

On the existence and stability properties of positive solution for some p-Laplacian Dirichlet problems*

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Abstract. This study concerns the existence and stability properties of positive weak solutions to classes of boundary value problems of the form

$$\begin{cases} -\Delta_p u = (\lambda a(x) f(u) - c), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\lambda, c > 0$ are parameters, Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2} \nabla z)$, $p > 1$, Ω is a bounded domain in R^N ($N > 1$) with smooth boundary $\partial\Omega$, the weight $a(x)$ satisfies $a(x) \in C(\Omega)$, $a(x) \geq a_0 > 0$ for all $x \in \Omega$, and f is a $C^2([0, \infty))$ function such that $f(0) = 0$. First by using the method of sub-super solution we study the existence of positive weak solution under the assumption that $f(u) > 0$ for $0 < u < \rho$ and $f(u) \leq 0$ for $u \geq \rho$. Next we study the stability properties of positive weak solution under the assumption that $t \mapsto \frac{f(t)}{t^{p-1}}$ is strictly increasing.

Keywords: positive weak solutions, p-Laplacian, sub-super solution, linearized stability

1 Introduction

In this paper we consider the existence and stability properties of positive weak solution to classes of boundary value problems of the form

$$-\Delta_p u = (\lambda a(x) f(u) - c), \quad x \in \Omega, \tag{1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{2}$$

where $\lambda, c > 0$ are parameters, Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2} \nabla z)$, $p > 1$, Ω is a bounded domain in R^N ($N > 1$) with smooth boundary $\partial\Omega$, the weight $a(x)$ satisfies $a(x) \in C(\Omega)$, $a(x) \geq a_0 > 0$ for all $x \in \Omega$, and f is a $C^2([0, \infty))$ function such that $f(0) = 0$.

Problems involving the ‘‘p-Laplacian’’ arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [9]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

In this paper we first study the existence of positive weak solution under the the following assumptions:

(H1) $f \in C^2([0, \infty))$, $f(0) = 0$, $f(u) > 0$ for $0 < u < \rho$ and $f(u) \leq 0$ for $u \geq \rho$ for some $\rho > 0$.

When $c = 0$ it is easy to establish the existence of a positive solution for large $\lambda > 0$. Here we consider the challenging semipositone case $c > 0$. Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see

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[1, 6]). Also most of the results established to date are for the case when $p = 2$. Here we establish an extend result for $p > 1$.

See [2] where the authors study such a problem in the case when $p = 2$. In particular, when c is very small, they establish an existence of a positive solution for $\tilde{\lambda}$ near the first eigenvalue λ_1 and then extend the existence for $\lambda \geq \tilde{\lambda}$. In this paper, we establish the existence of a positive solutions directly for λ large.

We then study the stability of positive weak solutions under the additional assumption $t \mapsto \frac{f(t)}{t^{p-1}}$ is strictly increasing.

2 Existence results

In this section we establish positive constants $c_0(\Omega, \rho)$ and $\lambda_0(\Omega, \rho, c)$ such that the problem (1) – (2) has a positive solution when $c \leq c_0$ and $\lambda \geq \lambda_0$. We employ the method of sub-super solutions to obtain the result (see [3]).

Let $W_0^{1,s} = W_0^{1,s}(\Omega)$, $s > 1$, denote the usual Sobolev space. We give the definition of weak solution and sub-super solution of (1) – (2).

Definition 1. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution to (1) – (2) if for any $v \in W_0^{1,p}$ with $v \geq 0$ we have

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} (\lambda a(x) f(u) - c) v \, dx.$$

However in this paper, we in fact study the existence of $C^1(\overline{\Omega})$ solutions that strictly positive in Ω .

Definition 2. We say that $\psi \in W_0^{1,p}(\Omega)$ is a subsolution to (1) – (2) if

$$\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla v \, dx \leq \int_{\Omega} (\lambda a(x) f(\psi) - c) v \, dx,$$

hold for all $v \in W_0^{1,p}$ with $v \geq 0$.

Definition 3. We say that $z \in W_0^{1,p}(\Omega)$ is a supersolution to (1) – (2) if

$$\int_{\Omega} |\nabla z|^{p-2} \nabla z \cdot \nabla v \, dx \geq \int_{\Omega} (\lambda a(x) f(z) - c) v \, dx,$$

hold for all $v \in W_0^{1,p}$ with $v \geq 0$.

Now if there exists sub and super solutions ψ and z respectively such that $0 \leq \psi \leq z$ for $x \in \Omega$, then (1) has a positive solution $u \in W_0^{1,p}(\Omega)$ such that $\psi \leq u \leq z$ (see [5]). We shall obtain the existence of positive weak solution to problem (1) – (2) by constructing a positive subsolution ψ and super solution z .

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta_p \phi = \lambda |\phi|^{p-2} \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \quad (3)$$

Let $\phi_1 \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue λ_1 of (3) such that $\phi_1(x) > 0$ in Ω , and $\|\phi_1\|_{\infty} = 1$. It can be shown that $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial\Omega$ and hence, depending on Ω , there exist positive constants k, η, μ such that

$$\lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -k, \quad x \in \overline{\Omega}_{\eta}, \quad (4)$$

$$\phi_1 \geq \mu, \quad x \in \Omega_0 = \Omega \setminus \overline{\Omega}_{\eta}, \quad (5)$$

with $\overline{\Omega}_{\eta} = \{x \in \Omega \mid d(x, \partial\Omega) \leq \eta\}$.

We will also consider the unique solution, $\zeta \in C^1(\overline{\Omega})$, of the boundary value problem

$$\begin{cases} -\Delta_p \zeta = 1, & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $\zeta > 0$ in Ω and $\frac{\partial \zeta}{\partial n} < 0$ on $\partial\Omega$. Our existence result is formulated in the following theorem.

Theorem 1. *Let (H1) hold. Then there exist positive constants $c_0 = c_0(\Omega, \rho)$ and $\lambda_0 = \lambda_0(\Omega, \rho, c)$ such that the problem (1) – (2) has a positive solution when $c \leq c_0$ and $\lambda \geq \lambda_0$.*

Proof. First we construct a positive subsolution of (1) – (2). For this, we let

$$\psi = \left(\frac{p-1}{p}\right) \rho \phi_1^{p/(p-1)}.$$

Note that $\|\psi\|_\infty < \rho$. Let the test function $w(x) \in W_0^{1,p}(\Omega)$ with $w \geq 0$. Since

$$\nabla \psi = \rho \phi_1^{1/(p-1)} \nabla \phi_1,$$

a calculation shows that

$$\begin{aligned} \int_\Omega |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w &= \rho^{p-1} \int_\Omega \phi_1 |\nabla \phi_1|^{p-2} \nabla \phi_1 \cdot \nabla w \, dx \\ &= \rho^{p-1} \left\{ \int_\Omega |\nabla \phi_1|^{p-2} \nabla \phi_1 \nabla(\phi_1 w) \, dx - \int_\Omega |\nabla \phi_1|^p w \, dx \right\} \\ &= \rho^{p-1} \int_\Omega (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w \, dx. \end{aligned}$$

Thus ψ is a subsolution if

$$\rho^{p-1} \int_\Omega (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) w \, dx \leq \int_\Omega (\lambda a(x) f(\psi) - c) w \, dx.$$

Now

$$\lambda_1 \phi_1^p - |\nabla \phi_1|^p \leq -k \quad \text{in } \bar{\Omega}_\eta,$$

and therefore

$$\rho^{p-1} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq -k \rho^{p-1} \leq (\lambda a(x) f(\psi) - c), \quad \text{if } c \leq c_0 = k \rho^{p-1},$$

since $\lambda a(x) f(\psi) \geq 0$.

Furthermore, we note that $\phi_1 \geq \mu > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, also in Ω_0 we have

$$\rho^{p-1} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq \lambda_1 \rho^{p-1}.$$

Hence if

$$\lambda \geq \lambda_0 = \frac{\lambda_1 \rho^{p-1} + c}{\alpha a_0},$$

where

$$\alpha = \inf \left\{ f(t) : \left(\frac{p-1}{p}\right) \rho \mu^{p/p-1} \leq t \leq \left(\frac{p-1}{p}\right) \rho \right\},$$

we have

$$\rho^{p-1} (\lambda_1 \phi_1^p - |\nabla \phi_1|^p) \leq \lambda_1 \rho^{p-1} \leq \lambda a_0 \alpha - c \leq (\lambda a(x) f(\psi) - c).$$

therefore if $c \leq c_0$ and $\lambda \geq \lambda_0$, then ψ is subsolution.

Next, we construct a supersolution z of (1) – (2) such that $z \geq \psi$. We denote $z = A\zeta(x)$, where the constant $A > 0$ is large and to be chosen later. We shall verify that z is a supersolution of (1) – (2). To this end, let $w(x) \in W_0^{1,p}(\Omega)$ with $w \geq 0$. Then we have

$$\int_{\Omega} |\nabla z|^{p-2} \nabla z \cdot \nabla w \, dx = A^{p-1} \int_{\Omega} |\nabla \zeta|^{p-2} \nabla \zeta \cdot \nabla w \, dx = A^{p-1} \int_{\Omega} w \, dx.$$

Thus ψ is a subsolution if

$$A^{p-1} \int_{\Omega} w \, dx \geq \int_{\Omega} (\lambda a(x) f(z) - c) w \, dx,$$

and therefore if

$$A \geq (A(\lambda))^{1/(p-1)} \quad \text{where} \quad A(\lambda) = \sup_{[0,\rho]} f(t),$$

we have

$$\int_{\Omega} |\nabla z|^{p-2} \nabla z \cdot \nabla w \, dx \geq \int_{\Omega} (\lambda a(x) f(z) - c) w \, dx,$$

and hence z is supersolution of (1) – (2). Since $\zeta > 0$ and $\partial\zeta/\partial n < 0$ on $\partial\Omega$, we can choose A large enough so that $\psi \leq z$ is also satisfied. Hence Theorem 1 is proven.

3 Stability results

In this section, we shall prove the instability of positive solution u (obtained in Theorem 2.4) by showing that the principal eigenvalue μ_1 , of the equation linearized about u is negative, the instability of u then follows from the well-known principle of linearized stability (see [7]).

We recall that, if u be any nonnegative solution of

$$\begin{cases} -\Delta_p u = g(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

then the linearized equation about u is

$$\begin{cases} -(p-1) \operatorname{div} (|\nabla u|^{p-2} \nabla \phi) - g_u(x, u) \phi = \mu \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

where $g_u(x, u)$ denotes the partial derivative of $g(x, u)$ with respect to u . Eq. (7) obtained from the formal derivative of the operator Δ_p .

Let μ_1 be the principal eigenvalue of (7) and ψ be the corresponding eigenfunction. We make take ψ such that $\psi > 0$ in Ω (see [3]).

Definition 4. We call a solution u of (6) a linearly stable solution if all eigenvalues of (7) are strictly positive, which can be inferred if the principal eigenvalue $\mu_1 > 0$. Otherwise u is linearly unstable.

Our stability result is formulated in the following theorem.

Theorem 2. If $u \mapsto f(u)/u^{p-1}$ be strictly increasing, then the positive solution u that obtained in Theorem 1 is linearly unstable.

Proof. Let u be positive solution of (1) – (2) that obtained in Theorem 2.4, then from (7) the linearized equation about u is

$$-(p - 1) \operatorname{div}(|\nabla u|^{p-2} \nabla \phi) - [\lambda a(x) f'(u)] \phi = \mu \phi, \quad x \in \Omega, \tag{8}$$

$$\phi = 0, \quad x \in \partial\Omega. \tag{9}$$

Let μ_1 be the principal eigenvalue and let $\psi(x) (\geq 0)$ be a corresponding eigenfunction (see [3]). Multiplying (1) by $(p - 1)\psi(x)$ and (8) by u , then subtracting and integrating over Ω , we obtain

$$\begin{aligned} & (p - 1) \int_{\Omega} [u \operatorname{div}(|\nabla u|^{p-2} \nabla \psi) - \psi(x) \operatorname{div}(|\nabla u|^{p-2} \nabla u)] dx \\ & + \lambda \int_{\Omega} a(x) \psi(x) [u f'(u) - (p - 1) f(u)] dx + (p - 1) c \int_{\Omega} \psi(x) dx \\ & = -\mu_1 \int_{\Omega} \psi(x) u(x) dx. \end{aligned} \tag{10}$$

But by green’s first identity

$$\begin{aligned} & \int_{\Omega} u \operatorname{div}(|\nabla u|^{p-2} \nabla \psi) dx = \int_{\Omega} u |\nabla u|^{p-2} (\Delta \psi) dx + \int_{\Omega} u \nabla \psi \nabla (|\nabla u|^{p-2}) dx \\ & = - \int_{\Omega} \nabla (u |\nabla u|^{p-2}) \nabla \psi(x) dx + \int_{\Omega} u \nabla \psi \nabla (|\nabla u|^{p-2}) dx + \int_{\partial\Omega} u |\nabla u|^{p-2} \left(\frac{\partial \psi}{\partial n}\right) ds \\ & = - \int_{\Omega} |\nabla u|^{p-2} (\nabla u \nabla \psi) dx, \end{aligned} \tag{11}$$

and similarly

$$\begin{aligned} \int_{\Omega} \psi(x) \operatorname{div}(|\nabla u|^{p-2} \nabla u) dx & = \int_{\Omega} \psi(x) |\nabla u|^{p-2} (\Delta u) dx + \int_{\Omega} \psi(x) \nabla u \nabla (|\nabla u|^{p-2}) dx \\ & = - \int_{\Omega} |\nabla u|^{p-2} (\nabla u \nabla \psi) dx. \end{aligned} \tag{12}$$

By using (11) – (12) in (10) we get

$$-\mu_1 \int_{\Omega} \psi(x) u(x) dx = \lambda \int_{\Omega} m(x) \psi(x) [u f'(u) - (p - 1) f(u)] dx + (p - 1) c \int_{\Omega} \psi(x) dx. \tag{13}$$

Now, since $u \mapsto f(u)/u^{p-1}$ is strictly increasing, we have

$$\left(\frac{f(u)}{u^{p-1}}\right)' = \frac{u f'(u) - (p - 1) f(u)}{u^p} > 0 \quad (u \in R^+), \tag{14}$$

$$\psi > 0, \quad a(x) > a_0, \quad x \in \Omega. \tag{15}$$

Thus, by using (14) – (15) in (13) we obtain

$$-\mu_1 \int_{\Omega} \psi(x) u(x) dx > 0. \tag{16}$$

Hence, it is easy to see that $\mu_1 < 0$ and the result follows (see [7]).

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