ux} - 2U_{uu})X + 5X_uU_u - 10X_uX_x = 0
\end{aligned}$$

Through (48), we obtain

$$X_u = 0 \tag{49}$$

Substituting (47), (48) and (49) into other equations, we can obtain the determining equations for the nonclassical symmetries of the BBM equation, and it reads as follows:

$$\begin{aligned}
&X_u = 0\\
&U_x = 0\\
&U_{uu} = 0\\
&X_t - XX_x = 0\\
&U_t + X_xU + X_xU_{xx} + X_{xx}U_x = 0\\
&-U_{tu} + 2X_{tx} - XX_{xx} + X_xU_u - 2X_x^2 = 0\\
&-X_t + U + X_{txx} - XX_x + X_{xx}U_u - 2X_{xx}X_x = 0
\end{aligned}$$

So the determining equations for the nonclassical symmetries of the BBM equation can be obtained by the compatibility between the original equation and the invariant surface condition (35). The determining equation is same to the determining equations obtained by use vector fields and its prolongations.

5 Conclusion

The nonclassical method, first introduced by Bluman and Cole, is based on a generalization of the method of Lie, which seeks invariance of a given partial differential equation under a group of infinitesimal transformations. In this paper we have considered a method of deriving the determining equations for the nonclassical symmetries of evolutionary partial differential equations. We study a new method to obtain the determining equation of the nonclassical of the partial differential equations. We show nonclassical symmetries can be easily and quickly derived through the compatibility between the original equation and the invariant surface condition (35). A simple partial differential equation and BBM equation serve as examples illustrating the feasibility of this method. Our future work is to consider whether conditional symmetry and plential symmetry can be obtained by this method.

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