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A reduction algorithm for sublinear reaction-diffusion dirichlet problems*

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Abstract. We consider a sublinear Reaction-Diffusion elliptic BVP on the unit square and recall proofs for the existence of five solutions. The fifth solution follows from an application of the Lyapunov-Schmidt reduction method. We present a new algorithm for approximating this solution.

Keywords: reduction algorithm, sublinear problems, lyapunov- schmidt method, dirichlet boundary value problem

1 Section heading

In this paper we want to study the stationary (time-independent) solutions to the Reaction-Diffusion $u_t(x) = \Delta u(x) + f(u(x))$, i.e., the solutions of elliptic boundary value problem

$$\begin{cases} \Delta u(x) + f(u(x)) = o & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \tag{1}$$

where Δ is the standard Laplacian operator, Ω is a bounded domain in R^N with smooth boundary and $f \in C^1(R,R)$ such that $f(0) = 0$. Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 < \cdots \to \infty$ be the eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω and $\{\psi_i\}$ the corresponding eigenfunctions, normalized in $L^2 = L^2(\Omega)$ and of course orthogonal in both Sobolev space (see [1]) $H = H_0^{1,2}(\Omega)$ and in L^2, with inner product $\langle u,v\rangle_H = \int_\Omega \nabla u.\nabla v dx$ and $\langle u,v\rangle_2 = \int_\Omega u.v dx$ respectively. It is a chief goal of ours to demonstrate the importance of these eigenfunctions, which form an orthonormal basis, to the theory and numerical solutions of related nonlinear elliptic equations.

We assume that $f'(0) < \lambda_1$, this is necessary to find one-sign solutions, and for some $k \geq 1$ there exist $\gamma > 0$ so that $f'(\infty) := \lim_{|t|\to\infty} \frac{f(t)}{t} \in (\lambda_k, \lambda_{k+1})$ and $f'(t) \leq \gamma < \lambda_{k+1}\ \forall t \in R.$

Under the above hypotheses on f, in [2] an existence proof provides five solutions when $k \geq 2$. In particular, one is the trivial solution and is of Morse index (MI) zero, two are of one sign and are of MI one, and a fourth solution which we now refer to as the CCN solution, which change sign exactly once is of Morse index 2 (if nondegenerate). The fifth solution is of Morse index k and is the specific solution that algorithm A approximates. The portion of the proof in [3] providing for this solution utilizes the Lyapunov-schmidt reduction method. The lemma in next section is the key to proof of the existence of this "reduction solution". Additionally, we observe that if we allow $k = 1$, then the reduction solution coincides with (either of) the one-sign solutions.

In order to find solutions to (1), we find critical points of the functional $J : H \to R$ defined by $J(u) = \int_\Omega \left\{\frac{1}{2}|\nabla u|^2 - F(u)\right\} dx$ where $F(u) = \int_0^u f(t)dt$. By regularity theory for elliptic boundary value problems u is solution to (1) if and only if u is critical point of the action functional J.

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We point out that in our two experimental examples, the eigenvalues and eigenfunctions are explicity known. Specifically, in the ODE case when $\Omega = [0,1]$ then $\lambda_i = (i\pi)^2$ and $\psi_i(x) = \sqrt{2}\sin(i\pi x)$, and in the PDE case when $\Omega = [0,1]\times[0,1]$ then $\lambda_{ij} = (i^2+j^2)\pi^2$ and $\psi_{ij}(x,y) = 2\sin(i\pi x)\sin(i\pi y)$. We often order the basis in this second case to be singly indexed. Note that (in H) we have $\langle \psi_i, \psi_j\rangle = \lambda_i\delta_{ij}$, where we have used the kronecker delta function.

2 Existence of reduction solution

For the sake of completeness we recall a global version of the Lyapunov-schmidt method. Since we are applying the following lemma to our functional J, it is useful to note that in our application we have

$$X = \text{span}\{\psi_1, ...\psi_k\} \quad \text{and} \quad Y = X^{\perp} = \text{span}\{\psi_{k+1}, \cdots\} \tag{2}$$

Lemma 1. *Let H be a real separable Hilbert space. Let X and Y be closed subspace of H such that $H = X\bigoplus Y$. Let $J : H \to R$ be a functional of class C^1. If there exist $m > 0$ such that for all $x \in X$ and $y, y_1 \in Y$ we have*

$$\langle \nabla J(x+y) - \nabla J(x+y_1), y-y_1\rangle \geq m\|y-y_1\|^2 \tag{3}$$

then the following hold:

(i) There exists a continuous function $\phi : X \to Y$ such that $J(x+\phi(x)) = \min_{y\in Y} J(x+y)$. Moreover, $\phi(x)$ is the unique member of Y such that

$$\langle \nabla J(x+\phi(x)), y\rangle = 0 \quad \textit{for all} \quad y \in Y. \tag{4}$$

The function $\tilde{J} : X \to R$ defined by $\tilde{J}(x) = J(x+\phi(x))$ is of class C^1, and

$$\langle \nabla \tilde{J}(x), x_1\rangle = \langle \nabla J(x+\phi(x)), x_1\rangle \quad \textit{for all} \quad x, x_1 \in X. \tag{5}$$

(iii) An element $x \in X$ is a critical point of $\tilde{J}$ if and only if $x+\phi(x)$ is a critical point of J.

(iv) If $-\tilde{J}$ is weakly lower semicontinuous and $J(x) \to -\infty$ as $\|x\| \to \infty$ ($x \in X$). Then there exists $u_0 \in H$ such that $\nabla J(u_0) = 0$ and $J(u_0) = \max_{x\in X}\min_{y\in Y} J(x+y)$.

In [3] it is shown that our functional J satisfies all the hypotheses to this lemma with X and Y defined as in (2).henceforth we refer to the solution u_0 as the "reduction solution".We now provide a sketch of proof of lemma 1.

Proof. For each $x \in X$ define $J_x : Y \to R$ by $J_x(y) = J(x+y)$. Using condition (3) it is easy to show that J_x is weakly lower semicontinuous and coercive. Thus J_x has a unique minimum $\phi(x) \in Y$. Therefore,

$$J(x+\phi(x)) = \min_{y\in Y} J(x+y). \tag{6}$$

Because $J \in C^1(H,R)$, it follows that $J_x \in C^1(Y,R)$, and $\phi(x)$ is the only element of Y such that $0 = \langle \nabla J_x(\phi(x)), y\rangle = \langle \nabla J(x+\phi(x)), y\rangle$. We can show that $\phi : X \to Y$ is a continuous function. This proves (*i*).

Let $x, x_1 \in X$ and $t > 0$. Since ∇J and ϕ are continuous,using (6) we can see that $\lim_{t\to 0}\frac{\tilde{J}(x+tx_1)-\tilde{J}(x)}{t} = \langle \nabla J(x+\phi(x)), x_1\rangle$. This shows that $\hat{J}$ has a continuous Gateaux derivative and hence is of class C^1.

From above we have $\langle \nabla \tilde{J}(x), x_1\rangle = \langle \nabla J(x+\phi(x)), x_1\rangle, \forall x, x_1 \in X$. This proves part (*ii*).

Part (*iii*) follows from (4) and (5).

Since $-\tilde{J}(x) = -J(x+\phi(x)) \geq -J(x)$ and $J(x) \to -\infty$ as $\|x\| \to \infty$, it follows that $-\tilde{J}(x) \to +\infty$ as $\|x\| \to \infty (x \in X)$.

Therefore $-\hat{J}$ is weakly lower semicontinuous and coercive, and hence $-\hat{J}$ has a minimum. Consequently, there exist $x_0 \in X$ such that

$$\hat{J}(x_0) = \max_{x \in X} \hat{J}(x). \tag{7}$$

Since $\tilde{J}(x) = J(x + \phi(x)) = \min_{y \in Y} J(x + y)$, we see that $J(x_0 + \phi(x_0)) = \max_{x \in X} \min_{y \in Y} J(x + y)$. Also, since $\tilde{J}$ is of class C^1, from (7) we have

$$\langle \nabla \hat{J}(x_0), x \rangle = 0 \quad \forall x \in X. \tag{8}$$

Let $x \in X$ and $y \in Y$

$$\langle \nabla J(x_0 + \phi(x_0)), x + y \rangle = \langle \nabla J(x_0 + \phi(x_0)), x \rangle + \langle \nabla J(x_0 + \phi(x_0)), y \rangle \tag{9}$$

Using (4), (5), and (8) we see that the first term and the second term of the right hand side of (9) are equal zero. Thus if $u_0 = x_0 + \phi(x_0)$ we have $\nabla J(u_0) = 0$ and $J(u_0) = \max_{x \in X} \min_{y \in Y} J(x + y)$. This proves part *(iv)*, which concludes the proof of Lemma 1.

3 The algorithms

By the Riesz Representation theorem there is a unique z such that $J'(u)(v) = \langle z, v \rangle_2$ then $z = \nabla_2 J(u)$. The function $\nabla_2 J(u)$ is said to be the L^2 gradient of $J(u)$, also $J'(u)(v) = \langle z, v \rangle_H$ then $z = \nabla_H J(u)$ is said to be the Sobolev gradient of $J(u)$. In our algorithm we will use the Sobolev gradient $\nabla_H J(u)$. The L^2 gradient $\nabla_2 J(u)$ is only densely defined. Not surprisingly, numerical approximation of $\nabla_2 J(u)$ behave poorly(see [5]). If $u \in C^2$, $\nabla_2 J(u)$ is defined and $\nabla_2 J(u) = \sum_{j=1}^{\infty} J'(u)(\psi_i)\psi_i$.

In this case, $J'(u)(v) = \langle \nabla_H J(u), v \rangle_H = \langle \nabla_2 J(u), v \rangle_2$. Since integrating by parts yields $\nabla_2 J(u) = -\Delta(\nabla_H J(u))$ and also we have $-\Delta\psi_i = \lambda_i \psi_i$, $\Delta_H J(u) = -\Delta^{-1}(\nabla_2 J(u)) = -\Delta^{-1} \sum_{j=1}^{\infty} J'(u)(\psi_i)\psi_i = \sum_{j=1}^{\infty} J'(u)(\psi_i)\frac{\psi_i}{\lambda_i}$. Also,using the Fourier expansion $u = \sum_{j=1}^{\infty} a_j \psi_j$ and that $\langle \psi_i, \psi_i \rangle_H = \langle -\Delta\psi_i, \psi_i \rangle_2 = \lambda_i$. $J'(u)(\psi_i) = \langle u, \psi_i \rangle_H - \int \psi_i f(u) dx = a_i \lambda_i - \int \psi_i f(u) dx$.

For the one-sign algorithm, in each iteration u is projected onto the codimension one submanifold of H (see for example [3]) $S = \{u \in H - 0 : J'(u)(u) = 0\}$, after which one takes a step in the $-\nabla J(u)$ direction. For the sign-changing "CCN" algorithm, u is projected onto $S_1 = \{u \in S : u_+ \in S, u_- \in S\}$, after which one follows $-\nabla J(u)$. In our reduction algorithm we perform steepest ascent in the X direction and steepest descent in Y direction , where $X = span\{\psi_1, \psi_2, ... \psi_k\}$, and $Y = span\{\psi_{k+1}, \psi_{k+2}, \cdots\}$. It turns out that this method allow us to find $\max_{x \in X} \min_{y \in Y} J(x + y)$.

Algorithm A (Reduction Algorithm).

Choose a function f which is sublinear and stepsize δ. Let k be the crossing eigenvalue number (e.g $f'(\infty) \in (\lambda_k, \lambda_{k+1})$).

Choose $a = a^0 \in R^M$ to be initial Fourier coefficients. M is the number of basis elements, so that our approximating subspace is $G = \{\psi_1, \psi_2, \cdots, \psi_M\} \approx X \bigoplus Y = H$. In the ODE case the singly indexed basis has size $\hat{M} = M$, whereas for convenience we refer to the size of the doubly basis for the PDE when $\Omega = [0, 1] \times [0, 1]$ as $\hat{M} = \sqrt{M}$.

Set $u = u^0 = \sum a_i \psi_i$.

Loop counter $n = 0$

$g = g^n = \{J'(u)(\psi_i)\}_{i=1,\cdots,M} \in R^M$, so that $P_G \nabla_2 J(u) = \sum_{i=1}^{M} g_i \psi_i$.

The numerical integration is accomplished by treating u as an array of values over a suitable grid on Ω.

For $i = 1$ to k, $\tilde{g}_i = -\frac{1}{\lambda_i} g_i$.

For $i = k + 1$ to M, $\tilde{g}_i = \frac{1}{\lambda_i} g_i$.

Set $a = a^{n+1} = a^n - \delta \tilde{g}$.

Set $u = u^{n+1} = \sum_{i=1}^{M} a_i \psi_i$.
Increment n.
If $|g| \approx \|\nabla_2 J(u)\|_2$ is small, exit loop.

We also use the two algorithms from [4] to find the one-sign solutions, and CCN solutions with only difference being the use of fourier approximation.

Algorithm B (One-Sign or Mountain Pass Algorithm).

Choose a function f which is sublinear and stepsize $\delta 1$ and $\delta 2$.
Choose $a = a^0 \in R^M$ to be initial Fourier coefficients.
Set $u = u^0 = \sum_{i=1}^{M} a_i \psi_i$.
Loop counter $n = 0$
Loop counter $m = 0$ (to project u onto S)
Calculate $t = \frac{\sum_{i=1}^{M} a_i^2 \lambda_i - \int_\Omega u f(u)}{\sum_{i=1}^{M} a_i^2 \lambda_i}$ so that $P_u \nabla J(u) = tu$.
Set $a = a^{m+1} = a^m + \delta_1 t a^n$ (steepest ascent in ray direction).
Increment m.
If $|a^{m+1} - a^m|$ is small, exit loop.
Set $g = g^n = \{J'(u)(\psi_i)\}_{i=1,\cdots,M} \in R^M$, so that $P_G \nabla_2 J(u) = \sum_{i=1}^{M} g_i \psi_i$.
For $i = 1$ to M (to take a step in the $-\nabla J(u)$ direction) $a_i = a_i^{n+1} = a_i^n - \frac{\delta 2}{\lambda_i} g_i^n$.
Set $u = u^{n+1} = \sum_{i=1}^{M} a_i \psi_i$.
Increment n.
If $|g| \approx \|\nabla_2 J(u)\|_2$ is small, exit loop.

The algorithm to produce the CCN solution is very similar to the one-sign algorithm B.The main difference is that instead of projection u onto S, one project u onto S_1, where $P_{S_1} u = P_S u_+ + P_S u_-$.

We found that our results in running this algorithm were not as good as in other article. This could be due to trying to estimate a function such as $\{\sin 2\pi x\}_+$ using a fourier expansion.

4 Numerical results

The particular solution that we most interested in is reduction solution. When $N = 1$ we take $\Omega = (0,1)$, whence the problem becomes the ODE $\begin{cases} u''(x) + f(u) = o & x \in (0,1) \\ u(x) = 0 & x \in \{0,1\} \end{cases}$. We require f to be sublinear. That is, $f'(\infty) = \lim_{|u| \to \infty} f'(u) < \infty$. In particular we use the following function: $f(x) = \begin{cases} ax + a\exp(-x) - a & x \geq 0 \\ ax - a\exp(x) + a & x < 0 \end{cases}$. Differentiating our function $f'(x) = \begin{cases} a - a\exp(-x) & x \geq 0 \\ a - a\exp(x) & x < 0 \end{cases}$. Where $f'(\infty) = a$ and $f'(0) = 0$. We performed all numerical integration using a left-hand Riemann sum, although certainly more sophisticated quadrature methods should be used. Unless otherwise noted, the algorithms stop when $\|\nabla_H J(u)\|_2 \leq 10^{-6}$. Using Algorithm,we numerically computed the solutions where $f'(\infty) \in (\lambda_1, \lambda_2)$ and the case where $f'(\infty) \in (\lambda_2, \lambda_3)$.In particular, for $k = 1$, $a = 2.5\pi^2$ and for $k = 2$, $a = 6.5\pi^2$. We also approximated and validated solution generated by Algorithm A for large values of k, but do not include those results

Clearly, since f is odd and $u(x)$ is a solution then $-u(x)$ is a solution.

In addition, we plotted several of the solutions in the ODE case. The graphs of the solutions for Algorithm A, for $k = 1$ and $k = 2$ are shown in Fig. 1 and Fig. 2. In PDE case, we will use Algorithm A to solve the problem $\begin{cases} \Delta u(x) + f(u) = o & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$.

We execute our algorithm for k=1, 3, 4 and approximate u in following tables. Then we plotted the graph of solutions in these cases.

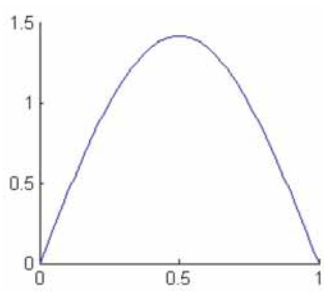

Fig. 1. $u, k = 1$

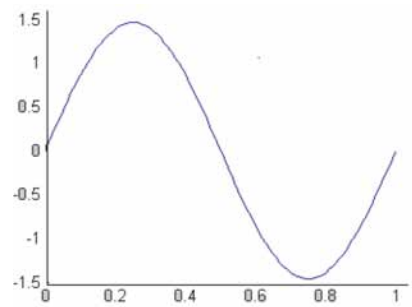

Fig. 2. $u, k = 2$

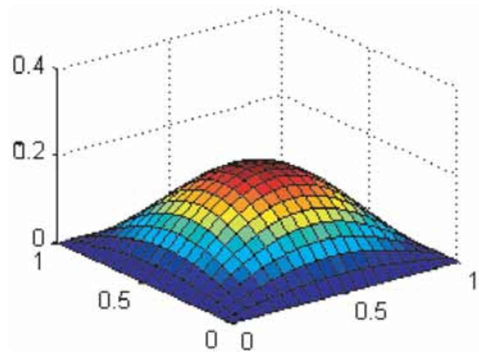

Fig. 3. $u, k = 2$

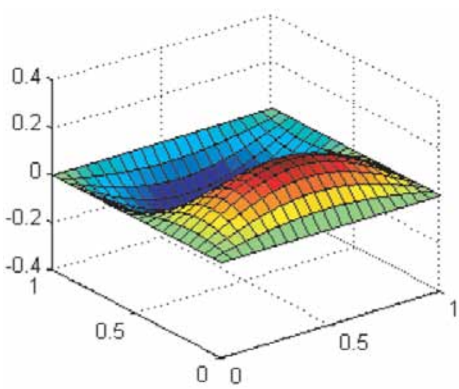

Fig. 4. $u, k = 1$

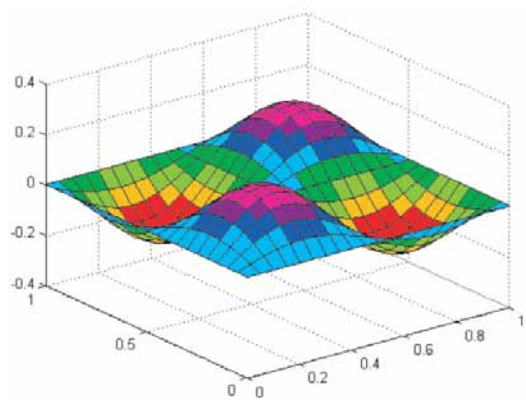

Fig. 5. $u, k = 2$

Table 1. Approximation of u

x	1	3	5	7	9
k=1	0.4335	1.1398	1.4118	1.1398	0.4335
k=2	0.8571	1.3899	0	-1.3899	-0.8571
k=3	1.1960	0.4588	-1.4691	0.4588	1.1960

Table 2. Approximation of u, Reduction Algorithm A, $k = 1$

	1	3	5	7	9
1	0.0191	0.0501	0.0620	0.0501	0.0191
3	0.0501	0.1313	0.1624	0.1313	0.0501
5	0.0620	0.1624	0.2008	0.1624	0.0620
7	0.0501	0.1313	0.1624	0.1313	0.0501
9	0.0191	0.0501	0.0620	0.0501	0.0191

Table 3. Approximation of u, Reduction Algorithm A, $k = 3$

	1	3	5	7	9
1	0.0363	0.0952	0.1177	0.0952	0.0363
3	0.0588	0.1541	0.1906	0.1541	0.0588
5	0	0	0	0	0
7	-0.0588	-0.1541	-0.1906	-0.1541	- 0.0587
9	-0.0363	- 0.0952	-0.1177	- 0.0952	-0.0363

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