ISSN 1 746-7233, England, UK
World Journal of Modelling and Simulation
Vol. 2 (2006) No. 3, pp. 143-149

Global behavior of a SEIR model in epidemiology with nonlinear incidence rates

Jing Li[1] , Jun Yu[2*]

[1] Department of mathematics, Linyi Teachers' University
[2] Department of mathematics, Nanjing University of science & technology

(*Received December 20 2005, Accepted March 20 2006*)

Abstract. The SEIR model with nonlinear incidence rates $\lambda I^p S^q$ in epidemiology is further studied. Global behavior of the SEIR model is divided into three parts under certain scope of parameters. We focus on the global analysis including the orbital stability of periodic orbits, one and two dimensional center manifolds, hopf bifurcation, steady switch phenomenon and so on.

Keywords: SEIR model, periodic orbit, center manifold

1 Introduction

It is traditionally postulated that the spread of an infection occurs according to the principle of mass action and associates with the nonlinear incidence rates. The incidence rate of the form $\lambda I^p S^q$, where S and I are respectively the number of susceptible and infective individuals in the population, and λ, p and q are positive constants, is the most common nonlinear incidence rate. In recent years, the models with this incidence rate were considered by several authors, for example Liu et al[6, 7], Hethcote et al[5], Hethcote and Van den Driessche[4], Derrick and Van den Driessche[1, 2], Glendinning and Perry[3], Lizana and Rivero[8], and others. With very few exceptions, these authors focused on the local properties and bifurcation of equilibrium states.

In this paper, we consider the global behavior of a SEIR model with the incidence rate of the form $\lambda I^p S^q$ for the peculiar case $p > 1$.

2 Model formulation and equilibrium discussion

The SEIR model with nonlinear incidence rates is described as the following system of differential equations:

$$\begin{aligned} S' &= -\lambda I^p S^q + b - \mu S, \\ E' &= \lambda I^p S^q - (\varepsilon + \mu)E, \\ I' &= \varepsilon E - (\gamma + \mu)I, \\ R' &= \gamma I - \mu R. \end{aligned} \tag{1}$$

Here the parameters $p, q, \varepsilon, \mu, \lambda, \gamma, b$ are all positive constants, and the total population size is divided into four classes of individuals which are susceptible, exposed, infectious and recovered (with permanent immunity), with symbols denoted by S, E, I, R respectively. μ is a non-negative constant and represents the death, b is the rate of birth, ε is the rate constant expressing the rate of an exposed individuals becoming

---

[*] *E-mail address*: judyujun@163.com.

infective, γ is the rate for recovery.

Throughout this paper, we assume that $p > 1$. The feasible region for (1) is R^4_+, the positive orthant of R^4. We have $\Gamma = \{(S,E,I,R) \in R^4_+ : S+E+I+R = 1\}$ that is positive invariant, on the simplex Γ: $R(t) = 1-S(t)-E(t)-I(t)$. By considering the birth equal to the death, (1) can be reduced to the following 3-dimensional system:

$$\begin{aligned} &S' = -\lambda I^p S^q + \mu - \mu S, \\ &E' = \lambda I^p S^q - (\varepsilon+\mu)\,E, \\ &I' = \varepsilon E - (\gamma+\mu)\,I. \end{aligned} \tag{2}$$

Let $\tau = (\gamma+\mu)t, \alpha = \frac{\mu}{\gamma+\mu}, \beta = \frac{\varepsilon}{\gamma+\mu}, a = \frac{\lambda}{\gamma+\mu}$ and $p = 2$, $q = 1$. The differential equations can be changed into:

$$\begin{aligned} &S' = -aI^2S + \alpha - \alpha S, \\ &E' = aI^2S - (\alpha+\beta)E, \\ &I' = \beta E - I. \end{aligned} \tag{3}$$

For the dynamic behavior of (1) on Γ is equivalent to that of (3), in the rest of the paper, we will study the system (3) in the region $T = \{(S,E,I) : 0 \leq S,E,I \leq 1, S+E+I \leq 1\}$ and formulate our results accordingly.

We can get

$$I^2 - \frac{\alpha\beta}{\alpha+\beta}I + \frac{\alpha}{a} = 0, \quad \Delta = \left(\frac{\alpha\beta}{\alpha+\beta}\right)^2 - \frac{4\alpha}{a}.$$

When

1^0 $\Delta = (\frac{\alpha\beta}{\alpha+\beta})^2 - \frac{4\alpha}{a} < 0$, there is only a disease-free equilibrium $P_0(1,0,0)$;

2^0 $\Delta = (\frac{\alpha\beta}{\alpha+\beta})^2 - \frac{4\alpha}{a} = 0$, there are a disease-free equilibrium $P_0(1,0,0)$ and a endemic- equilibrium $P^*(\frac{1}{2}, \frac{\alpha}{2(\alpha+\beta)}, \frac{\alpha\beta}{2(\alpha+\beta)})$;

Discuss the denotation of eigenvalues:

(i) when $2\alpha(\alpha+\beta+1)-(\alpha+\beta) > 0$, we have $\lambda_2 < 0$, $\lambda_3 < 0$, $\lambda_1 \equiv 0$;

(ii) when $2\alpha\alpha+\beta+1)-(\alpha+\beta) = 0$, we have $\lambda_2 = 0$, $\lambda_3 < 0$, $\lambda_1 \equiv 0$;

(iii) when $2\alpha(\alpha+\beta+1)-(\alpha+\beta) < 0$, we have $\lambda_2 > 0$, $\lambda_3 < 0$, $\lambda_1 \equiv 0$.

3^0 $\Delta = (\frac{\alpha\beta}{\alpha+\beta})^2 - \frac{4\alpha}{a} > 0$, there are a disease-free equilibrium $P_0(1,0,0)$ and two endemic equilibria $P\,(S_1,E_1,I_1)$, $P\,(S_2,E_2,I_2)$ and

$$I_{1,2} = \frac{\dfrac{\alpha\beta}{\alpha+\beta} \pm \sqrt{\left(\dfrac{\alpha\beta}{\alpha+\beta}\right)^2 - \dfrac{4\alpha}{a}}}{2}.$$

3 Global dynamics of the epidemical model

In this part, we will focus on parameters fulfilling $(\frac{\alpha\beta}{\alpha+\beta})^2 = \frac{4\alpha}{a}$ that is critical state.

Remark 1. There exists a one-dimensional center manifold if parameters satisfy $2\alpha(\alpha+\beta+1)-(\alpha+\beta) > 0$ while $(\frac{\alpha\beta}{\alpha+\beta})^2 = \frac{4\alpha}{a}$.

When the parameters satisfy the condition in remark 1, there exists a zero eigenvalue and two negative eigenvalues. According to the center manifold theorem, we choose $\alpha = \beta = 1$ and then can compute out $a = 16$. Thus system (3) can be changed into:

$$\begin{aligned} &S' = -16I^2S - 8I^2 - 8IS - 4I - 2S, \\ &E' = 16I^2S + 8I^2 + 8IS + 4I + S - 2E, \\ &I' = E - I. \end{aligned} \tag{4}$$

By the coordinate transform

$$\begin{pmatrix} S \\ E \\ I \end{pmatrix} = \begin{pmatrix} -2 & -4 & 2 \\ 1 & 0 & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ \omega \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u \\ v \\ \omega \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 1 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{12} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} S \\ E \\ I \end{pmatrix},$$

system (4) becomes

$$\begin{pmatrix} u' \\ v' \\ \omega' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \\ \omega \end{pmatrix} + \begin{pmatrix} f \\ 0 \\ -f \end{pmatrix}$$

where

$$f = 2(u+v+\omega)[2(u+v+\omega)(2\omega - 2u - 4v) - u - 3v + 3\omega].$$

The form of center manifold is: $v = h_1(u)$ and $\omega = h_2(u)$.

Letting

$$\begin{aligned} v &= h_1(u) = a_1u^2 + a_2u^3 + \cdots, \\ \omega &= h_2(u) = b_1u^2 + b_2u^3 + \cdots, \end{aligned}$$

we can calculate out

$$\begin{aligned} v &= h_1(u) = 0, \\ \omega &= h_2(u) = \frac{1}{2}u^2 + 2u^3 + \cdots. \end{aligned}$$

Finally, system (4) becomes

$$\begin{aligned} u' =& Au + f(u, h(u)) \\ =& 2(u+\omega)\left[4\left(\omega^2 - u^2\right) - u + 3\omega\right] \\ =& 2\left(u + \frac{1}{2}u^2 + 2u^3 + \cdots\right)\left[4\left(\frac{1}{2}u^2 + 2u^3 + \cdots\right)^2 - 4u^2 - u + 3\left(\frac{1}{2}u^2 + 2u^3 + \cdots\right)\right] \\ =& u^2(-2 - 6u + \cdots). \end{aligned}$$

In this situation, equilibrium is non-stable.

Remark 2. There exists a two-dimensional center manifold if parameters satisfy $2\alpha(\alpha+\beta+1) - (\alpha+\beta) = 0$ while $(\frac{\alpha\beta}{\alpha+\beta})^2 = \frac{4\alpha}{a}$.

We choose $\alpha = \frac{1}{4}$, then can compute out $\beta = \frac{3}{4}, a = (\frac{16}{3})^2$. Thus system (3) can be changed into:

$$\begin{aligned} S' &= -aI^2S - \frac{a}{2}I^2 - \frac{16}{3}IS - \frac{8}{3}I - \frac{1}{2}S, \\ E' &= aI^2S + \frac{a}{2}I^2 + \frac{16}{3}IS + \frac{8}{3}I + \frac{1}{4}S - E, \\ I' &= \frac{3}{4}E - I. \end{aligned} \tag{5}$$

By the coordinate transform

$$\begin{pmatrix} u \\ v \\ \omega \end{pmatrix} = -\frac{1}{1200} \begin{pmatrix} 12 & -96 & -208 \\ -30 & -60 & -80 \\ -12 & 96 & -192 \end{pmatrix} \begin{pmatrix} S \\ E \\ I \end{pmatrix} = \begin{pmatrix} -\frac{1}{100} & \frac{2}{25} & \frac{13}{75} \\ \frac{1}{40} & \frac{1}{20} & \frac{1}{15} \\ \frac{1}{100} & -\frac{2}{25} & \frac{4}{25} \end{pmatrix} \begin{pmatrix} S \\ E \\ I \end{pmatrix},$$

System (5) becomes

$$\begin{pmatrix} u' \\ v' \\ \omega' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\dfrac{5}{2} \end{pmatrix} \begin{pmatrix} u \\ v \\ \omega \end{pmatrix} + g \begin{pmatrix} \dfrac{9}{100} \\ \dfrac{1}{40} \\ -\dfrac{9}{100} \end{pmatrix} \tag{6}$$

where

$$g = (3u+3\omega)\left[a(3u+3\omega)(-16u+32v+4\omega) + \frac{a}{2}(3u+3\omega) + \frac{16}{3}(-16u+32v+4\omega)\right].$$

The form of center manifold is: $\omega = h(u,v)$.

Letting

$$\omega = h(u,v) = a_1u^2 + a_2uv + a_3v^2 + \cdots,$$

we can get

$$\omega = h(u,v) = \frac{576}{125}\left(u^2 - \frac{24}{5}uv + \frac{48}{25}v^2\right) + \cdots.$$

Thus, system (6) becomes lower dimension

$$\begin{aligned} u' &= v + \frac{144}{25}(-2u^2 + 8uv + \cdots), \\ v' &= \frac{8}{5}(-2u^2 + 8uv + \cdots), \end{aligned} \tag{7}$$

By making one-one map

$$u \to x, \quad v + \frac{144}{25}(-2u^2 + 8uv) \to y$$

system (7) becomes

$$\begin{aligned} x' &= y, \\ y' &= -2C_2x^2(1 + 8C_1x) + (8C_2 - 4C_1)xy\left(1 + \frac{16C_1^2 + 64C_1C_2}{8C_2 - 4C_1}x\right), \end{aligned}$$

where $a_r = -2C_2 < 0$, $r = 2$, $b_n = 8C_2 - 4C_1 < 0$, $C_1 = \frac{144}{25}$, $C_2 = \frac{8}{5}$. In this case, equilibrium (0,0) is duplicate, $n = 1$, $m = 1$, so it is degenerate and its exponent is zero.

4 Analysis of the trajectory towards

When the parameters satisfy $2\alpha(\alpha+\beta+1) - (\alpha+\beta) < 0$ while $(\frac{\alpha\beta}{\alpha+\beta})^2 = \frac{4\alpha}{a}$, there are a zero eigenvalue, a negative and a positive eigenvalue. The endemic-equilibrium P^* is non-stable. We now discuss the existence of special periodic orbits in the positive invariant region $B^1 = \{(S,E,I)|\, 0 \le S \le 1, 0 \le E \le \frac{a\beta^2}{\alpha+\beta}, 0 \le I \le \beta, \beta \le 1\}$. Dividing it into eight sub-regions as following:

$$\begin{aligned} B_1 &= \left\{(S,E,I) \in B^1 \,|S < S^*, E < E^*, I < I^*\right\}; \\ B_2 &= \left\{(S,E,I) \in B^1 \,|S > S^*, E < E^*, I < I^*\right\}; \\ B_3 &= \left\{(S,E,I) \in B^1 \,|S > S^*, E > E^*, I < I^*\right\}; \\ B_4 &= \left\{(S,E,I) \in B^1 \,|S > S^*, E > E^*, I > I^*\right\}; \\ B_5 &= \left\{(S,E,I) \in B^1 \,|S < S^*, E > E^*, I > I^*\right\}; \\ B_6 &= \left\{(S,E,I) \in B^1 \,|S < S^*, E < E^*, I > I^*\right\}; \\ B_7 &= \left\{(S,E,I) \in B^1 \,|S > S^*, E < E^*, I > I^*\right\}; \\ B_8 &= \left\{(S,E,I) \in B^1 \,|S < S^*, E > E^*, I < I^*\right\}. \end{aligned}$$

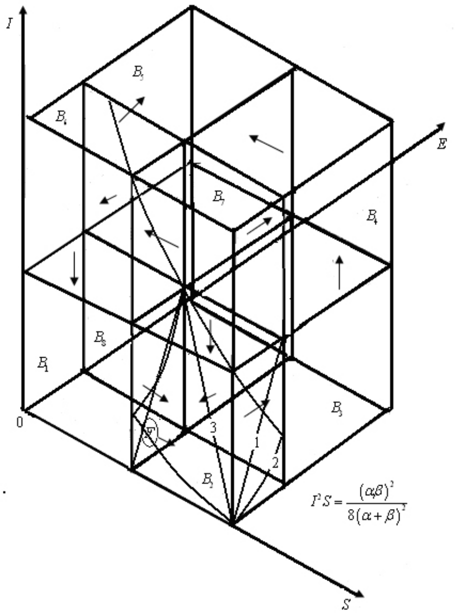

Fig. 1.

Remark 3. Plane (1) denoted by $\frac{\mathrm{d}I}{\mathrm{d}\tau}=0$; graphic-plane (2) denoted by $\frac{\mathrm{d}E}{\mathrm{d}\tau}=0$; beeline (3) denoted by $\frac{\mathrm{d}I}{\mathrm{d}\tau}=0$ and $\frac{\mathrm{d}E}{\mathrm{d}\tau}=0$ (We have from Fig. 1).

For B_7: there are three vector spaces of inner-boundary point to B_2, B_4, B_6. For B_8: there are three vector spaces of inner-boundary point to B_1, B_3, B_5.

- In $B_1, \forall q \in F \subset (B_1 \cap B_2)$, there is $\frac{\mathrm{d}S}{\mathrm{d}\tau}=-aI^2S+\alpha-\alpha S>0$, because $S=S^*, I<I^*, E<E^*$.
- in B_2, there is $\exists$ a curve $I^2S=\frac{(\alpha\beta)^2}{8(\alpha+\beta)^2} \in B_2 \cap B_3$ that satisfies $\frac{\mathrm{d}E}{\mathrm{d}\tau}=aI^2S-(\alpha+\beta)E^*=0$. When the trajectory is upon the curve, we can get $\frac{\mathrm{d}E}{\mathrm{d}\tau}>0$, thus $B_2 \rightarrow B_3$. When the trajectory is below the curve, we can get $B_3 \rightarrow B_2$.

And there is a graphic-plane (2), when trajectory enters B_2. On the graphic-plane, it follows $\frac{\mathrm{d}E}{\mathrm{d}\tau}>0$, and under the graphic-plane, it follows $\frac{\mathrm{d}E}{\mathrm{d}\tau}<0$. There is a plane (1). When trajectory is on the plane, we can get $\frac{\mathrm{d}I}{\mathrm{d}\tau}<0$. When under the plane, we can get $\frac{\mathrm{d}I}{\mathrm{d}\tau}>0$.

So when trajectory enter B_2, it can be divided into two parts:

1^0 graphic-plane (2) is above plane (1). When trajectory upside the line $I=\beta E$ on the interface $B_2 \cap B_3$ enters B_2, it first reaches graphic-plane $\frac{\mathrm{d}E}{\mathrm{d}\tau}=0$. Due to $\frac{\mathrm{d}I}{\mathrm{d}\tau}<0$, trajectory descends along I and satisfies $\frac{\mathrm{d}E}{\mathrm{d}\tau}<0$ and $\frac{\mathrm{d}S}{\mathrm{d}\tau}>0$. It is attracted by free-equilibrium $P_0(1,0,0)$;

2^0 plane (1) is above graphic-plane (2). When trajectory under the line $I=\beta E$ enters B_2, it first reaches graphic-plane $\frac{\mathrm{d}E}{\mathrm{d}\tau}=0$. Due to $\frac{\mathrm{d}I}{\mathrm{d}\tau}>0$, trajectory ascends along I, so it satisfies $\frac{\mathrm{d}E}{\mathrm{d}\tau}>0$ and $\frac{\mathrm{d}S}{\mathrm{d}\tau}>0$, thus $B_2 \rightarrow B_3$. If it first reaches plane (1), the analysis is as the same as 1^0.

- In B_3, there are $S>S^*, I<I^*, E>E^*$, so $\frac{\mathrm{d}I}{\mathrm{d}\tau}=\beta E-I>0$ and trajectory becomes $B_3 \rightarrow B_4$. When trajectories enter B_3 from B_8, they can be divided into two parts:

1^0 When trajectory is upon graphic-plane (2), that is $\frac{\mathrm{d}E}{\mathrm{d}\tau}>0$, it will go into B_4 along with the trajectory that is from B_2 to B_3;

2^0 When trajectory is under graphic-plane (2), that is $\frac{\mathrm{d}E}{\mathrm{d}\tau}<0$, the analysis is as the same as B_2.

- In B_4, there are $S>S^*, I>I^*, E>E^*$, so $\frac{\mathrm{d}S}{\mathrm{d}\tau}=-aI^2S+\alpha-\alpha S<0$;
- In B_5, there are analysis is as the same as in B_2;
- In B_6, there are $S<S^*, I>I^*, E<E^*$, so $\frac{\mathrm{d}I}{\mathrm{d}\tau}=\beta E-I<0$, trajectory is $B_6 \rightarrow B_1$; The trajectories from B_7 to B_6 partly go into B_1 and partly go into B_5, but there is no attractor in B_5, so it will return to B_6 and finally to B_1.

Repeat the reasoning $\varGamma_q$: trajectories are spiring forward following $B_2 \rightarrow B_3 \rightarrow B_4 \rightarrow B_5 \rightarrow B_6 \rightarrow B_1 \rightarrow B_2$, and there may be periodic orbits.

Theorem 1. *A sufficient condition for a periodic orbit* $\gamma = \{P(t) : 0 \leq t \leq \omega\}$ *of (3) to be asymptotically orbitally stable with asymptotic phase is that the linear system*

$$Z'(t) = \frac{\partial f^{[2]}}{\partial x}(P(t))Z(t) \tag{8}$$

be asymptotically stable.

Equation (8) is called the second compound equation of $y'(t) = \frac{\partial f}{\partial x}(x(t, x_0))y(t)$ and $\frac{\partial f^{[2]}}{\partial x}$ is the second compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$ of f.

Theorem 2. *Any periodic solution of the system (3), if it exists, is asymptotically orbitally stable.*

Proof. We can write the linear system with respect to the solution $(S(t), E(t), I(t))$ of (3) as the following 3×3 system:

$$\begin{aligned} X' &= -(aI^2 + 2\alpha + \beta)X + 2aIS(Y + Z), \\ Y' &= \beta X - (aI^2 + \alpha + 1)Y, \\ Z' &= aI^2 Y - (\alpha + \beta + 1)Z. \end{aligned} \tag{9}$$

To show the asymptotic stability of the system (9), we consider the following function:

$$V(X, Y, Z; S, E, I) = \sup\{|X|, \frac{E}{I}(|Y| + |Z|).$$

Suppose that the solution $(S(t), E(t), I(t))$ is periodic with least period $\omega > 0$, as to the orbit γ remains at a positive distance from the boundary of T. There exists constant c satisfying

$$V(X, Y, Z; S, E, I) \geq c \sup\{|X|, |Y|, |Z|\}$$

for all $(X, Y, Z) \in R^3_+$ and $(S, E, I) \in \gamma$.

The right-hand derivative of $V(t)$ exists and the direct calculation yields

$$\begin{aligned} D_+|X(t)| &\leq -(aI^2 + 2\alpha + \beta)|X| + 2aIS(|Y| + |Z|) \\ &= -(aI^2 + 2\alpha + \beta)|X| + \frac{2aI^2 S}{E}\left\{\frac{E}{I}(|Y| + |Z|)\right\} \end{aligned} \tag{10}$$

and

$$D_+|Y(t)| \leq \beta|X| - (aI^2 + \alpha + 1)|Y|, \quad D_+|Z(t)| \leq aI^2|Y| - (\alpha + \beta + 1)|Z|,$$

and thus

$$\begin{aligned} D_+\frac{E}{I}(|Y| + |Z|) &= \left(\frac{E'}{I} - \frac{I'}{E}\right)\frac{E}{I}(|Y| + |Z|) + \frac{E}{I}D_+(|Y| + |Z|) \\ &\leq \left(\frac{E'}{E} - \frac{I'}{I}\right)\frac{E}{I}(|Y| + |Z|) + \frac{E}{I}(\beta|X| - (\alpha + 1)t|Y| + (\alpha + \beta + 1)|Z|) \\ &\leq \frac{\beta E}{I}|X| + \left(\frac{E'}{E} - \frac{I'}{I} - \alpha - 1\right)\frac{E}{I}(|Y| + |Z|). \end{aligned} \tag{11}$$

We claim that (10) and (11) lead to

$$D_+V(t) \leq \sup\{g_1(t), g_2(t)\}V(t) \tag{12}$$

where

$$g_1(t) = -(aI^2 + 2\alpha + \beta) + \frac{2aI^2 S}{E}, \quad g_2(t) = \frac{\beta E}{I} + \frac{E'}{E} - \frac{I'}{I} - \alpha - 1.$$

Using (3), we find that

$$\frac{I'}{I}+1=\frac{\beta E}{I},\quad \frac{E'}{E}+\alpha+\beta=\frac{aI^2S}{E},$$

and can get

$$D_+V(t)\le \sup\left\{\frac{E'}{E}-\alpha,\frac{2E'}{E}+\beta-aI^2\right\}V(t).$$

Because $(S(0),E(0),I(0))\in \overset{0}{T}$, there exists suitable parameter β and a that satisfies $\beta-aI^2<0$, and thus

$$\int_0^{\omega}\sup\{g_1(t),g_2(t)\}\,\mathrm{d}t<0$$

which, together with (12), implies that $V(t)\to 0$ as $t\to\infty$, and in turn that $(X(t),Y(t),Z(t))\to 0$ as $t\to\infty$. As a result, the linear system (9) is asymptotically stable and the periodic solution $(S(t),E(t),I(t))$ is asymptotically orbitally stable with asymptotic phase.

5 Conclusions and discussion

We focus on the critical state that enduced three types, including one and two dimensional center manifolds, the stability of periodic orbits. If there exist periodic orbits, there maybe break out of the infection and will be persistence. It is important for the prevention and cure and also it is valuable to open put the orderliness of infections.

It is significative to study higher-dimensional periodic orbits further, and whether there is only a periodic orbit is not settled.

References

[1] W. R. Derrick, P. V. Driessche. empha disease transmission model in a nonconstant population. *J. Math. Biol.*, 1993, **31**: 495–512.

[2] W. R. Derrick, P. V. Driessche. Homoclinic orbits in a disease transmission model with nonlinear incidence and nonconstant population. *Driscreate and Contin. Dyn. Syst. Ser. B*, 2003, **3**: 299–309.

[3] P. Glendinning, L. P. Perry. Melnikov analysis of chaos on a simple epidemiological model. *J. Math. Biol.*, 1997, **35**: 359–373.

[4] H. W. Hethcote, P. V. Driessche. Some epidemiological models with nonlinear incidence. *J. Math. Biol.*, 1991, **29**: 271–287.

[5] H. W. Hethcote, M. A. Lewis, P. V. Driessche. An epidemiological models with delay and a nonlinear incidence rate. *J. Math. Biol.*, 1989, **27**: 49–64.

[6] W. M. Liu, H. W. Hethcote, S. A. Levin. Dynamical behaviorof epidemiological models with nonlinear incidence rates. *J. Math. Biol.*, 1987, **25**: 359–380.

[7] W. M. Liu, S. A. Levin, Y. Isawa. Influence of nonlinear incidence rates upon the behavior of SIRS epidemiological models. *J. Math. Biol.*, 1986, **23**: 187–204.

[8] M. Lizana, J. Rivero. Multiparametertric bifurcationa for a model in epidemiology. *J. Math. Biol.*, 1996, **35**: 21–36.