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Stability criteria of stochastic partial differential equations with variable delays*

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Abstract. Some criteria for the asymptotic stability of nonlinear stochastic partial differential equations with variable delays are presented. A coercivity condition plays the role of an exponential stability criterion. Consequently, under the coercivity condition almost all the trajectories of the nonstationary solutions of the given stochastic system finally tend exponentially to zero. Two examples are studied to illustrate our theory.

Keywords: stochastic partial differential equations with variable delays, almost sure exponential stability, mean square exponential stability.

1 Introduction

In 1892, Lyapunov introduced the concept of stability of dynamical systems and created a very powerful tool known as Lyapunov's second method in the study of stability. On the other hand, the theory of stochastic differential equations in infinite dimensions has developed rapidly over the last years[5]. For instance, consider the following equation

$$\begin{cases} \mathrm{d}X_t = A(X_t)\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t, \quad t \geq 0, \\ X_0 = x_0, \end{cases} \tag{1}$$

where $A(\cdot)$ and $B(\cdot)$ are families of operators (linear or nonlinear) in Hilbert space and W_t is a Hilbert space-valued Wiener process. We should mention that under various circumstances Da Prato and Zabczyk[5], Ichikawa[7] and Pardoux[9] (amongst others) have established results on the existence and uniqueness of strong, weak and mild solutions for the stochastic differential equation (1). In the meanwhile, Lyapunov's methods or ideas have been developed to deal with stochastic stability of infinite dimensional dynamical systems by many authors. In our opinion, the most original work on this aspect goes back at least to the well-known studies by Haussmann[6] of the linear case of the stochastic evolution equations (1). Subsequently, Ichikawa[7] considered similar problems for the mild solutions of a class of semilinear stochastic evolution equations. At the same time, Chow[4] obtained asymptotic stability for the sample paths of (1) when $A(\cdot)$ and $B(\cdot)$ satisfy a coercivity condition. In order to improve the results of asymptotic stability in Chow[4] so as to be able to include more general situations, Caraballo and Liu[3] obtained exponential stability criteria of general non-autonomous stochastic partial differential equations which contain as a special case the results from Chow[4] and at the same time improve relevant ones from Caraballo and Real[2].

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The aim of this paper is to investigate exponential stability for general nonlinear stochastic partial differential equations with variable delays. To this end, we firstly state the abstract framework where our analysis will be carried out.

Let V be a reflexive Banach space and H, K be two real, separable Hilbert spaces such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V',$$

where the injections are continuous and dense.

We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V' respectively; by $<\cdot,\cdot>$ the duality product between V', V, and by $(\cdot,\cdot)$ the scalar product in H. Let W_t be a Wiener process defined on a certain complete probability space (Ω, F, P) with filtration $\{F_t\}_{t\geq 0}$ and take values in the separable Hilbert space K, with increment covariance operator Q which has finite trace.

We shall consider the following infinite dimensional nonlinear stochastic differential equation:

$$\begin{cases} X_t = X_0 + \int_0^t A(s, X_s, X_{s-\tau(s)})\,\mathrm{d}s + \int_0^t B(s, X_s, X_{s-\tau(s)})\,\mathrm{d}W_s, \quad \forall t \geq 0, \\ X_t = \phi(t) \in H, \quad t \in [-h, 0], \end{cases} \tag{2}$$

where $\phi(t) : [-h, 0] \times \Omega \to H$, $h \geq 0$, is an initial datum such that $\phi(t)$ is F_0-measurable and $\sup_{-h\leq s\leq 0} E|\phi(s)|^2 < \infty$. $\tau : [0, \infty) \to [0, h]$, $h \geq 0$, is a certain differentiable function with $\tau'(t) \leq 0$, playing the role of time delays. $A(t, \cdot, \cdot) : V \times V \to V'$ is a family of nonlinear operators defined a.e.t. and $B(t, \cdot, \cdot) : V \times V \to L(K, H)$, the family of all bounded linear operators from K into H, satisfies:

(b.1) There exists $k > 0$ such that

$$\|B(t, u, v) - B(t, \tilde{u}, \tilde{v})\|_{L(K,H)} \leq k\Big(\|u - \tilde{u}\| + \|v - \tilde{v}\|\Big), \quad \forall u, \tilde{u}, v, \tilde{v} \in V, \ a.e.t;$$

(b.2) $t \in (0, T) \to B(t, u, v) \in L(K, H)$ is Lebesgue-measurable $\forall u, v \in V$, $\forall T > 0$.

Definition 1. *Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be the stochastic basis and W_t a K-valued Wiener process with finite trace covariance operator Q. Suppose that $\phi(t)$, $t \in [-h, 0]$, is a H-valued random function such that $\sup_{-h\leq r\leq 0} E|\phi(r)|^2 < \infty$. A process X_t is said to be a strong solution to the SDE (2) on Ω for $t \in [0, T]$ if the following are satisfied:*

(a) X_t is a V-valued F_t-measurable random variable;

(b) $X_t \in I^p(-h, T; V) \cap L^2(\Omega; C(-h, T; H))$, $p > 1$, $h > 0$ and $T > 0$, where $I^p(-h, T; V)$ denotes the space of all V-valued processes $(X_t)_{t\in[-h,T]}$ (we will write X_t for short) measurable (from $[-h, T] \times \Omega$ into V), and satisfying

$$E \int_{-h}^{T} \|X_t\|^p \,\mathrm{d}t < \infty,$$

with $C(-h, T; H)$ denoting the space of all continuous functions from $[-h, T]$ into H;

(c) The SDE (2) is satisfied for all $t \in [0, T]$ and almost all $\omega \in \Omega$.

If T is replaced by ∞, X_t is called a global strong solution of (2).

Since we are mainly concerned about exponential stability for strong solutions, one always assumes that for each given initial datum $\phi \in I^p(-h, 0; V) \cap L^2(\Omega; C(-h, 0; H))$, there exists a process

$$X_t \in I^p(-h, T; V) \cap L^2(\Omega; C(-h, T; H)), \ \forall T > 0,$$

which is the strong solution of the following problem:

$$\begin{cases} \mathrm{d}X_t = A(t, X_t, X_{t-\tau(t)})\,\mathrm{d}t + B(t, X_t, X_{t-\tau(t)})\,\mathrm{d}W_t, \\ X_t = \phi(t), \quad t \in [-h, 0]. \end{cases}$$

In other words, X_t satisfies the following integral equation (in V'):

$$X_t = \phi(0) + \int_0^t A(s, X_s, X_{s-\tau(s)})\,\mathrm{d}s + \int_0^t B(s, X_s, X_{s-\tau(s)})\,\mathrm{d}W_s, \quad t > 0, \ P-\text{a.s.}, \tag{3}$$

and $X_t = \phi(t)$, $t \in [-h, 0]$. Observe that, in the particular case in which A and B are given by $A(t,x,y) = A_1(t,x) + f_1(t,y) + f_2(t)$, $B(t,x,y) = B_1(t,y) + g(t)$ with A_1 and B_1 satisfying, mainly, some suitable coercivity and monotonicity assumptions, one can get existence and uniqueness of the strong solution for the equation (3), as was shown in Caraballo[1] (see also Real [10] for the linear case). In our general situation, the technique used by Pardoux[9] can be properly adapted to establish a result on the existence and uniqueness of the strong solution to (3). In consideration of the fact that, until now, we have not found in the literature a general result on existence and uniqueness of the strong solutions of the nonlinear stochastic delay differential equations (2), we are including an Appendix containing a result on this aspect and a sketch of the proof.

For this purpose, for instance, one can suppose the following assumptions hold (see Pardoux [9] and Appendix)

(a.1) (Coercivity). There exist $\alpha > 0$, $p > 1$ and θ, λ, $\gamma \in \mathbf{R}^1$ such that

$$2 < A(t,x,y), x > + \|B(t,x,y)\|_2^2 \le -\alpha \|x\|^p + \lambda |x|^2 + \theta |y|^2 + \gamma, \quad \forall x, y \in V, \ a.e.t.$$

where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm of a nuclear operator, i.e.,

$$\|B(t,x,y)\|_2^2 = tr(B(t,x,y)QB(t,x,y)^*);$$

(a.2) (Boundedness).There exists $c > 0$ such that

$$\|A(t,x,y)\|_* \le c\|x\|^{p-1} + c\|y\|^{p-1}, \quad \forall x, y \in V, \ a.e.t;$$

(a.3) (Measurability).

$$t \in (0,T) \mapsto A(t,x,y) \in V' \text{ is Lebesgue-measurable } \forall x, y \in V, \text{ a.e.t}, \ \forall T > 0;$$

(a.4) (Hemicontinuity).

$$\xi \in R^1 \mapsto \langle A(t, x + \xi y, z), v \rangle \in \mathbf{R}^1 \text{ is continuous for all } x, y, z, v \in V, \text{ a.e.t;}$$

(a.5) (Monotonicity). For all x_1, x_2, y_1, $y_2 \in V$, and a.e.t,

$$\begin{aligned} 2 < A(t,x_1,y_1) - A(t,x_2,y_2), x_1 - x_2 > &+ \|B(t,x_1,y_1) - B(t,x_2,y_2)\|_2^2 \\ &\le \lambda\Big(|x_1 - x_2|^2 + |y_1 - y_2|^2\Big). \end{aligned}$$

We also note that there exists a positive constant $\beta > 0$ such that

$$|x| \le \beta \|x\|, \quad \forall x \in V. \tag{4}$$

2 The main results

For our stability purpose, we need the following coercivity condition:

(H1) There exist constants $\alpha > 0$, $\mu > 0$, $\theta \in \mathbf{R}_+$, $\lambda \in \mathbf{R}^1$, and a nonnegative function $\gamma(t)$, $t \in \mathbf{R}_+$, such that

$$2 < A(t,x,y), x > + \|B(t,x,y)\|_2^2 \le -\alpha \|x\|^p + \lambda |x|^2 + \theta |y|^2 + \gamma(t) e^{-\mu t}, \quad x, y \in V, \tag{5}$$

where $p > 1$ and $\gamma(t)$ satisfies the condition that for arbitrary $\delta > 0$, $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$, i.e., $\lim_{t\to\infty} \gamma(t)/e^{\delta t} = 0$.

Theorem 1. *Suppose conditions (H1) and (b.1) hold. Then, if X_t is a global strong solution of the equation (2), there exist constants $\varepsilon > 0$, $C = C(\phi) > 0$ such that*

$$E|X_t|^2 \le C \cdot e^{-\varepsilon t}, \qquad \forall t \ge 0, \tag{6}$$

if either one of the following hypotheses holds

(i) $\lambda < 0$, $-\lambda > \theta$, $(\forall p > 1)$;

(ii) More sharply, $\nu > \theta$ with $\nu := \alpha/\beta^2 - \lambda$, (for p = 2).

Proof. We only prove case (ii). Case (i) can be similarly proved. Firstly, from (4) and (5) it is easy to deduce that

$$2 < A(t,x,y), x > + \|B(t,x,y)\|_2^2 \leq -\nu|x|^2 + \theta|y|^2 + \gamma(t)e^{-(\mu\wedge\nu)t}, \quad x, y \in V. \tag{7}$$

Since $\nu > \theta$, we can find a suitable positive constant $\varepsilon \in (0, \frac{\nu\wedge\mu}{2})$ such that

$$\frac{\theta e^{\varepsilon h}}{\nu - \varepsilon} < 1.$$

We claim that there exists a positive constant $K = K(h, \nu, \theta, \mu, \varepsilon) < \infty$ such that

$$\int_0^\infty e^{\varepsilon t} E|X_t|^2 \, \mathrm{d}t \leq K.$$

Indeed, applying Itô's formula to the strong solution yields that

$$\begin{aligned} e^{\nu t}|X_t|^2 - |\phi(0)|^2 =& \nu \int_0^t e^{\nu s}|X_s|^2 \, \mathrm{d}s + 2\int_0^t e^{\nu s}\langle A(s, X_s, X_{s-\tau(s)}), X_s\rangle \, \mathrm{d}s \\ &+ 2\int_0^t e^{\nu s}(X_s, B(s, X_s, X_{s-\tau(s)}) \, \mathrm{d}W_s) \\ &+ \int_0^t e^{\nu s} tr(B(s, X_s, X_{s-\tau(s)})QB(s, X_s, X_{s-\tau(s)})^*) \, \mathrm{d}s \end{aligned}$$

which, in addition to (7), implies

$$\begin{aligned} & e^{\nu t}E|X_t|^2 \\ &\leq E|\phi(0)|^2 + \theta\int_0^t e^{\nu s}E|X_{s-\tau(s)}|^2 \, \mathrm{d}s + \int_0^t \gamma(s)e^{[\nu-(\mu\wedge\nu)]s} \, \mathrm{d}s \\ &\leq E|\phi(0)|^2 + \theta\int_0^t e^{\nu s}E|X_{s-\tau(s)}|^2 \, \mathrm{d}s + e^{[\nu-(\mu\wedge\nu)]t}\cdot e^{(\mu\wedge\nu)t/2}\int_0^t \gamma(s)e^{-(\mu\wedge\nu)s/2} \, \mathrm{d}s \\ &\leq E|\phi(0)|^2 + \theta\int_0^t e^{\nu s}E|X_{s-\tau(s)}|^2 \, \mathrm{d}s + e^{\left[\nu-\frac{(\mu\wedge\nu)}{2}\right]t}\int_0^t \gamma(s)e^{-(\mu\wedge\nu)s/2} \, \mathrm{d}s \end{aligned}$$

for all $t \geq 0$. Consequently,

$$E|X_t|^2 \leq E|\phi(0)|^2 \cdot e^{-\nu t} + \theta e^{-\nu t}\int_0^t e^{\nu s}E|X_{s-\tau(s)}|^2 \, \mathrm{d}s + e^{-(\mu\wedge\nu)t/2}\int_0^t \gamma(s)e^{-(\mu\wedge\nu)s/2} \, \mathrm{d}s\,.$$

Therefore, we have for arbitrary $T > 0$ large enough

$$\begin{aligned} \int_0^T e^{\varepsilon t}E|X_t|^2 \, \mathrm{d}t \leq& E|\phi(0)|^2 \cdot \int_0^T e^{-(\nu-\varepsilon)t} \, \mathrm{d}t + \theta\int_0^T e^{-(\nu-\varepsilon)t}\int_0^t e^{\nu s}E|X_{s-\tau(s)}|^2 \, \mathrm{d}s \, \mathrm{d}t \\ &+ \int_0^T e^{\varepsilon t}e^{-(\mu\wedge\nu)t/2}\int_0^\infty \gamma(s)e^{-(\mu\wedge\nu)s/2} \, \mathrm{d}s \, \mathrm{d}t \\ \leq& \frac{1}{\nu-\varepsilon}E|\phi(0)|^2 + \frac{\theta}{\nu-\varepsilon}\int_0^T e^{\varepsilon s}E|X_{s-\tau(s)}|^2 \, \mathrm{d}s + \frac{2k_1}{\mu\wedge\nu-2\varepsilon} \end{aligned} \tag{8}$$

where $k_1 = \int_0^\infty \gamma(s)e^{-(\mu\wedge\nu)s/2} \, \mathrm{d}s < \infty$.

However, as the function $\rho(t) = t - \tau(t)$ is strictly increasing with $\lim_{t\to+\infty}\rho(t) = +\infty$, there exists $\delta_1 \in (0, h]$ such that $\rho(\delta_1) = 0$, $\rho(t) \in [-h, 0]$ for all $t \in [0, \delta_1]$ and $\rho(t) > 0$ for all $t > \delta_1$. Thus, taking into account the change of variables $u = s - \tau(s)$, it follows for arbitrary $T > 0$ large enough

$$\begin{aligned}\int_0^T e^{\varepsilon s}E|X_{s-\tau(s)}|^2\,\mathrm{d}s \le &\int_0^{\delta_1} e^{\varepsilon s}E|X_{s-\tau(s)}|^2\,\mathrm{d}s + e^{\varepsilon h}\int_{\delta_1}^T e^{\varepsilon(s-\tau(s))}E|X_{s-\tau(s)}|^2\,\mathrm{d}s\\ \le&\delta_1\cdot e^{\varepsilon\delta_1}\sup_{-h\le r\le 0}E|\phi(r)|^2 + e^{\varepsilon h}\int_0^T e^{\varepsilon s}E|X_s|^2\,\mathrm{d}s\\ \le& h\cdot e^{\varepsilon h}\sup_{-h\le r\le 0}E|\phi(r)|^2 + e^{\varepsilon h}\int_0^T e^{\varepsilon s}E|X_s|^2\,\mathrm{d}s \end{aligned}\tag{9}$$

which, together with (8), immediately implies that

$$\int_0^T e^{\varepsilon t}E|X_t|^2\,\mathrm{d}t \le \Big(\frac{1}{\nu-\varepsilon}+\frac{\theta\cdot h\cdot e^{\varepsilon h}}{\nu-\varepsilon}\Big)\sup_{-h\le r\le 0}E|\phi(r)|^2+\frac{2k_1}{\mu\wedge\nu-2\varepsilon}+\frac{\theta e^{\varepsilon h}}{\nu-\varepsilon}\int_0^T e^{\varepsilon t}E|X_t|^2\,\mathrm{d}t,$$

i.e., there exists a positive constant $\tilde{C}=\tilde{C}(\mu,\nu,\theta,h)<\infty$ such that

$$\int_0^T e^{\varepsilon t}E|X_t|^2\,\mathrm{d}t \le \tilde{C}\tag{10}$$

where

$$\tilde{C}=\frac{1}{1-\frac{\theta e^{\varepsilon h}}{\nu-\varepsilon}}\Big[\Big(\frac{1}{\nu-\varepsilon}+\frac{\theta\cdot h\cdot e^{\varepsilon h}}{\nu-\varepsilon}\Big)\sup_{-h\le r\le 0}E|\phi(r)|^2+\frac{2}{\mu\wedge\nu-2\varepsilon}k_1\Big],$$

which, letting $T>0$ tend to infinity in (10) and using Fatou's lemma, immediately implies

$$\int_0^\infty e^{\varepsilon t}E|X_t|^2\,\mathrm{d}t \le \tilde{C}.\tag{11}$$

Now, we can obtain our main result. Applying once again Itô's formula to the strong solution X_t, $t\ge 0$, as above and using (9) and (11), we derive

$$\begin{aligned}e^{\varepsilon t}E|X_t|^2 \le& E|\phi(0)|^2+\theta\int_0^t e^{\varepsilon s}E|X_{s-\tau(s)}|^2\,\mathrm{d}s+\int_0^t\gamma(s)e^{[\varepsilon-(\mu\wedge\nu)]s}\,\mathrm{d}s\\ \le& E|\phi(0)|^2+\theta\Big(h\cdot e^{\varepsilon h}\sup_{-h\le r\le 0}E|\phi(r)|^2+e^{\varepsilon h}\tilde{C}\Big)+\int_0^\infty\gamma(s)e^{-[(\mu\wedge\nu)-\varepsilon]s}\,\mathrm{d}s\\ :=&C<\infty,\end{aligned}$$

i.e.,

$$E|X_t|^2\le C\cdot e^{-\varepsilon t}$$

for all $t\ge 0$. In other words, the solution is mean square exponentially stable, and the proof is now complete.

Theorem 2. *Assume the hypotheses in Theorem 1 hold. Then there exist positive constants K, r and a subset $\Omega_0\subset\Omega$ with $P(\Omega_0)=0$ such that, for each $\omega\notin\Omega_0$, there exists a positive random number $T(\omega)$ such that*

$$|X_t|^2\le K\cdot e^{-rt},\qquad \forall t\ge T(\omega).\tag{12}$$

Proof. As in the last proof, we only prove case (ii). Firstly, we can choose a natural number N_0, large enough, such that $N_0-\tau(N_0)\ge 0$ and moreover for all $N\ge N_0$ it has $N-\tau(N)\ge 0$. Now, applying Itô's formula to 2 immediately yields for $t\ge N\ge N_0$

$$\begin{aligned}|X_t|^2-|X_N|^2=&2\int_N^t < A(s,X_s,X_{s-\tau(s)}),X_s>\mathrm{d}s+\int_N^t\|B(s,X_s,X_{s-\tau(s)})\|_2^2\,\mathrm{d}s\\ &+2\int_N^t(X_s,B(s,X_s,X_{s-\tau(s)})\,\mathrm{d}W_s),\end{aligned}\tag{13}$$

which, together with the coercivity condition (5), implies

$$
\begin{aligned}
|X_t|^2 \leq & |X_N|^2 + |\lambda| \int_N^t |X_s|^2 \,\mathrm{d}s + \theta \int_N^t |X_{s-\tau(s)}|^2 \,\mathrm{d}s \\
& + \int_N^t \gamma(s) e^{-\mu s} \,\mathrm{d}s + \Big| 2 \int_N^t (X_s, B(s, X_s, X_{s-\tau(s)}) \,\mathrm{d}W_s) \Big|. \qquad (14)
\end{aligned}
$$

Let I_N denote the interval $I_N = [N, N+1]$, then from (14) we can get

$$
\begin{aligned}
\sup_{t \in I_N} |X_t|^2 \leq & |X_N|^2 + |\lambda| \int_N^{N+1} |X_s|^2 \,\mathrm{d}s + \theta \int_N^{N+1} |X_{s-\tau(s)}|^2 ds \\
& + \int_N^{N+1} \gamma(s) e^{-\mu s} \,\mathrm{d}s + \sup_{t \in I_N} \Big| 2 \int_N^t (X_s, B(s, X_s, X_{s-\tau(s)}) \,\mathrm{d}W_s) \Big|. \qquad (15)
\end{aligned}
$$

In particular, by using Chebychev's inequality, it is not difficult to obtain that for large enough $N \geq N_0$ and for $\epsilon_N^2 = e^{-\varepsilon N/2}$, we have

$$
\begin{aligned}
P\Big\{ \sup_{t \in I_N} |X_t|^2 \geq \epsilon_N^2 \Big\} \leq & P\Big\{ |X_N|^2 \geq \epsilon_N^2/5 \Big\} + P\Big\{ |\lambda| \int_N^{N+1} |X_s|^2 \,\mathrm{d}s \geq \epsilon_N^2/5 \Big\} \\
& + P\Big\{ \theta \int_N^{N+1} |X_{s-\tau(s)}|^2 \,\mathrm{d}s \geq \epsilon_N^2/5 \Big\} \\
& + P\left\{ \left[\sup_{t \in I_N} \Big| 2 \int_N^t (X_s, B(s, X_s, X_{s-\tau(s)}) \,\mathrm{d}W_s) \Big| \right] \geq \epsilon_N^2/5 \right\} \\
\leq & 5\epsilon_N^{-2} E|X_N|^2 + 5|\lambda| \epsilon_N^{-2} \int_N^{N+1} E|X_s|^2 \,\mathrm{d}s + 5\theta \epsilon_N^{-2} \int_N^{N+1} E|X_{s-\tau(s)}|^2 \,\mathrm{d}s \\
& + 10\epsilon_N^{-2} E \left[\sup_{t \in I_N} \Big| \int_N^t (X_s, B(s, X_s, X_{s-\tau(s)}) \,\mathrm{d}W_s) \Big| \right]. \qquad (16)
\end{aligned}
$$

Now, according to Burkholder-Davis-Gundy's lemma, we can estimate the last term in (16) (from now on, $K_1, K_2, ...$ denote some proper positive constants).

$$
\begin{aligned}
& E\left[\sup_{t \in I_N} \Big| \int_N^t (X_s, B(s, X_s, X_{s-\tau(s)}) \,\mathrm{d}W_s) \Big| \right] \\
& \leq K_1 E\left[\int_N^{N+1} |X_s|^2 \|B(s, X_s, X_{s-\tau(s)})\|_2^2 \,\mathrm{d}s \right]^{1/2} \\
& \leq K_1 E\left[\sup_{t \in I_N} |X_t| \left(\int_N^{N+1} \|B(s, X_s, X_{s-\tau(s)})\|_2^2 \,\mathrm{d}s \right)^{1/2} \right] \\
& \leq \frac{1}{4} E\left[\sup_{t \in I_N} |X_t|^2 \right] + K_2 \int_N^{N+1} E\|B(s, X_s, X_{s-\tau(s)})\|_2^2 \,\mathrm{d}s\,.
\end{aligned}
$$

However, taking expectations in (15) it easily deduces

$$
\begin{aligned}
E\left[\sup_{t \in I_N} |X_t|^2 \right] \leq & E|X_N|^2 + |\lambda| \int_N^{N+1} E|X_s|^2 \,\mathrm{d}s + \theta \int_N^{N+1} E|X_{s-\tau(s)}|^2 \,\mathrm{d}s \\
& + \int_N^{N+1} \gamma(s) e^{-\mu s} \,\mathrm{d}s + E\left[\sup_{t \in I_N} \Big| 2 \int_N^t (X_s, B(s, X_s, X_{s-\tau(s)}) \,\mathrm{d}W_s) \Big| \right], \qquad (17)
\end{aligned}
$$

then one can get

$$
\begin{aligned}
\frac{1}{2} E\left[\sup_{t \in I_N} |X_t|^2 \right] \leq & E|X_N|^2 + |\lambda| \int_N^{N+1} E|X_s|^2 \,\mathrm{d}s + \theta \int_N^{N+1} E|X_{s-\tau(s)}|^2 \,\mathrm{d}s \\
& + \int_N^{N+1} \gamma(s) e^{-\mu s} \,\mathrm{d}s + 2K_2 \int_N^{N+1} E\|B(s, X_s, X_{s-\tau(s)})\|_2^2 \,\mathrm{d}s\,. \qquad (18)
\end{aligned}
$$

Now, we can derive from (16) and (18) that there exists a positive constant $K_3 > 0$ such that

$$P\left[\sup_{t\in I_N}|X_t|^2 \geq \epsilon_N^2\right] \leq K_3\epsilon_N^{-2}\bigg[E|X_N|^2 + \int_N^{N+1} E|X_s|^2\,\mathrm{d}s + \int_N^{N+1} E|X_{s-\tau(s)}|^2\,\mathrm{d}s + \int_N^{N+1}\gamma(s)e^{-\mu s}\,\mathrm{d}s + \int_N^{N+1} E\|B(s,X_s,X_{s-\tau(s)})\|_2^2\,\mathrm{d}s\bigg]. \tag{19}$$

Evaluating the terms on the right-hand side of (19), taking into account (6) and $\varepsilon < \mu$, we can easily deduce that there exists $K_4 > 0$ such that

$$E|X_N|^2 + \int_N^{N+1} E|X_s|^2\,\mathrm{d}s + \int_N^{N+1} E|X_{s-\tau(s)}|^2\,\mathrm{d}s + \int_N^{N+1}\gamma(s)e^{-\mu s}\,\mathrm{d}s \leq K_4 e^{-\varepsilon N}, \tag{20}$$

and for the last term, assume that the following claim which will be proved below holds:

Claim. There exists a positive constant $K_5 > 0$ such that for all $N \geq N_0$

$$\int_N^{N+1} E\|B(s,X_s,X_{s-\tau(s)})\|_2^2\,\mathrm{d}s \leq K_5 e^{-\varepsilon N}, \tag{21}$$

then (19)-(21) imply that there exists $K_6 > 0$ such that

$$P\left[\sup_{t\in I_N}|X_t|^2 \geq \epsilon_N^2\right] \leq K_6\epsilon_N^{-2}e^{-\varepsilon N} = K_6 e^{-\varepsilon N/2},$$

and a Borel-Cantelli's lemma type argument completes the proof.

Let us finally prove our claim (21). Indeed, for the parameter $\varepsilon > 0$ in Theorem 1 and using (5), we obtain from Itô's formula once again,

$$e^{\varepsilon t}E|X_t|^2 \leq E|\phi(0)|^2 + \varepsilon\int_0^t e^{\varepsilon s}E|X_s|^2\,\mathrm{d}s - \alpha\int_0^t e^{\varepsilon s}E\|X_s\|^2\,\mathrm{d}s + \lambda\int_0^t e^{\varepsilon s}E|X_s|^2\,\mathrm{d}s + \theta\int_0^t e^{\varepsilon s}E|X_{s-\tau(s)}|^2\,\mathrm{d}s + \int_0^t \gamma(s)e^{-(\mu-\varepsilon)s}\,\mathrm{d}s, \tag{22}$$

which, in addition to (11), implies that there exists $K_7 > 0$ such that

$$\int_0^t e^{\varepsilon s}E\|X_s\|^2\,\mathrm{d}s \leq \frac{1}{\alpha}\bigg[E|\phi(0)|^2 + \varepsilon\int_0^t e^{\varepsilon s}E|X_s|^2\,\mathrm{d}s + \lambda\int_0^t e^{\varepsilon s}E|X_s|^2\,\mathrm{d}s + \theta\int_0^t e^{\varepsilon s}E|X_{s-\tau(s)}|^2\,\mathrm{d}s + \int_0^t \gamma(s)e^{-(\mu-\varepsilon)s}\,\mathrm{d}s\bigg] \leq K_7 < \infty.$$

and, consequently,

$$\int_s^t E\|X_u\|^2\,\mathrm{d}u \leq \int_s^t e^{\varepsilon(u-s)}E\|X_u\|^2\,\mathrm{d}u \leq K_7 e^{-\varepsilon s}, \quad \text{for} \quad 0 \leq s \leq t.$$

Now, taking into account (b.1), (5) and that $N - \tau(N) \geq 0$ for $N \geq N_0$, we can get for $t \in I_N$

$$\begin{aligned}\int_N^t E\|B(u,X_u,X_{u-\tau(u)})\|_2^2\,\mathrm{d}u \leq& 2\int_N^t E\|B(u,X_u,X_{u-\tau(u)}) - B(u,0,0)\|_2^2\,\mathrm{d}u \\ &+ 2\int_N^t E\|B(u,0,0)\|_2^2 du \\ \leq& K_8\left(\int_N^t E\|X_u\|^2\,\mathrm{d}u + \int_N^t E\|X_{u-\tau(u)}\|^2\,\mathrm{d}u\right) \\ &+ \int_N^t \gamma(u)e^{-\mu u}\,\mathrm{d}u \leq K_9 e^{-\varepsilon N},\end{aligned}$$

and for $t = N + 1$ the claim is finally proved.

Remark 1. Observe that in the case $B(t,\cdot,\cdot) : H \times H \to L(K;H)$ and condition (b.1) turns out

$$\|B(t,u,v) - B(t,\tilde{u},\tilde{v})\|_{L(K,H)} \le k\Big(|u-\tilde{u}| + |v-\tilde{v}|\Big), \quad \forall u,\tilde{u},v,\tilde{v} \in H,\ a.e.t.,$$

the last claim follows immediately since the upper bound on $\int_N^t E\|X_s\|^2\,\mathrm{d}s$ follows easily from (b.1) and (6).

Remark 2. The exponential decay term appearing on the right hand side of (5) is of the essence for our stability purposes. In fact, to illustrate this, let us simply consider the following one dimensional linear Itô equation (notice that on this occasion $V = H$):

Example 1. Assume X_t satisfies the following

$$\mathrm{d}X_t = -pX_t\,\mathrm{d}t + (1+t)^{-q}\,\mathrm{d}W_t, \qquad t \ge 0$$

with initial datum $X_0 = 0$, where p, $q > 0$ are two positive constants and W_t is a one-dimensional standard Brownian motion.

Clearly, the coercivity type condition (5) now turns out

$$2 < -pX_t, X_t > + \Big[(1+t)^{-q}\Big]^2 = -2pX_t^2 + (1+t)^{-2q}.$$

where $< \cdot,\cdot >$ denotes the standard inner product in $\mathbf{R}^1$. However, under this condition the solution is exponentially unstable. Indeed, it is easy to obtain the explicit solution

$$X_t = e^{-pt}\int_0^t e^{ps}\cdot(1+s)^{-q}\,\mathrm{d}W_s \equiv e^{-pt}M_t, \qquad t \ge 0$$

which immediately implies that for arbitrarily given $q > 0$ Lyapunov exponent

$$\lim_{t\to\infty}\frac{\log E|X_t|^2}{t} = 0.$$

In the meantime, noticing the law of the iterated logarithm

$$\limsup_{t\to\infty}\frac{M_t}{\sqrt{2 < M_t > \log\log < M_t >}} = 1 \qquad a.s.$$

and

$$\limsup_{t\to\infty}\frac{\log\Big(\int_0^t e^{2ps}(1+s)^{-2q}\,\mathrm{d}s\Big)}{t} = 2p,$$

we therefore get Lyapunov exponent

$$\limsup_{t\to\infty}\frac{1}{t}\log|X_t| = 0 \qquad a.s.$$

That is, in spite of the typical stability of ordinary differential equation

$$\mathrm{d}X_t = -pX_t\,\mathrm{d}t,$$

the polynomial type decay of the noise term is not sufficient to ensure the exponential stability of its stochastically perturbed system.

As an immediate application of Theorem 1 and Theorem 2, we shall next derive a couple of useful corollaries.

Firstly, consider the following nonlinear stochastic diffusion equation:

$$X_t = X_0 + \int_0^t A(s,X_s)\,\mathrm{d}s + \int_0^t B(s,X_s)\,\mathrm{d}W_s\,. \tag{23}$$

Here $A(t,\cdot) : V \to V'$ is a family of nonlinear operators defined a.e.t. for all $t \in \mathbf{R}_+$ and $B(t,\cdot) : V \to L(K,H)$, the family of all bounded linear operators from K into H, both satisfying some corresponding properties as in Section 1 (see [3] for further details) and the following:

(c.1) There exists $k > 0$ such that

$$\|B(t,y) - B(t,x)\| \le k\|y - x\|, \quad \forall x, y \in V,\ a.e.t.$$

We also suppose the following coercivity condition holds, that is, there exist constants $\alpha > 0$, $\mu > 0$, $\lambda \in \mathbf{R}^1$, and a nonnegative function $\gamma(t)$, $t \in \mathbf{R}_+$, such that

$$2 < A(t,x), x > + \|B(t,x)\|_2^2 \le -\alpha\|x\|^p + \lambda|x|^2 + \gamma(t)e^{-\mu t}, \quad x \in V, \tag{24}$$

where $p > 1$ and $\gamma(t)$ satisfies that for arbitrary $\delta > 0$, $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$, i.e., $\lim_{t\to\infty} \gamma(t)/e^{\delta t} = 0$.

As a direct consequence, simply letting $\theta = 0$ in (5) immediately yields the following important null variable delay result Corollary 1, (in fact, Theorem 1 from [3]) which could be regarded as a special case of a stochastic delay differential equation.

Corollary 1. *Assume (c.1) and coercivity condition (24) hold. Suppose that X_t is a global strong solution to (23). Then, there exist positive constants $\varepsilon > 0$, $C = C(X_0) > 0$ such that*

$$E|X_t|^2 \le C \cdot e^{-\varepsilon t}, \quad \forall t \ge 0,$$

if either one of the following hypotheses holds:

(a) $\lambda < 0$, $(\forall p > 1)$;

(b) More sharply, $\nu := \alpha/\beta^2 - \lambda > 0$, (for $p = 2$).

Furthermore, under the same conditions the solution is almost surely stable. That is, there exist positive constants $r > 0$, $K > 0$ and a subset $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$ such that, for each $\omega \notin \Omega_0$, there exists a positive random number $T(\omega)$ such that the following holds:

$$|X_t(\omega)|^2 \le K \cdot e^{-rt}, \quad \forall t \ge T(\omega).$$

As the second application, we shall impose the following fractional power type coercivity condition which is in essence more nonlinear type:

(H2) There exist constants $\alpha > 0$, $\lambda \in \mathbf{R}^1$, $\mu > 0$, $\theta > 0$, $0 \le \sigma \le 1$ and nonnegative functions $\gamma(t)$, $\zeta(t)$, $t \in \mathbf{R}_+$, such that

$$2 < A(t,u,v), v > + \|B(t,u,v)\|_2^2 \le -\alpha\|u\|^p + \lambda|u|^2 + \zeta(t)e^{-\theta t}|v|^{2\sigma} + \gamma(t)e^{-\mu t}, \quad u, v \in V, \tag{25}$$

where $p > 1$, $\gamma(t)$, $\zeta(t)$ satisfy that for arbitrary $\delta > 0$, $\zeta(t) = o(e^{\delta t})$, $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$.

Corollary 2. *Suppose that (H2) and (b.1) hold. Let X_t be a global strong solution to the equation (2). Then there exist constants $\varepsilon > 0$, $C > 0$ such that*

$$E|X_t|^2 \le C \cdot e^{-\varepsilon t}, \qquad \forall t \ge 0, \tag{26}$$

if either one of the following hypotheses holds:

(i) $\lambda < 0, (\forall p > 1)$;

(ii) More sharply, $\nu > 0$ with $\nu := \alpha/\beta^2 - \lambda$, (for $p = 2$).

Remark 3. Notice that letting $\sigma = 0$ or $\sigma = 1$ in Corollary 2, we simply obtain Corollary 1 or Theorem 1, respectively.

Proof. Observe that the cases $\sigma = 0$ and $\sigma = 1$ are trivial. For $0 < \sigma < 1$, by virtue of Young's inequality

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for any} \quad a \geq 0,\ b \geq 0,\ p,\ q > 1 \text{ with } 1/p + 1/q = 1,$$

we have that for arbitrary $\varepsilon > 0$, the third term on the right hand side of (25) turns out

$$\zeta(t)e^{-\theta t}|v|^{2\sigma} \leq \sigma\varepsilon^{1/\sigma}|v|^2 + (1-\sigma)\varepsilon^{\frac{1}{1-\sigma}}\zeta(t)^{\frac{1}{1-\sigma}} \cdot e^{-\frac{\theta}{1-\sigma}t}$$

which, together with (25) , implies that

$$\begin{aligned} 2 < A(t,u,v), v > + \|B(t,u,v)\|_2^2 \leq & -\alpha\|u\|^p + \lambda|u|^2 + \sigma\varepsilon^{1/\sigma}|v|^2 \\ & + \Big(\gamma(t) + (1-\sigma)\varepsilon^{\frac{1}{1-\sigma}}\zeta(t)^{\frac{1}{1-\sigma}}\Big)e^{-(\frac{\theta}{1-\sigma}\wedge\mu)t}, \quad u, v \in V. \end{aligned}$$

Hence, in view of Theorem 1 and Theorem 2, it is easy to deduce that if $\nu > \sigma\varepsilon^{1/\sigma}$ the solution is mean square exponentially stable and at the same time almost sure exponentially stable. Observe $\varepsilon > 0$ is an arbitrary constant, the proof of (ii) is therefore complete. (i) can be proved in a similar way.

In a similar manner as in the proof of Theorem 2, we could also prove the following result.

Corollary 3. *Assume the hypotheses in Corollary 2.2 hold. Then there exist positive constants $K > 0$, $r > 0$ and a subset $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$ such that, for each $\omega \notin \Omega_0$, there exists a positive random number $T(\omega)$ such that*

$$|X_t|^2 \leq K \cdot e^{-rt}, \qquad \forall t \geq T(\omega). \tag{27}$$

3 Examples

In this section, we consider two stochastic partial differential equations to illustrate our theory.

Example 2. Firstly, we consider the following semilinear stochastic partial differential equation:

$$\begin{cases} \mathrm{d}Y_t(x) = 2\alpha\frac{\partial^2}{\partial x^2}Y_t(x)\,\mathrm{d}t + Y_{t-h}(x)\,\mathrm{d}t + 2t^3e^{-t}\cdot g(Y_t(x))\,\mathrm{d}W_t, \quad t>0, \quad x\in(0,\pi), \\ Y_0(x) = y_0(x), \quad Y_t(0) = Y_t(\pi) = 0, \quad t \geq 0. \end{cases} \tag{28}$$

Here $h > 0$, $\alpha > 0$ are two positive real numbers, $g(\cdot) : \mathbf{R}^1 \to \mathbf{R}^1$ is a bounded, Lipschitz continuous function and W_t is a real standard Wiener process (so, $K = \mathbf{R}^1$ and $Q = 1$). We can set this problem in our formulation by taking $H = L^2[0,\pi]$, $V = W_0^{1,2}([0,\pi])$ (Sobolev spaces with elements satisfying the boundary conditions above), $K = \mathbf{R}^1$, $A(t,u,v) = 2\alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + v(x)$ and $B(t,u,v) = 2t^3e^{-t}g(u(x))$.

Clearly, the diffusion term B satisfies (b.1). On the other hand, it is easy to deduce (for arbitrary $u, v \in V$)

$$2 < A(t,u,v), u > + \|B(t,u,v)\|_2^2 \leq -4\alpha|u|^2 + |u|^2 + |v|^2 + 4Kt^6e^{-2t}, \tag{29}$$

where K is a certain positive constant.

Therefore, whenever $\alpha > 1/2$, we easily deduce that the hypotheses in Theorems 2.1 and 2.2 are fulfilled, that is, the strong solution of our equation is mean square exponentially stable and also almost surely stable.

Example 3. Consider the following semilinear stochastic partial differential equation:

$$\begin{cases} \mathrm{d}Y_t(x) = \frac{\partial^2}{\partial x^2}Y_t(x)\,\mathrm{d}t + e^{-t/2}(Y_{t-\tau(t)}(x))^{\frac{1}{3}}\,\mathrm{d}t + \sqrt{\mu}\frac{Y_t(x)}{1+|Y_{t-\tau(t)}(x)|}\,\mathrm{d}W_t, \quad t>0, \quad x\in(0,1), \\ Y_0(x) = y_0(x), \quad Y_t(0) = Y_t(1) = 0, \quad t \geq 0, \end{cases} \tag{30}$$

where $\mu \geq 0$ is a non-negative real number and $\tau(t) : \mathbf{R}^1 \to [0,h]$, is a certain differentiable function with $\tau'(t) \leq 0$. W_t is a real standard Wiener process (so, $K = \mathbf{R}^1$ and $Q = 1$). We can set this problem in our formulation by taking $H = L^2[0,1]$, $V = W_0^{1,2}([0,1])$ (Sobolev spaces with elements satisfying the boundary conditions above), $K = \mathbf{R}^1$, $A(t,u,v) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) + e^{-t/2}v(x)^{\frac{1}{3}}$ and $B(t,u,v) = \sqrt{\mu}u(x)/(1+|v(x)|)$.

It is easy to deduce that for arbitrary $\delta > 0$ small enough and $u, v \in V$

$$2 < A(t,u,v), u > + \|B(t,u,v)\|_2^2 \leq -2\pi^2|u|^2 + (\delta+\mu)|u|^2 + 1/\delta \cdot e^{-t}|v|^{2/3}. \tag{31}$$

Therefore, whenever $2\pi^2 > \delta + \mu \geq 0$, or equivalently, $2\pi^2 > \mu \geq 0$ (notice $\delta > 0$ is an arbitrary positive number), we easily deduce from Corollary 2.2 and Corollary 2.3 that for arbitrary delay interval $[-h, 0]$, $h > 0$, the strong solution of our equation is mean square exponentially stable and also almost surely stable.

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Appendix

In this section, we shall establish a theorem on the existence and uniqueness of solutions to (2).

Theorem 3. *In addition to (a.1)–(a.5), (b.1), (b.2), assume that ϕ and τ satisfy the hypotheses in Section 1. Then, there exists a unique strong solution to* (2) *on* $[0,T]$, *for all* $T > 0$.

Proof. Uniqueness. Suppose that X_t and Y_t are two strong solutions of (2) on $[0,T]$. Then, denoting $\rho(t) = t - \tau(t)$, $t \geq 0$, it follows

$$\begin{aligned} X_t - Y_t = &\int_0^t \Big(A(s, X_s, X_{\rho(s)}) - A(s, Y_s, Y_{\rho(s)})\Big)\,\mathrm{d}s \\ &+ \int_0^t \Big(B(s, X_s, X_{\rho(s)}) - B(s, Y_s, Y_{\rho(s)})\Big)\,\mathrm{d}W_s, \quad \forall t \in [0,T]. \end{aligned}$$

Now, Itô's formula, (a.5) and the fact $X_{\rho(t)} = Y_{\rho(t)}$ as $\rho(t) \leq 0$ yield that

$$
\begin{aligned}
E|X_t - Y_t|^2 =& 2\int_0^t E\Big(A(s, X_s, X_{\rho(s)}) - A(s, Y_s, Y_{\rho(s)}), X_s - Y_s\Big)\,\mathrm{d}s \\
&+ \int_0^t E\|B(s, X_s, X_{\rho(s)}) - B(s, Y_s, Y_{\rho(s)})\|_2^2\,\mathrm{d}s \\
\le& \lambda\Big[\int_0^t E|X_s - Y_s|^2\,\mathrm{d}s + \int_0^t E|X_{\rho(s)} - Y_{\rho(s)}|^2\,\mathrm{d}s\Big] \\
\le& \lambda\Big[\int_0^t E|X_s - Y_s|^2\,\mathrm{d}s + \int_{\rho(0)}^{\rho(t)} E|X_u - Y_u|^2\,\mathrm{d}u\Big] \\
\le& \lambda\Big[\int_0^t E|X_s - Y_s|^2\,\mathrm{d}s + \int_0^t E|X_u - Y_u|^2\,\mathrm{d}u\Big], \quad \forall t \in [0, T],
\end{aligned}
$$

and then a Gronwall's lemma type argument yields the required uniqueness.

Existence. First of all, notice that since $\tau'(t) \le 0$ and $\tau(t) \in [0, h]$ for all $t \ge 0$, there exist only three possible situations:

Case 1. $\lim_{t\to+\infty} \tau(t) = \delta > 0$.

Case 2. $\lim_{t\to+\infty} \tau(t) = 0$ but $\tau(t) > 0$ for all $t \ge 0$.

Case 3. There exists $T^* > 0$ such that $\tau(t) > 0$ for $t \in [0, T^*)$ and $\tau(t) = 0$ for $t \ge T^*$.

Let us analyze each of them separately:

Case 1. As $\tau(t) \ge \delta$ for all $t \ge 0$, we get that $\rho(t) \le t - \delta$ for all $t \ge 0$. So, $\rho(t) \le t - \delta \le 0$ for $t \in [0, \delta]$ and therefore the problem on $[0, \delta]$ can be rewritten as

$$
\begin{cases}
X_t = \phi(0) + \int_0^t A(s, X_s, \phi(\rho(s)))\,\mathrm{d}s + \int_0^t B(s, X_s, \phi(\rho(s)))\,\mathrm{d}W_s, \quad \forall t \in [0, \delta], \\
X_t = \phi(t), \quad t \in [-h, 0],
\end{cases}
$$

which is a nondelay problem. Now, observe that in the case without delays considered by Pardoux in [9], existence of strong solutions is proved under the following similar assumptions to (a.1)–(a.5) and (b.1)–(b.2). In fact, consider $A(t, \cdot) : V \to V'$, a family of nonlinear operators defined a.e.t., and $B(t, \cdot) : V \to L(K, H)$, satisfying

(a.1)' (Coercivity). There exist $\alpha > 0$, $p > 1$ and λ, $\gamma \in \mathbf{R}^1$ such that:

$$2 < A(t, x), x > + \|B(t, x)\|_2^2 \le -\alpha\|x\|^p + \lambda|x|^2 + \gamma, \quad \forall x \in V, \; a.e.t;$$

(a.2)' (Boundedness). There exists $c > 0$ such that

$$\|A(t, x)\|_* \le c\|x\|^{p-1}, \quad \forall x \in V, \; a.e.t;$$

(a.3)' (Measurability).

$$t \in (0, T) \mapsto A(t, x) \in V' \text{ is Lebesgue-measurable } \forall x \in V, \text{ a.e.t}, \forall T > 0;$$

(a.4)' (Hemicontinuity).

$$\xi \in \mathbf{R}^1 \mapsto \langle A(t, x + \xi y), v\rangle \in \mathbf{R}^1 \text{ is continuous for all } x, y, v \in V, \text{a.e.t};$$

(a.5)' (Monotonicity). For all $x, y \in V$, and a.e.t,

$$2\langle A(t, x) - A(t, y), x - y\rangle + \|B(t, x) - B(t, y)\|_2^2 \le \lambda|x - y|^2.$$

(b.1)' There exists $k > 0$ such that

$$\|B(t, x) - B(t, y)\|_2^2 \le k\|x - y\|^2, \quad \forall x, y \in V, \; a.e.t;$$

(b.2)' $t \in (0, T) \to B(t, x) \in L(K, H)$ is Lebesgue-measurable $\forall x \in V, \forall T > 0$.

However, it is not difficult to check that the proofs in [9] are also valid if one assumes some special integral versions of hypotheses (a.1)', (a.2)', (a.5)' and (b.1)' (see also Real [10] for the linear case with variable delays). In fact, it is sufficient to make the following assumptions instead of (a.1)', (a.2)', (a.5)' and (b.1)':

(A.1) There exist $\alpha > 0$, $p > 1$ and $\lambda, \gamma \in \mathbf{R}^1$ such that for all $u \in L^p(\Omega \times (0,T); V) \cap L^2(\Omega \times (0,T); H)$ and all $t \in [0,T]$,

$$2\int_0^t E < A(s,u_s), u_s > \mathrm{d}s + \int_0^t E\|B(s,u_s)\|_2^2 \,\mathrm{d}s \le -\alpha \int_0^t E\|u_s\|^p \,\mathrm{d}s + \lambda \int_0^t E|u_s|^2 \,\mathrm{d}s + \gamma t;$$

(A.2) There exist positive constants $c_1, c_2 > 0$ such that for all $u \in L^p(\Omega \times (0,T); V) \cap L^2(\Omega \times (0,T); H)$,

$$\int_0^T E\|A(t,u_t)\|_*^{p/(p-1)} \,\mathrm{d}t \le \int_0^T (c_1 E\|u_t\|^p + c_2) \,\mathrm{d}t;$$

(A.5) For all $u, v \in L^p(\Omega \times (0,T); V) \cap L^2(\Omega \times (0,T); H)$ and all $t \in [0,T]$,

$$2\int_0^t E\langle A(s,u_s) - A(s,v_s), u_s - v_s\rangle \,\mathrm{d}s + \int_0^t E\|B(s,u_s) - B(s,v_s)|_2^2 \,\mathrm{d}s \le \lambda \int_0^t E|u_s - v_s|^2 \,\mathrm{d}s\,.$$

(B.1) There exists $k > 0$ such that for all $u, v \in L^p(\Omega \times (0,T); V) \cap L^2(\Omega \times (0,T); H)$ and all $t \in [0,T]$,

$$\int_0^t E\|B(s,u_s) - B(s,v_s)\|_2^2 \,\mathrm{d}s \le k \int_0^t E\|u_s - v_s\|^2 \,\mathrm{d}s\,.$$

Denoting $A_1(t,u_t) = A(t,u_t,\phi(\rho(t)))$ and $B_1(t,u_t) = B(t,u_t,\phi(\rho(t)))$ for $u \in L^p(\Omega \times (0,T); V) \cap L^2(\Omega \times (0,T); H)$ and $t \in [0,\delta]$, our existence result will hold if we could prove that A_1 and B_1 satisfy (A.1), (A.2), (A.5) and (B.1). But this follows immediately from assumptions (a.1), (a.2), (a.5) and (b.1).

Let us prove (A.1), for instance. Indeed, for $x_\cdot \in L^p(\Omega \times (0,\delta); V) \cap L^2(\Omega \times (0,\delta); H)$ and $t \in [0,\delta]$, we obtain

$$\begin{aligned}
&2\int_0^t (E < A_1(s,x_s), x_s > + E\|B_1(s,x_s)\|_2^2) \,\mathrm{d}s \\
=&2\int_0^t (E < A(s,x_s,\phi(\rho(s))), x_s > + E\|B(s,x_s,\phi(\rho(s)))\|_2^2) \,\mathrm{d}s \\
\le& -\alpha \int_0^t E\|x_s\|^p \,\mathrm{d}s + \lambda \int_0^t E|x_s|^2 \,\mathrm{d}s + \theta \int_0^t E|\phi(\rho(s))|^2 \,\mathrm{d}s + \gamma t \\
\le& -\alpha \int_0^t E\|x_s\|^p \,\mathrm{d}s + \lambda \int_0^t E|x_s|^2 \,\mathrm{d}s + (\beta^2|\theta| \sup_{-h \le s \le 0} E|\phi(s)|^2 + \gamma)t.
\end{aligned}$$

Now, since (A.2), (A.5) and (B.1) can be similarly proved, we are then in a position to obtain existence of the strong solution on $[0,\delta]$. By recurrence, the problem can be solved on $[n\delta, (n+1)\delta]$ for all natural number $n \ge 0$ and therefore on $[0,+\infty)$.

Case 2. In this case, we can choose a strictly increasing sequence $\{\delta_n\}$ such that $\delta_n \to +\infty$, $\delta_0 = 0$, $\rho(\delta_{n+1}) = \delta_n$ and $\rho(t) \in [\delta_{n-1}, \delta_n]$ for all $t \in [\delta_n, \delta_{n+1}]$ and for all $n \ge 0$, where we denote $\delta_{-1} = -h$. Consequently, our equation can be solved on each $[\delta_n, \delta_{n+1}]$ exactly as in Case i), and further on $[0,+\infty)$.

Case 3. Firstly, we can prove existence of the strong solution on $[0,T^*)$ in the same way as Case ii). Then, it is not difficult to show that the solution X_t tends to certain $X_{T^*} \in L^2(\Omega, F_{T^*}, P; H)$, as $t \to T^*$. Now, on $[T^*,+\infty)$ the problem becomes

$$X_t = X_{T^*} + \int_{T^*}^t A(s,X_s,X_s) \,\mathrm{d}s + \int_{T^*}^t B(s,X_s,X_s) \,\mathrm{d}W_s,$$

which obviously has a unique strong solution and our proof is now complete.