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Smooth fitting of B-spline curve with constrained length

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Abstract. In this paper, the authors present a method to construct a smooth B-spline curve which fairly fits 3D points and at the same time satisfies the length constraints. By using the Lagrange multiplier's conditional extreme, resorting to Broyden method, and then finding the least square solution of the control points, we can obtain the fair quasi-fitting modeling B-spline curve.

Keywords: B-spline, length constraint, curve fitting, smoothing of data

1. Introduction

In real situation and application, Beyond interpolation, approximation and continuity, the shape of a curve should satisfy some certain features, that is constraints. There're two methods to achieve these constraints. First, we can adjust the existing curve to satisfy the requirements [1], [2]. Another method is to limit and control, when generating the curve, to satisfy different requirements. In this paper, we use the later method to discuss the fair quasi-fitting curve construction which has length constraints. Cubic B-spline can achieve C^2 continuity, and has the advantage of local revision and easy control of boundary condition, so it's convenient to resolve this kind of problem by using B-spline. Considering that amending smooth fitting always results global revision, we use curve approach instead of curve interpolation.

2. Discuss

2.1. The Question

Giving a set of data points $\left\{\vec{q}_i\right\}_{i=0}^{m}$ in R3, we need a global smooth fitting cubic uniform B-spline curve S_m :

$$\vec{p}(t)=\sum_{i=0}^{m+2} B_i(t)\vec{v}_i(0\le t\le 1)\cdots(1)$$

Fit the giving points $\left\{\vec{q}_i\right\}_{i=0}^{m}$,or $\vec{p}(t_i)=\vec{q}_i$, and satisfying $L_i=l_i$, here l_i is a constant, L_i is the curve length of i th segment of curve S_m . Here the B-spline basis fuctions B_i are even knot vectors.

$U=\{u_0=0,u_0,\cdots,u_{m+5},u_{m+6}=1\}$ can be deduced by DeBoor expressions. During approach process, parameters are calculated by accumulated chord length, namely:

$$t_0=0, t_k=\sum_{i=1}^{k}\left|\vec{q}_i-\vec{q}_{i-1}\right|, k=1,2,\ldots,m+2$$

2.2. Cubic B-spline curve with constrains

The expression of i th segment of cubic uniform B-spline curve has the form below:

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$$\vec{r}_i(u) = (1, u, u^2, u^3)\frac{1}{6}\begin{pmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ 1 & 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} \overrightarrow{v_i} \\ \overrightarrow{v_{i+1}} \\ \overrightarrow{v_{i+2}} \\ \overrightarrow{v_{i+3}} \end{pmatrix} (0 \le u \le 1, i = 0, \cdots, m-1) \cdots (2)$$

here $\vec{v}_i (i = 0,1,\cdots, m+2)$ is the control point of cubic B-spline curve. (2) and (1) has the relation below:

when $t_i \le t \le t_{i+1}$, $u = \dfrac{t - t_i}{t_i - t_{i-1}}$.

The end point $\vec{p}_i, \vec{p}_{i+1} (i = 0,1,\cdots m-1)$ of the curve $\vec{r}_i(u)$ is :

$$\vec{p}_i = \vec{r}_i(0) = \frac{1}{6}(\vec{v}_i + 4\vec{v}_{i+1} + \vec{v}_{i+2})$$
$$\vec{p}_{i+1} = \vec{r}_i(1) = \frac{1}{6}(\vec{v}_{i+1} + 4\vec{v}_{i+2} + \vec{v}_{i+3}) .$$

The curve length of the curve $\vec{r}_i(u)$ is :

$$L_i = \int_0^1 (\vec{r}_i^{\,'}(u), \vec{r}_i^{\,'}(u)) du \cdots (3)$$

where (g, g) indicates an inner product.

For the fixed curve length of every given segment, the constraints is expressed as

$$L_i = l_i .$$

And also to ensure that S_m is adequately smooth curve, we usually resort to the minimum of energy integral according to physical theory namely:

$$E = \sum_{i=0}^{m-1} E_i = \sum_{i=0}^{m-1} (E_{2i} + E_{3i}) = \sum_{i=0}^{m-1} (\beta \int_0^1 \left(\frac{d^2 \overrightarrow{r_i(u)}}{du^2} \right)^2 du + \gamma \int_0^1 \left(\frac{d^3 \overrightarrow{r_i(u)}}{du^3} \right)^2 du) \cdots (4)$$

taking its minimum, where $0 \le \beta, \gamma \le 1$ are known weight coefficients and $\alpha + \beta = 1$.

In order to ensure curve S_m approaching to the given point $\left\{ \vec{q}_i \right\}_{i=0}^{m}$, one need to control the distance between point $\vec{p}_i$ and $\vec{q}_i$ in the curve. Therefore, let

$$D = \sum_{i=0}^{m} \alpha_i (\overrightarrow{p_i} - \overrightarrow{q_i})^2$$

here weights α_i are given. Curve S_m is achieved when normal function $J = E + D$ taking minimum under the constraint condition $L_i = l_i$.

2.3. Algorithm Design

2.3.1 The expression of E

Matrix mark can be used here to represent the constraint factor and normal function of smooth Measurement. (2) can be expressed as

$$\overrightarrow{r_i(u)} = BV_i$$

where

$$V_i = \begin{pmatrix} \overrightarrow{v_i} \\ \overrightarrow{v_{i+1}} \\ \overrightarrow{v_{i+2}} \\ \overrightarrow{v_{i+3}} \end{pmatrix} \qquad \mathrm{B}=\frac{1}{6}[1,u,u^2,u^3]\begin{pmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

and we can deduce the derived function

$$\overrightarrow{r_i^{\,p}(u)} = \frac{d^p B}{du^p} B_p V_i\,(p=1,2,3)$$

combining with（4）we obtain

$$E_{2i} = \beta\int_0^1 v_i^T B_2^T B_2 v_i du = \beta v_i^T M_2 v_i$$

$$E_{3i} = \gamma\int_0^1 v_i^T B_3^T B_3 v_i du = \gamma v_i^T M_3 v_i$$

where

$$M_2 = \frac{1}{6}\begin{pmatrix} 2 & -3 & 0 & 1 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 1 & 0 & 3 & 2 \end{pmatrix} \quad \mathrm{M}_3 = \begin{pmatrix} 1 & -3 & 3 & -1 \\ -3 & 9 & -9 & 3 \\ 3 & -9 & 9 & -3 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

2.3.2 The expression of L_i

For

$$L_i = \int_0^1 (\overrightarrow{r_i^{'}(u)}, \overrightarrow{r_i^{'}(u)})du\,.$$

Considering (2), and using $S\,\mathrm{i}\,mpson$ formula/expressions, calculate L_i:

$$L_i = \frac{1}{6}g_i(0) + \frac{4}{6}g_i(\frac{1}{2}) + \frac{1}{6}g_i(1)$$

where

$$g_i(0) = \{(-\frac{1}{2}x_i + \frac{1}{2}x_{i+2})^2 + (-\frac{1}{2}y_i + \frac{1}{2}y_{i+2})^2\}^{\frac{1}{2}}$$

$$g_i(\frac{1}{2}) = \{(-\frac{1}{8}x_i - \frac{5}{8}x_{i+1} + \frac{5}{8}x_{i+2} + \frac{1}{8}x_{i+3})^2 + (-\frac{1}{8}y_i - \frac{5}{8}y_{i+1} + \frac{5}{8}y_{i+2} + \frac{1}{8}y_{i+3})^2\}^{\frac{1}{2}}$$

$$g_i(1) = \{(-\frac{1}{2}x_{i+1} + \frac{1}{2}x_{i+3})^2 + (-\frac{1}{2}y_{i+1} + \frac{1}{2}y_{i+3})^2\}^{\frac{1}{2}}$$

2.3.3 Minimizing function

Based on Lagrange Multiplier Method, the normal function of smooth fitting curve with constraints length $L_i = l_i$ can be expressed as:

$$J = D + E + \sum_{i=0}^{m-1}\lambda_i(L_i - l_i)\,.$$

And through fixed $\overrightarrow{v_i}(i=0,1,\cdots,m+2), \lambda_i(i=0,1,\cdots m-1)$, minimize normal function J. According to variation theory,

$$\begin{cases} \dfrac{\partial J}{\partial \overrightarrow{v_i}} = 0(i = 0,1,\cdots m+2) \\ \dfrac{\partial J}{\partial \lambda_j} = 0(j = 0,1,\cdots m-1) \end{cases} \cdots(5)$$

(5) can be expressed as

$$\begin{cases} \dfrac{\partial D}{\partial v_i} + \dfrac{\partial E}{\partial v_i} + \sum_{i=0}^{m-1} \lambda_i \dfrac{\partial L_i}{\partial v_i} = 0(i = 0,2,\cdots m+2) \\ L_j - l_j = 0 \qquad (j = 0,1,\cdots m-1) \end{cases} \ldots(6)$$

for the convenience of calculation ,denote D_x, D_y as the components in the x direction and y direction of vector D respectively. Similarly, $E_x, E_y, v_{i,x}, v_{i,y}$ indicate the different components of E and v_i . Finally (6) can be expanded as

$$f_1 = \frac{\partial D_x}{\partial v_{1,x}} + \frac{\partial E_x}{\partial v_{1,x}} + \sum_{i=0}^{m-1} \lambda_i \frac{\partial L_i}{\partial v_{1,x}} = 0$$

$$f_2 = \frac{\partial D_x}{\partial v_{2,x}} + \frac{\partial E_x}{\partial v_{2,x}} + \sum_{i=0}^{m-1} \lambda_i \frac{\partial L_i}{\partial v_{2,x}} = 0$$

$$\vdots$$

$$f_{m+3} = \frac{\partial D_x}{\partial v_{m+3,x}} + \frac{\partial E_x}{\partial v_{m+3,x}} + \sum_{i=0}^{m-1} \lambda_i \frac{\partial L_i}{\partial v_{m+3,x}} = 0$$

$$f_{m+4} = \frac{\partial D_y}{\partial v_{1,y}} + \frac{\partial E_y}{\partial v_{1,y}} + \sum_{i=0}^{m-1} \lambda_i \frac{\partial L_i}{\partial v_{1,y}} = 0$$

$$\vdots$$

$$f_{2m+6} = \frac{\partial D_y}{\partial v_{m+3,y}} + \frac{\partial E_y}{\partial v_{m+3,y}} + \sum_{i=0}^{m-1} \lambda_i \frac{\partial L_i}{\partial v_{m+3,y}} = 0$$

$$f_{2m+7} = L_1 - l_1 = 0$$

$$\vdots$$

$$f_{3m+6} = L_m - l_m = 0$$

Broyden method can be adopted here to get the unknown $\overrightarrow{v_i}(i = 0,1,\cdots m+2)$ in the 3m+6 nonlinear equations above. And through (2), one can construct the curves which satisfy the condition. Steps are:

1. Give a initial control point $(v_0^{(0)}, v_1^{(0)}, \cdots, v_{m+2}^{(0)})$ and Lagrange coefficients $\lambda_0^{(0)}, \lambda_1^{(0)}, \cdots \lambda_{m-1}^{(0)}$, and denote $X_0 = \{v_{0,x}^{(0)}, v_{1,x}^{(0)}, \cdots, v_{m+2,x}^{(0)}, v_{0,y}^{(0)}, v_{1,y}^{(0)}, \cdots v_{m+2,y}^{(0)}, \lambda_0^{(0)}, \lambda_1^{(0)}, \cdots \lambda_{m-1}^{(0)}\}$

2. Calculation

$$F' = \begin{pmatrix} \dfrac{\partial f_1}{\partial v_{0,x}}, \cdots, \dfrac{\partial f_1}{\partial v_{m+2,x}}, \dfrac{\partial f_1}{\partial v_{0,y}}, \cdots \dfrac{\partial f_1}{\partial v_{m+2,y}}, \dfrac{\partial f_1}{\partial \lambda_0}, \cdots, \dfrac{\partial f_1}{\partial \lambda_{m-1}} \\ \dfrac{\partial f_2}{\partial v_{0,x}}, \cdots, \dfrac{\partial f_2}{\partial v_{m+2,x}}, \dfrac{\partial f_2}{\partial v_{0,y}}, \cdots \dfrac{\partial f_2}{\partial v_{m+2,y}}, \dfrac{\partial f_2}{\partial \lambda_0}, \cdots, \dfrac{\partial f_2}{\partial \lambda_{m-1}} \\ \vdots \qquad \vdots \qquad \vdots \\ \dfrac{\partial f_{3m+6}}{\partial v_{0,x}}, \cdots, \dfrac{\partial f_{3m+6}}{\partial v_{m+2,x}}, \dfrac{\partial f_{3m+6}}{\partial v_{0,y}}, \cdots \dfrac{\partial f_{3m+6}}{\partial v_{m+2,y}}, \dfrac{\partial f_{3m+6}}{\partial \lambda_0}, \cdots, \dfrac{\partial f_{3m+6}}{\partial \lambda_{m-1}} \end{pmatrix}$$

3. $H = F^{'(-1)}(X_0)$
4. $X_1 = X_0 - HF(X_0) \quad S = X_1 - X_0$

$$Y = F(X_1) - F(X_0)$$

$$H = H + (S - HY)\frac{S^T H}{S^T HY}$$

5. If $\|s\|$ is less than error tolerance, then stop. Else set $X_1 = X_0$, continue step 4.

2.4. Experiments

Here, we use our algorithm to implement the Smooth Fitting to unit-semicircle with curve length constraints. Select three point in the circumference, (-1,0),(0,1),(1,0). First, we implement the common least squares fitting using cubic B-spline curve（Fig. 1）. Then treat with curve length constraint conditions（Fig. 2）.

Choose $\alpha = 300, \beta = \gamma = 0.5, l_1 = l_2 = \frac{\pi}{2}$, we can get control points:

$$-1.74,-2.81507$$
$$-1.1074,0.311528$$
$$-7.1054\times 10^{-6},1.41913$$
$$-1.10737,0.311491$$
$$-1.74018,-2.81479$$

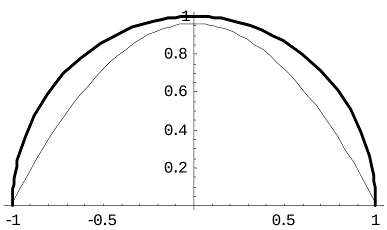

Fig. 1. without Constrained Length

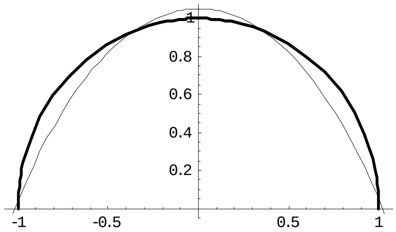

Fig. 2. with Constrained Length

i	1	2
$l_i - L_i$	-1.59641×10^{-7}	0.0000105612

One can compare the method of least squares (Figure 1) with the new method (Figure 2) . The results show that we can get better performance using the algorithm presented here. Figure 3 shows the smooth fitting B-spline curves under different curve length constraints（ L_i choose 3, pi, 3.3, 3.5 respectively）.

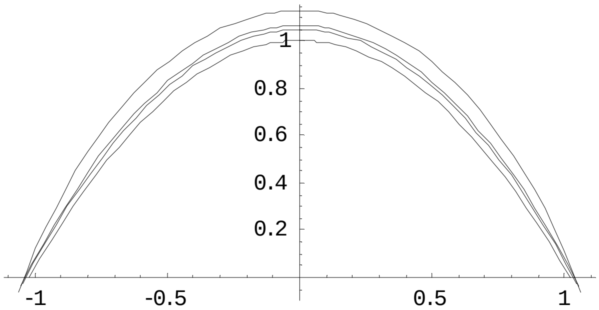

Fig. 3. B-spline under different Constrained Length

Further, we can select the key points during approximation as constraint conditions, and we can construct more ideal curves.

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